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Gazing at r-star: A Hysteresis Perspective

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Abstract

Many explanations for the decline in real interest rates over the last 30 years point to the role that population aging or rising income inequality plays in increasing the long-run aggregate demand for assets. Notwithstanding the importance of such factors, the starting point of this paper is to show that the major change driving household asset demand over this period is instead an increased desire—for a given age and income level—to hold assets. We begin by presenting a simple explanation for this pattern that relies on integrating retirement and inter-temporal substitution motives in saving decisions. We then show how the interaction of these two saving motives can have profound implications in terms of the shape of asset demands, the possibility of multiple steady state real interest rates, and a potential role for monetary policy to influence the long-run evolution of real rates. The framework highlights how an inflationary episode followed by a strong monetary response, as we are currently witnessing, can have long-term implications for real interest rates.

Topics: Monetary policy; Interest rates; Inflation targets; Monetary policy framework; Inflation and prices; Fiscal policy; Economic models

JEL codes: E21, E52, E31, E43, E58, E62, G51, H6

Résumé

Bon nombre des raisons avancées pour expliquer le recul des taux d'intérêt lors des 30 dernières années soulignent le rôle joué par le vieillissement de la population ou la montée des inégalités de revenu dans la croissance de la demande agrégée d'actifs sur le long terme. Sans nier l'importance de ces facteurs, cette étude montre tout d'abord que le principal changement à l'origine d'une hausse de la demande d'actifs au cours de cette période a plutôt été le désir accru des ménages (selon leur classe d'âge et catégorie de revenu) d'acquérir des actifs. Nous commençons par présenter une explication simple de cette tendance en nous appuyant sur les besoins d'épargne motivés par la retraite et la substitution intertemporelle. Nous montrons ensuite que l'interaction de ces deux motifs d'épargne peut avoir des conséquences majeures sur la forme des différentes demandes d'actifs, sur la multiplicité des taux d'intérêt envisageables à l'état stationnaire et sur l'influence potentielle de la politique monétaire dans l'évolution à long terme des taux réels. Le cadre met en lumière les retombées que peut générer à long terme sur les taux d'intérêt réels un épisode inflationniste suivi d'une forte réaction des autorités monétaires (comparable à la situation actuelle).

Sujets : Politique monétaire; Taux d'intérêt; Cibles en matière d'inflation; Cadre de la politique monétaire; Inflation et prix; Politique budgétaire; Modèles économiques

Codes JEL : E21, E52, E31, E43, E58, E62, G51, H6

1 Introduction

In most advanced economies, prior to the pandemic, real interest rates had been trending down since the mid to late 1980s (see Figure 1).[1] The most common explanation for this trend is that economies had been experiencing an increased demand for assets that pushed down interest rates and increased the price of other assets, such as stocks and real estate. Important forces cited for inducing such an increase in asset demand include population aging and increased inequality.[2]

While these factors are certainly relevant, we begin this paper by showing that a key element driving the increased demand for assets over the last thirty years comes from households' desire to hold more assets for given age and income levels. Notably, we document that the increase in the wealth-to-income ratio observed over this period is largely a within group phenomenon as opposed to resulting mainly from changes in demographics or income distribution. Furthermore, we show that saving behavior supports an interpretation of the observed higher wealth holdings as reflecting desired increases as opposed to temporarily above target levels due to valuation effects. In contrast, commonly used models of savings suggest that, if the aggregate demand for savings changes due a change in the composition of the population, then the desired assets holdings of households of a given age and income level should actually decrease. Most importantly, this implication remains true even in the presence of valuation effects induced by lower interest rates which increase the price of assets. Given the tension between this empirical observations and standard theories, this paper is aimed at both offering an explanation of this household level observation and highlighting its general equilibrium implications with a special emphasis on its implication for monetary policy.

When looking to explain why households with similar income and demographic characteristics may have increased their desired asset-to-income ratios as interest rates fell, retirement needs in a low income environment come quickly to mind.[3] This is nicely ex-

[1]The influential empirical study by Laubach and Williams (2003) provides estimates showing that the natural rate of interest r^* has been declining.

[2]A vast literature examines the sources of the decreasing trend in real interest rates. Borio et al. (2017) provide an excellent survey of the literature on these issues. Several hypotheses about these sources have been proposed: demographics (Summers (2014), Eggertsson and Mehrotra (2014), Eichengreen (2015)), and Goodhart and Pradham (2020); a productivity slowdown (Gordon (2017)); a global saving glut and/or lack of safe assets (Bernanke (2005), Caballero et al. (2008), Gourinchas et al. (2020), and Acharya and Dogra (2022)); a decline in desired investment (Rachel and Smith (2017)); a rise in inequality (Mian et al. (2020), Auclert and Rognlie (2020), Fagereng et al. (2019), and Rachel and Smith (2017)).

[3]While we focus on retirement motives to help explain asset holdings, bequest motives likely play a similar role. See Beaudry and Meh (2021).

pressed by Raghuram Rajan, former governor of the Reserve Bank of India:[4]

"...savers put more money aside as interest rates fall in order to meet the savings they think they will need when they retire."

With this in mind, we begin by presenting a model of asset accumulation in a continuous time overlapping generations (OLG) environment that allows for inter-temporal substitution and retirement motives to compete. The model builds on Blanchard (1985) and Yaari (1965), and is closest to Gertler (1999).[5] The framework is sufficiently tractable to allow the relationship between desired asset-to-income ratios and interest rates to be derived analytically. In particular, we show that if the inter-temporal elasticity of substitution is less than 1 (which is the more empirically plausible case), then long-run asset demands become C-shaped, with lower interest rates motivating households to increase their asset-to-income ratios in line with what we observed in the data over the 1989-2019 period.

After laying out the partial-equilibrium setting, we move to explore general-equilibrium (GE) implications. The main GE implication we focus upon is the possibility of multiple steady state real interest rates (r^*). Given C-shaped long-run asset demands, the possibility of two or more steady state real interest rates is easily understood as the supply for assets can readily intersect such a demand curve more than once. In order to examine the dynamic properties of such settings, and especially highlight why monetary policy may affect which real interest arises in the long run, we embed our OLG households in a sticky price environment where monetary authority sets the nominal interest rate with a Taylor rule, subject to the effective lower bound (ELB) constraint.

Our main results with respect to monetary policy is that, even if money is neutral in the long run, monetary policy can nevertheless have important long-run effects by influencing the stability properties and the basins of attraction of different steady state real interest rates.[6] In particular, an aggressive monetary policy regime can make a high-real-rate environment fragile to small negative inflation shocks and favor the convergence to a low-real-rate environment. In such a case, large inflation shocks or large increases in public debt can help get the economy out of the low-inflation, low-real-rate trap. Specifically, we show why large increases in public debt could lead to a discontinuous jump in the long run equilibrium real rate r^*, while large inflation shocks could move the economy away from a low r^* basin of attraction, to a high r^* basin of attraction, both features being potentially relevant for understanding the current post pandemic period.

[4] See Rajan (2013).

[5] Galí (2021) introduces retirement in a similar fashion in a New Keynesian model with bubbles.

[6] Fernández-Villaverde et al. (2021) and Rungcharoenkitkul and Winkler (2022) also consider how monetary policy can affect the long-run level of real interest rates.

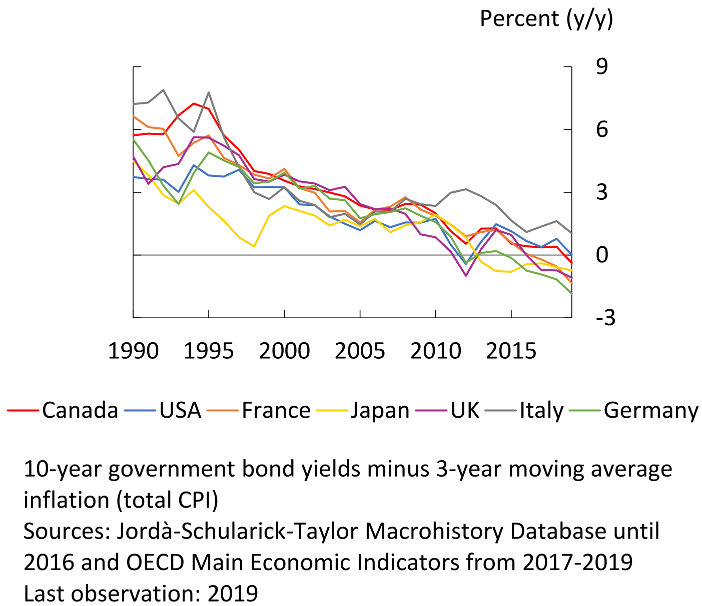


Figure 1
Long-term interest rates for G7 countries from 1990 to 2019

Our results regarding the role of monetary policy in affecting long-run real interest rate outcomes give novel support to the view of many market commentators that the long-term downward trend in real interest rates may have reflected central banks' willingness to decrease interest rates aggressively in every downturn, but being hesitant to increase them as rapidly in upturns. This view has also been advanced by policymakers such as Borio et al. (2017), who provide evidence that over a long history persistent changes in real interest rates coincide with changes in monetary regimes. Recently, Bianchi et al. (2022) estimate that two-thirds of the fall in the real interest rate since the early 1980s may be due to shifts in the parameters of the monetary policy rule. All this points to a possibly underrated role of nominal factors and monetary policy in affecting real interest rates over long horizons.[7]

Our paper is also related, but distinct, to the literature on equilibrium multiplicity and the lower bound on nominal interest rates. From the influential work of Benhabib et al. (2001a,b, 2002), and related literature, we know that an ELB constraint can give rise to multiple equilibria.[8] However, most of this literature is not aimed at explaining changes

[7]Gourinchas et al. (2020)'s focus on financial cycles, especially the leveraging cycle that accompanied the boom and bust in the 1930s and 2000s, for explaining the short-term real interest rate movements is consistent with the role of monetary policy.

[8]Expectations-driven liquidity traps have also been applied to fiscal policy, optimal monetary policy and open economy issues. See for example, Mertens and Ravn (2014), Bilbiie (2018), Nakata and Schmidt (2021), Aruoba et al. (2018), and Kollmann (2018).

in real interest rates, as the long-run real interest rate in the ELB regime is the same as the one in the non-ELB regime. In contrast, in our set-up, we show that the stable real interest rate that emerges when the ELB constraint is binding is lower than when it is not binding. Therefore, a shift from a non-ELB-constrained equilibrium toward an ELB-constrained equilibrium is associated with a fall in real interest rates.

The remainder of the paper is organized as follows. Section 2 exploits household level data to examine how asset positions changed over the 30 years prior to the pandemic. We show that for households with similar income and age, asset holding increased substantially even as interest rates fell significantly. Section 3 presents an OLG model—similar in spirit to that of Gertler (1999)—that integrates both inter-temporal substitution forces and retirement preoccupations in a manner capable of explaining the household level observations by giving rise to C-shaped asset demands. Section 4 embeds this OLG structure in a GE setting. The section begins with an environment without nominal rigidities to show how and when the real side of this economy generates more than one steady state real interest rate. Then the section introduces sticky wages/prices to highlight how monetary policy can affect long term real rates in such en environment. Section 5 enriches the environment by including a claim on a productive asset—where the price of the asset increases when interest rates decrease—in order to highlight the robustness of our results to asset valuation effects. Section 6 offers a more general formulation of the model to offer further robustness analysis. Section 7 concludes.

2 The Between versus Within Household Decomposition of Aggregate Asset Holdings over 30 years: 1989-2019

While real interest rates were declining over the last several decades (as seen in Figure 1), Figure 2 indicates that the aggregate wealth-to-income ratio in the US increased significantly and the aggregate saving rate mildly decreased.[9] The question we want to address is how best to interpret such observations; should they mainly be interpreted as reflecting between group (composition) effects or do they instead largely reflect within group choices. In particular, we want to ask if measured within group changes in asset holdings over the period appear consistent with asset demands that are monotonically increasing in real interest rates or if they place in doubt such monotonicity.

Before moving to the empirical analysis, it is important to note that such analysis will

[9] Mian et al. (2021b) provide an extensive analysis of the evolution of household saving behavior using the ratio of saving to national income since 1950, including over the period from 1995 to 2019, when the natural rate of interest fell to an extremely low level.

not in itself provide any direct evidence of multiple steady state equilibria. Instead it is aimed at shedding light on whether long-run asset demands are likely monotonically increasing in real interest rates — which would make multiple steady states very unlikely — or whether they point to an alternative configuration where asset demands are decreasing in interest rates at least over a range of values — which makes multiple steady state equilibria much more likely. The latter case is a necessary condition for the sort of multiple steady state equilibria we will study in the theory sections.

Between group (compositional effect) explanation. A common explanation for the rise in aggregate wealth-to-income ratio is that it reflects increase in demand for assets induced by changes in demographics and income distribution. As the population aged, and more income was concentrated in higher income groups, the demand for wealth increased. This put downward pressure on interest rates, which through valuation effects among others, raised the effective supply of wealth. The higher savings of the older and richer population was compensated by a decreased incentive to save by the population at large due to lower interest rates, leaving the overall savings rate relatively flat. Such narrative is essentially a "between" group narrative which relies on compositional changes in types of individuals to explain the increased demand for wealth. In particular, it suggests that, for similar age and income levels, as interest rates fell, households at large may have saved less and accumulated less wealth, but due to the changes in the age and income distributions of the population, the aggregates behaved very differently from the prevailing individual level outcomes.[10]

Within group explanation. At the other end of the spectrum, a multiple steady state equilibrium story suggests that the joint pattern of increased aggregate wealth, lower interest rates and slightly decreasing aggregate saving rates potentially reflects a "within" group phenomenon. In this alternative view, we still have that as interest rates decrease, the effective supply of wealth increases through valuation effects. However, now the endogenous increase in supply does not need to be primarily driven by an exogenous increase in demand. Instead, it can be accompanied by a simultaneous endogenous response of demand whereby households choose to hold more wealth at lower interest rates due to the income effect that is associated with interest rate movements. Note that such a multiple steady state equilibrium story would not negate the possibility of between group effects arising from demographics and inequality, but it does not rely on them. In fact, a change in demographics or income inequality could complement this type of multiple steady state equilibrium story by helping

[10] In this type of scenario, the between group effect should actually explain more than 100 percent of the increase in asset demand as the within component should be negative.

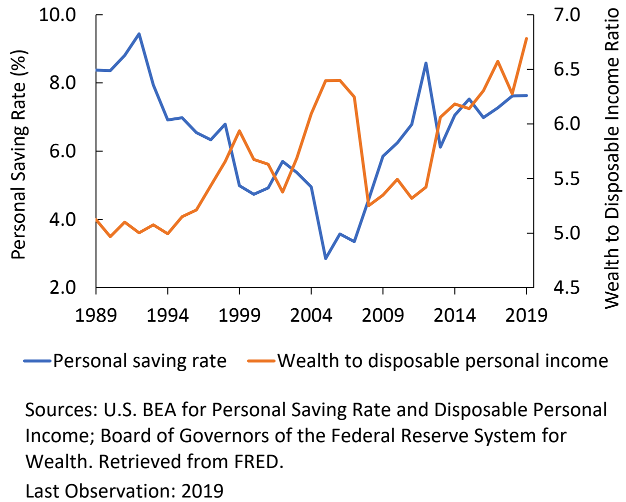


Figure 2
Household saving rates and aggregate wealth-to-income ratios in the US from 1989 to 2019

to explain why the economy may have switched from a high real interest rate to a low real interest rate at this time in history.

The above discussion underlines the relevance of understanding the relative roles of within versus between group effects in explaining the increased wealth holdings in the US over the last three decades. To do so, we use the Survey of Consumer Finances (SCF) and focus on the difference in asset holdings across household groups between 1989 and 2019. We choose this period for our analysis as it corresponds quite closely to the period of decreasing real interest rates presented in Figure 1. Furthermore, by looking at this thirty-year difference, we hope to minimize higher frequency movements in wealth accumulation dynamics associated with business cycles forces and crises.

The SCF is the most comprehensive source of data on household-level wealth and its components in the United States. It also has a consistent sampling methodology, over-sampling the rich, in all the survey waves between 1989 and 2019, which is useful for our analysis. The survey has between 3 and 5.5 thousand households, depending on the year, and our results use weights throughout. For our baseline definition of wealth, given the importance of retirement considerations in the theory sections, we supplement the SCF data with the estimates on defined benefit (DB) pensions of households from Sabelhaus and Volz (2020), which have been widely adopted in the related literature.[11] Thus, our

[11] SCF only directly measures pensions in defined contribution plans. Defined benefit pension entitlements

measurement of wealth, including DB pension wealth and excluding social security wealth, also lines up well with that reported in Financial Accounts of the Federal Reserve (FA).[12] In this section, we primarily report findings using the SCF (plus DB pensions) data, which allows us to establish our results using several approaches that require micro-level data.[13] These data show similar upward movement in the dynamics of the total household wealth-to-income ratio as in the aggregate accounts of the United States (FA and NIPA), although the magnitudes are somewhat smaller for the latter.

The aggregate wealth-to-income ratios in 1989 and 2019 we use for our decompositions are calculated from the SCF as the ratio of the sum of the wealth of each household to the sum of incomes of each household, respectively denoted $\left(\frac{w}{y}\right)_{89}$ and $\left(\frac{w}{y}\right)_{19}$. In our baseline, we include all household wealth either directly reported in or constructed from the SCF (including estimates of DB pensions from Sabelhaus and Volz (2020)) in our measure of wealth.[14] To explore robustness, we also provide calculations where we exclude wealth in a primary residence from the baseline measure of wealth.[15] Our measure of income is the total of components available in SCF, and does not vary with the definition of wealth used either in the baseline or the robustness scenarios.[16]

calculated by Sabelhaus and Volz (2020) represent their termination value, which is the legal obligation of employer plans, and corresponds to the measure of defined benefit pension entitlements (both funded and unfunded) in Financial Accounts. We thank the authors for sharing their estimates with us.

[12]Sabelhaus and Volz (2020) also provide estimates of social security wealth using SCF, in addition to defined benefit pension wealth, using both termination and expected values of such wealth. However, similar to Auclert et al. (2021) conceptually we think of social security wealth as a future transfer, and do not include it in our measure of household wealth.

[13]Later in the section we also report results from micro data scaled to Financial Accounts and National Income and Product Account (NIPA) aggregates.

[14]In particular, we do not exclude vehicles as a measure of consumer durables from household wealth in the SCF. Consistent with this approach, our measure of saving rates in the next section also includes consumer durables, as the FA concept of saving rate, but unlike the NIPA measure of saving rate. However, the difference in saving rates implied by inclusion of vehicles is very small. There is some difference associated with using NIPA or FA saving, as the FA saving is more noisy. However, the dynamics over time of these saving rates are quite similar.

[15]Our preferred measure of wealth includes housing. An additional reason for this has to do with the subsequent analysis of group-wise changes in wealth and saving rates in Section 2.1. We construct saving rates by components of household wealth following the approach in Mian et al. (2021b). Thus, while it is also possible to exclude the components of housing wealth – both in assets and liabilities – from the construction of saving rates, the relevance of this measure in comparison to other studies using standard measures of saving from the data is less clear. Saving in housing expressed as the net new housing also represents a non-trivial component of saving.

[16]Following Fagereng et al. (2019) and Eika et al. (2020) we have also examined the case where we include in the SCF definition of income a measure of imputed housing rents of homeowners, constructed by distributing NIPA reported rents according to the value of housing of SCF respondents. When applying this definition of income with the baseline measure of wealth, we find that the contribution of the within-group component to the overall change in the wealth-to-income ratio is largely unchanged. For this reason, we

The aggregate wealth-to-income ratio in the SCF increased from 5.61 in 1989 to 8.43 in 2019, which is an increase of about 2.82. This is the increase associated with an inclusive wealth measure from the SCF. When we exclude net housing wealth from this measure, the increase in the ratio is of similar magnitude at 2.65. The increases are all substantial relative to 1989 levels.[17] To examine the within versus between components of increased wealth holdings, we apply a simple shift-share methodology in the main text, and report robustness results using a regression based decomposition in Appendices B.2 and B.3.

For the shift share analysis, we place households in I bins, with N_i households in a bin $i = 1, ..., I$. The change in the aggregate wealth-to-income ratio can be decomposed as follows:

$$\left(\frac{w}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89} = \underbrace{\sum_i \left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{89} \left[\left(\frac{y_i}{y}\right)_{19} - \left(\frac{y_i}{y}\right)_{89}\right]}_{\text{between group or compositional effect}} + \underbrace{\sum_i \left(\frac{y_i}{y}\right)_{19} \left[\left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{19} - \left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{89}\right]}_{\text{within group}}, \quad (1)$$

where the first summation term represents the between group component, using 1989 as the base year for income and wealth profiles, and the second one represents the within group component.[18] In this expression, y_i is the total income in bin i, $\bar{y}_i$ is the average income in bin i, $\bar{w}_i$ is the average wealth in bin i and finally y is the total income across all bins. All nominal variables are converted into real variables indexed in 2019 dollars. As can be seen from Equation (1), the changes in the total wealth-to-income ratio can be divided into the between group component determined by the shift in the share of income going to each of the individual groups (y_i/y) and the within group component determined by changes in the (average) wealth-to-income ratio of each group $(\bar{w}_i/\bar{y}_i)$. If the wealth-to-income ratios of individual groups were stable across time (e.g., $(\bar{w}_i/\bar{y}_i)_{19} = (\bar{w}_i/\bar{y}_i)_{89}$ for all groups i), the change in the aggregate wealth-to-income ratio would need to be fully explained by the between group component (i.e., by the change in income shares alone). However, at the other extreme, if the income and age distributions remained stable across time (e.g., if $(y_i/y)_{19} = (y_i/y)_{89}$ for all groups i), then the within group components would need to

keep the original measure of income from SCF in our baseline results.

[17] In the scaled wealth and income data, the ratio changes by 171pp from 4.27 in 1989 to 5.98 in 2019. The literature has also used other definitions of wealth ratios, for example, normalized by GDP. While the exact changes in these ratios may depend on what goes into their numerator/denominator, they all have increased substantially over time.

[18] In Appendix B.3, we clarify the close relationship between the shift-share and the regression based decomposition. This discussion helps to highlight under what conditions the first component represents the between-group component and the second component represents a within-group component.

account for all the change in the aggregate wealth-to-income ratio.[19]

We start by dividing the population households into age groups, defined by the age of the head of the household, to look narrowly at the effects of demographic changes in isolation. Then, we divide the population of households into income groups to examine only the effects of changes in the current income distribution. Finally, in our preferred specification, we combine the two and place households into age-income specific bins.[20] The results of the shift share analysis for these different groupings are presented in Tables 1 and 2.

Table 1
Shift Share Decomposition of the Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019

Groups	Total Change	Between	Within	Fraction due to Within (%)
		Age Groups		
5 age groups	2.819	0.944	1.875	66.5
12 age groups	2.819	0.984	1.835	65.1
		Income Groups		
6 income groups	2.819	0.175	2.644	93.8
12 income groups	2.819	0.179	2.640	93.6

Note: The 5 age groups are: 18-34, 34-35, 35-44, 45-54, 54-64, 65+ and the 12 age groups are: <25, 25-29, 30-34, 35-39, 40-44, 45-49, 50-54, 55-59, 60-64, 65-69, 70-74, and 75+. The 6 income groups (in thousands of real 2019 dollars) are: 0-20, 20-40, 40-60, 60-80, 80-120, 120+ and the 12 income groups: 0-20, 20-40, 40-60, 60-80, 80-120, 120-160, 200-250, 250-500, 500-750, 750-1250, 1250+. Wealth includes defined benefit pensions.

In Table 1, we report results for the more narrow focus on either only age or income

[19]Work by Auclert et al. (2021) is the closest to this paper in terms of quantifying the contribution of population aging, i.e. between-group component with 5-year age groups in place of i, to the change in the wealth-to-income ratio in the US between 1950 and 2016.

[20]In their shift-share analysis of the changes in saving to national income ratio, Mian et al. (2021b) test the relative importance of aging versus income inequality drivers over the 1950-2019 period. For this reason, they choose to focus separately on age groups and within-birth-cohort income distribution groups defined by 10th, 50th, and 90th income percentiles. Feiveson and Sabelhaus (2019) also look at within-birth-cohort permanent income groups which are only available for the 1995-2019 period. When using normal income for the formation of income groups and income measure itself, we find that over the period between 1995 and 2019, the within-group component is responsible for 55% of the change in our benchmark measure of the wealth-to-income ratio, including defined benefit plans.

groups.[21] With respect to the results based on demographics the table presents two breakdowns: one based on 5 age groups and one based on 12 age groups. For these two breakdowns, we get very similar results: the within component explains about 65 percent of the change in the wealth-to-income ratio.[22] Then, we look at two groupings based only income: one based on 6 groups and one based on 12 groups. In both of these cases, the between component only explains about 7 percent of the change, leaving 93 percent of the change to the within-group component.[23]

In Table 2, we present results for our preferred approach, where we allow for 30 groups as the product of 5 age groups and 6 income groups. These results use two different measures of wealth: our baseline measure inclusive of all wealth and the baseline measure less net housing wealth (primary residence).[24] For comparison between the survey and aggregate data, we also report in Table 2 the results of the shift-share analysis when rescaling SCF estimates of wealth and income to match the FA and NIPA aggregates ("scaled" estimates). The latter approach is used in the literature, such as Feiveson and Sabelhaus (2019), Mian et al. (2020), and Bauluz and Meyer (2019). It builds each group's wealth using its shares of different assets and liability classes in SCF and values of their counterpart FA classes.[25] The same is done on the income side where SCF reports income from different sources, which are matched to their corresponding aggregates in NIPA.[26] As shown in Panels A and B, the two sets of results are quite similar. The within component — that is, the component

[21] Appendix B.1 also presents the results of the decomposition of the changes in wealth-to-income ratio between 1989 and 2019 into within and between-group components with additional income groups at the higher end of the income distribution. The results using additional groups are similar to the benchmark 6 and 12 income groups.

[22] The compositional age effect in Auclert et al. (2021) computed for the 1950-2016 period and using 2016 for base profiles of labor earnings and wealth is responsible for 105 out of 118 percentage points increase in the wealth-to-GDP ratio. Over the period studied in this paper the compositional effect in Auclert et al. (2021) is about half of that in the full period, while the actual change in wealth-to-GDP ratio is similar between 1989 and 2016 and 1950 and 2016. In what follows, we also discuss the results of using 2019 as the base year for the between-group component calculation, which is closer to 2016 used in Auclert et al. (2021).

[23] As mentioned earlier, this definition of income groups differs from that used in Mian et al. (2021b) analysis of the changes in aggregate saving rate, which also controls for life-cycle effects by looking at top, middle and bottom parts of income distribution within each cohort. We similarly condition on age differences, when looking at the product of income and age groups in what follows.

[24] In the remainder of the sections pertaining to empirical analysis, we would refer to SCF data plus DB pensions as simply raw SCF data, for ease of the exposition.

[25] We use the aggregates as reported directly in FA, as opposed to combining aggregates for some asset classes from SCF and others from FA. But the results do not substantially change when we use a combination approach instead.

[26] In the benchmark, we use the definition of gross NIPA income less imputations for owner-occupied housing rents. However, we have also used other measures of income, including with imputed owner-occupied rents, and the results using these other measures are similar.

associated with changes in the wealth-to-income ratio of different groups — accounts for between 57 and 65 percent of the change with the between component explaining around 40 percent.[27]

The results in Table 2 are obtained using 1989 as the base year for each group i's initial profiles. In Appendix B.1 we also check the robustness of these results when changing the base year to 2019. When doing so, as suggested by Mian et al. (2021b), the importance of the between component increases, helping to explain some of the difference in our results relative to those reported in Auclert et al. (2021). However, even with the change in the base year the within-group component accounts for more than 50% of the change in the aggregate wealth-to-income ratio between 1989 and 2019.[28]

It must be immediately noted that these decomposition results — by themselves — do not imply that within group *desired* wealth holdings have necessarily gone up. Instead, if households' wealth holdings are sticky, it could be that these high levels of within-group increases in wealth holdings simply reflect the fact that falls in real interest rates have led to increased valuation of wealth, and that households in 2019 are holding much more wealth than they desire relative to similar households in 1989. This could be the case if households face constraints on adjusting their portfolios. This is especially likely for housing, which is why we also reported results excluding housing. As we saw, the results are not driven by housing wealth. Nonetheless, to explore the possibility that households hold wealth above their target level more thoroughly, we need to examine the changes in saving rates by age-income groups. We do so focusing on total saving rates.

[27] It is interesting to note that showing that the within group effects are important for explaining the rise in wealth-to-income ratios does not mean that these effects are solely due to interest rates. They could be due to other common time effects such as tax changes.

[28] It is worth noting that the sensitivity of results to base year choice tends to decrease as we increase the number of groups. However, it must also be recognized that some of the groups start to have rather few observations when we go above 30, explaining the choice of the number of groups for our baseline results. Nonetheless we did explore how our decomposition results change when we considerably increase the number of groups. For example, when we allow for 75 groups (15 income groups and 5 age groups), our within group estimate declines to around 48% when using 1989 as the base year, and this does not change much if we change the base year to 2019. Given this, we are comfortable interpreting our results as suggesting that both the within and between components are close to equally important in explaining the increase in wealth to income ratios.

Table 2
Total Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019 and the Fraction of the Change due to Within and Between Effects: Shift Share Decomposition

Definition	Total Change	Within (%)	Between (%)
Panel A: Raw SCF Data			
Wealth (baseline)	2.819	61.6	38.4
Wealth less housing	2.649	61.4	38.6
Panel B: Scaled SCF Data			
Wealth (baseline)	1.71	57.4	42.6
Wealth less housing	1.64	65.9	34.1

Note: The decomposition is done for 30 groups which are the product of 5 age groups and 6 income groups. The age groups are: 18-34, 34-35, 35-44, 45-54, 54-64, 65+ and the income groups (in thousands of real 2019 dollars) are: 0-20, 20-40, 40-60, 60-80, 80-120, 120+.

2.1 Within-group saving behavior: Are households in 2019 trying to shed their increased wealth?

In the previous section we documented that a large share of the increase in the aggregate wealth-to-income ratio in the US over the period 1989-2019 is accounted for by increases in wealth for given levels of age and income, that is, it is predominantly a within group phenomenon. There are at least two potential interpretations of such an observation. On the one hand, increases in wealth-to-income ratio could reflect increases in desired wealth holdings due to low expected returns on assets. On the other hand, such increases in wealth could reflect unanticipated valuation effects, where the observed higher wealth holdings reflect wealth holdings that are above their desired levels. To help discriminate between these two possibilities, in this section we look at the changes in within group saving patterns over the same period. In particular, if the observed within group increases in wealth-to-income ratios reflect wealth levels in 2019 that are above desired levels, then we should see household groups with large increases in wealth wanting to spend more and save less to get their wealth back down to its target level. Accordingly, we should see them decrease their savings rates.[29] Hence, the absence of a negative relationship between increased wealth and changes savings rates would indicate that the extra wealth holdings are likely desired not excessive.

In line with the previous section, we focus on within group changes in saving rates for the 30 groups we used for our analysis of changes in wealth-to-income ratios. We measure saving in the SCF using synthetic saving approach, widely adopted in the literature, which approximates saving by each group by netting out valuation effects from changes in their wealth between two SCF waves.[30] Our saving rates are calculated over a three-year window. Saving rates for 1989-92 and 2016-2019 periods, respectively, correspond to the start and the end of our 30-year period used to analyze changes in the aggregate wealth-to-income ratio in the US. In our robustness exercises using the SCF, we also exclude net inheritances from changes in wealth, which does not materially change the results.[31]

We follow the approach of the previous section in using both unscaled/raw SCF data, as well as scaled to the aggregates SCF data to construct group savings rates and their changes. For valuation effects we apply asset/debt inflation factors from Mian et al. (2020), which are aggregate in nature and are available until 2016 inclusive, and use their methodology to extend them to 2019. Appendix A provides further details of the saving rate construction.

Table 3
Correlation between Group Changes in Wealth-to-Income Ratios and Changes in Saving Rates: Raw and Scaled SCF Data, 30 Age-Income Groups

	Raw SCF	Scaled SCF
corr($\Delta(s/y)$, $\Delta(w/y)$)	-0.05	0.16

Note: Correlation is computed using 30 age-income groups constructed using SCF data as defined previously.

[29] Fagereng et al. (2019) ask a similar question whether households who experience capital gains sell off the assets subject to price increases to consume. They find evidence against such behavior and show that it is consistent with a model where asset price increases are driven by declining asset returns, as opposed to growing dividends.

[30] For other papers using synthetic saving approach see Mian et al. (2021b) and references therein. This approach of decomposing total changes in wealth into the component associated with capital gains and non-capital gains component is also used in the FA approach to calculating saving, with the latter conceptually corresponding to the measure of saving in NIPA.

[31] Accounting for inheritances has a zero net effect in the aggregate, as inheritances received should equal inheritances bequeathed, but within groups these inflows and outflows may not be equal, potentially affecting group-wise changes in saving rates.

Table 3 presents correlations between changes in wealth-to-income ratios and changes in saving rates using these two different approaches. Using raw SCF data to compute the correlation between group changes in wealth-to-income ratios and saving rates results in a coefficient of -0.05, and with scaled SCF data it is 0.16. Both of these numbers suggest that groups that faced greater increases in wealth-to-income ratios do not appear to systematically reverse this accumulation by decreasing their saving rates.[32] In Figure 3, we complement the evidence on correlations from Table 3 by plotting the changes in saving rates against the changes in log wealth for all the groups that experienced increases in wealth. Given that the saving rates constructed using raw SCF data were low relative to aggregate measures of saving rates in FA/NIPA, for this figure we are using results based on the scaled SCF measures of saving rates. The average change in savings rates for this subset is slightly positive. Moreover, as can be seen in the figure (and is confirmed by the correlation), higher increases in wealth are not on average associated with larger decreases in saving rates. It must be recognized that our measure of saving rates, which is common to the literature, is quite noisy. Accordingly, we witness substantial variation in saving rates. Nonetheless, we view these patterns as providing support to the notion that increases in within group wealth-to-income ratios documented in the previous section are more likely reflecting changes in desired wealth holdings as opposed to reflecting wealth holdings that exceed desired levels.

2.2 The Asset-demand interest-rate relationship holding income constant

In the previous section, we documented that when one compares households with the same real income in 1989 versus 2019, desired asset holdings appear to have increased as interest rates fell. In Appendix C, we show that such a pattern conflicts with a large class of models commonly used in macroeconomics to capture household savings behavior. Specifically, in the appendix we consider asset accumulation decisions of a household with preferences of the form

$$\max \int_0^\infty e^{-(\delta+\rho)t} \left[\frac{c_t^{1-\sigma_c}}{1-\sigma_c} + \Lambda \frac{a_t^{1-\sigma_a}}{1-\sigma_a} \right] dt, \qquad \sigma_c, \sigma_a > 0, \Lambda \geq 0$$

subject to the budget constraint

[32] Amongst our 30 benchmark groups, we find that all of the groups in the top income grouping, except one, did not decrease their saving rates, which is consistent with findings in Mian et al. (2021b) using averages for 1963-1982 and 1995-2019 periods and the top 10% of the within-cohort income distribution. However, the time periods and the income group definitions in the two studies are not fully comparable.

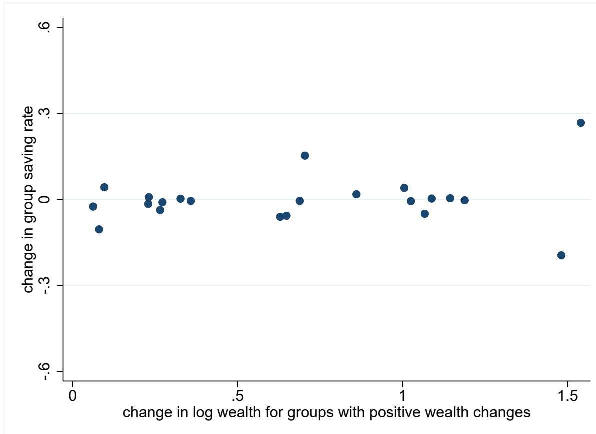


Figure 3
Change in saving rates vs. change in log wealth for age-income groups with wealth increases between 1989 and 2019

Sources: Survey of Consumer Finances values scaled using aggregates from the Financial Flow Accounts and National Income and Product Accounts.

$$\dot{a}_t = y_t - c_t = w_t + r_t a_t - c_t$$

where c_t is consumption, ρ is the discount rate, δ is a death rate, a_t is asset holdings, y_t is total income, r_t is the return on the asset a_t, w_t is non-asset income and $\Lambda \frac{a^{1-\sigma_a}}{1-\sigma_a}$ represents any potential additional gains—besides the consumption they allow—of holding assets. If $\Lambda = 0$, we have the standard representative agent setting. Allowing for $\Lambda \geq 0$ covers a wider range of models including those considered in Kumhof et al. (2015), Mian et al. (2021a), Michaillat and Saez (2018), De Nardi (2004), Straub (2019), and Michau (2022) which allow assets to directly affect utility.

For this class of models, one can easily derive the steady state asset demand function which is of the form

$$a^{ss} = A(r, y)$$

and verify that $\frac{\partial a^{ss}}{\partial r} > 0$ for all values of σ_a and σ_c, that is, desired assets are always increasing in r (holding total income y constant).[33]

[33] It may be worth noting that for this class of utility functions, the asset-to-income ratio will be increasing, invariant or decreasing in income depending on whether $\sigma_c > \sigma_a$, $\sigma_c = \sigma_a$ or $\sigma_c < \sigma_a$, respectively. The interesting work by Mian et al. (2021a) examines implications of asset holdings with non-linear Engel curves by considering the case where $\sigma_c > \sigma_a$ (non-homothetic preferences). Instead, our paper (as seen from Section 3 onwards) focuses on asset demands that are not monotonic in r holding income constant. Note that when $\sigma_c > \sigma_a$ as in Mian et al. (2021a), $\frac{\partial a^{ss}}{\partial r}$ is positive when holding income constant. This observation may appear to contradict Mian et al. (2021a) where it is shown that asset holdings can be

Note that in such an environment, if there is a change in the distribution of types, say for example due to a change in the distribution of non-asset income w_t, the general-equilibrium implication can be a fall in asset returns and valuations effects due to the repricing of fixed-income assets. In the new steady state equilibrium induced by such a change, agents would therefore need to hold—in aggregate—a portfolio of higher value because of the valuation effects. However, it will remain the case that despite these valuation effects, when comparing two households with the same income before and after the change, the household in the higher-interest-rate environment should be holding more assets. This is a direct consequence of $\frac{\partial a}{\partial r} > 0$. In other words, if we perform a between group and within group decomposition of the change in the aggregate asset-to-income ratio induced by a change in the distribution of types that lowers the real rate in this kind of environment, we should find that the between component explains more than 100 percent of the change in the asset-to-income ratio while the within component would be making a negative contribution. Such a pattern is clearly at odds with what we observed when decomposing the change in the asset-to-income ratio in the higher interest rate environment of 1989 relative to that of the lower interest rate environment of 2019. For this reason, we now turn to present an environment which is more compatible with the decomposition observations.

3 Model

The asset holding pattern presented in the previous section suggests that the quantity of asset holdings desired by households may be increasing with low frequency decreases in interest rates, at least over certain ranges. As indicated previously, such a property is not theoretically problematic as interest rate changes can have both income and substitution effects on desired wealth holdings. The question for us is how best to explain such an observation and what does it imply about the potential role of monetary policy in affecting long-run outcomes. The workhorse infinitely lived agent model is not a good starting point for asking these questions as it is not consistent with desired wealth holdings decreasing with higher interest rates. In contrast, the positive effect of lower interest rates on asset demands can in principle be easily captured in an OLG type framework. However, the perpetual youth OLG model of Blanchard (1985) and Yaari (1965), by omitting retirement savings needs, downplays precisely the potential income effects of interest rates which could

decreasing in r when $\sigma_c > \sigma_a$. However, the potential confusion comes from the fact that the relationships between asset holdings and interest rates depend on whether income is being held constant or not. For example, in Mian et al. (2021a), income is ra. If one replaces y by ra in our asset demand relationship $a^{ss} = A(r, y) = A(r, ra^{ss})$ and examines the implicit relationship between a^{ss} and r with income changing according to $y = ra$, then this will result in, as in Mian et al. (2021a), $\frac{\partial a^{ss}}{\partial r}$ being negative if $\sigma_c > \sigma_a$.

help explain the pattern of interest to us. For these reasons, in this section we build on a model similar to that of Gertler (1999) that integrates both inter-temporal substitution forces and retirement preoccupations in wealth accumulation.[34] In particular, these two forces will be shown to interact in a manner that gives rise to C-shaped wealth demands where desired wealth holdings increase when long term interest rate decrease at low levels. In the subsequent section, we will embed this household decision model into a general equilibrium set-up to show how it can lead to multiple steady state real interest rates(r^*) and how it creates a role for monetary policy to influence long run real interest rates. In particular, the model will highlight why aggressive inflation targeting monetary policy may have contributed to the fall in real interest rates over the last thirty years. It is worth noting that the mechanisms we will highlight are not driven by monetary policy simply affecting beliefs in a multiple equilibrium setting but are instead associated with monetary policy affecting the inherent dynamics of a system with multiple steady states.

3.1 The household's decision problem with both inter-temporal substitution and retirement motives

When thinking about consumption and wealth accumulation decisions, it is common to think about people in different states. As is standard in simple OLG models, we can think of a household in one of three states: an active work state, a retirement state and a death state. Following Blanchard (1985), Yaari (1965) and Gertler (1999) we want to think of these states as evolving stochastically. To be more precise, let us assume that a person starts life in a work state and transits out with instantaneous probability δ_1. In the absence of fixed retirement dates, this shock can be thought as a health shock. At this transition, with probability q, the person retires and with probability $(1-q)$, the health shock is severe, and the person dies. If the person retires, the person will die with instantaneous probability $\delta_2 \geq \delta_1$. If we denote the expected discounted utility of entering the retirement state at time t by V_t, we can express the utility of an active household, that is a household in the work state, as:

$$\int_0^\infty e^{-(\delta_1+\rho)t} \left[\frac{c_t^{1-\sigma_1}}{1-\sigma_1} + \delta_1 q V_t \right] dt, \qquad \sigma_1 > 0$$

[34] We depart from Gertler (1999) by maintaining the more common CRRA utility specification instead of adopting RINSE preferences. Carvalho et al. (2016) uses the model of Gertler to examine equilibrium rates. More recently, Galí (2021) introduces retirement in a New Keynesian model with logarithmic utility in which there are multiple steady state real rates that are related to the size of bubbles. In a two-period OLG model with nominal rigidities, Plantin (2022) also examines the case where a Taylor rule may create monetary bubbles. We do not explore the possibility of bubbles in our analysis.

where c_t is consumption, ρ is the subjective discount rate and $\sigma_1 > 0$ is the inverse of the elasticity of substitution $(1/\sigma_1)$, or alternatively the risk aversion parameter.

A retiree's decision problem. For the household in the retirement state, the preferences are given by:

$$\int_0^\infty e^{-(\delta_2+\rho)\tau} \frac{c_\tau^{1-\sigma_2}}{1-\sigma_2} d\tau, \qquad \sigma_2 > 0$$

We are allowing the parameters governing inter-temporal substitution, σ_1 and σ_2, to differ between the two states of life to illustrate important forces at play. Later, we will restrict attention to the more standard case where $\sigma_1 = \sigma_2$.

The budget constraint facing the retired household is given by:

$$\dot{a}_t = a_t r_t - c_t,$$

where a_t is the asset holding of a retired person at time t and r_t is the return on the asset a. As can be seen from the budget constraint of the retirees, moving into the retirement state is associated with the absence of labor income implying that households must rely only on asset income for consumption. The need to rely on asset income in retirement will play an important role in our results. Given this structure, the discounted expected utility of a household who retires at time t_1, V_{t_1}, can be solved explicitly and expressed as[35]

$$V_{t_1} = \frac{a_{t_1}^{1-\sigma_2}}{1-\sigma_2} \left[\int_{t_1}^\infty e^{-\int_{t_1}^t \frac{1}{\sigma_2}[(\rho+\delta_2)-(1-\sigma_2)r(\tau)]d\tau} dt \right]^{\sigma_2},$$

where a_{t_1} is the level of assets held by the household at time of retirement. For convenience, we will also express V_{t_1} as

$$V_{t_1} = V(a_{t_1}, \Gamma_{t_1}) = \frac{a_{t_1}^{1-\sigma_2}}{1-\sigma_2} \left[\Gamma_{t_1}\right]^{\sigma_2},$$

where

$$\Gamma_{t_1} = \int_{t_1}^\infty e^{-\int_{t_1}^t \frac{1}{\sigma_2}[(\rho+\delta_2)-(1-\sigma_2)r(\tau)]d\tau} dt,$$

with Γ_{t_1} being a function of the whole future path of returns $\{r_t\}_{t_1}^\infty$. Expressing utility as $V(a_{t_1}, \Gamma_{t_1}) = \frac{a_{t_1}^{1-\sigma_2}}{1-\sigma_2} \left[\Gamma_{t_1}\right]^{\sigma_2}$ makes it clear that the utility of someone who retires at time t_1

[35]The expected utility associated with the retirement state is found by first solving for the optimal consumption path, which is governed by the Euler equation $\frac{\dot{c}_t}{c_t} = \frac{r_t-\rho-\delta_2}{\sigma_2}$ and then integrating the implied utility flow over the expected duration of retirement.

depends on both the asset at the time of retiring and the entire path of asset returns over the retirement period. As we shall see, the degree of inter-temporal substitution $\frac{1}{\sigma_2}$ will play an important role in controlling how asset returns affect marginal value of assets.

For future reference, it is useful to note that Γ_{t_1} obeys the following differential equation

$$\dot{\Gamma}_t = -1 + \Gamma_t \left[\frac{\rho + \delta_2}{\sigma_2} - \frac{1 - \sigma_2}{\sigma_2} r_t \right]. \tag{2}$$

To see most easily how asset returns affect retirement utility, note that if the return on asset a is constant, $r_t = r$, then V_{t_1} can be expressed as

$$V_{t_1} = \frac{a_{t_1}^{1-\sigma_2}}{1 - \sigma_2} \left[\frac{\rho + \delta_2}{\sigma_2} - \frac{1 - \sigma_2}{\sigma_2} r \right]^{-\sigma_2}$$

Here we see that higher r increases utility in both the case where $\sigma_2 < 1$ or when $\sigma_2 > 1$, that is, retired individuals always like higher interest rates as this gives them a superior income stream . However, what will play an important role in our analysis is how higher r affects the marginal value of a_{t_1} to a retiree. This is given by the following key lemma.

Lemma 1. *At fixed r, the marginal value of assets to a retiree is decreasing in interest rates when $\sigma_2 > 1$ and is increasing in interest rates when $\sigma_2 < 1$ since* $\frac{\partial^2 V_{t_1}}{\partial a_{t_1} \partial r} = a_{t_1}^{-\sigma_2} (1 - \sigma_2) \left[\frac{\rho + \delta_2}{\sigma_2} - \frac{1 - \sigma_2}{\sigma_2} r \right]^{-\sigma_2 - 1}$.

In general, the effect of asset returns on the marginal value of assets for retirees depends on σ_2. As noted in Lemma 1,[36] this marginal value is decreasing in r when $\sigma_2 > 1$. In other words, when a retiree has limited opportunities to inter-temporally substitute consumption across time, the retiree will view assets at time of retirement to have a greater marginal value when interest rates are low than when they are high. This property will influence the wealth accumulation behavior of non-retirees as will be the focus below.[37] It is worth

[36] Lemma 1 can be trivially extended to include the case of log preferences. In this case, the marginal value of assets is independent of interest rates, i.e., $\frac{\partial^2 V_{t_1}}{\partial a_{t_1} \partial r} = 0$.

[37] When $\sigma_2 > 1$, a rise in interest rates causes the optimal path of post-retirement consumption to be higher at all dates and hence the marginal value of assets is lower. This is easily understood and intuitive. In contrast, when $\sigma_2 < 1$ different interest rates cause optimal paths of post-retirement consumption to cross; with retirees consuming initially less in a higher interest rates environment but having their consumption decline more slowly over time. Because of this crossing property, the effect of interest rates on the marginal value of assets is not straightforward when $\sigma < 1$. Lemma 1 indicates that the net effect is that higher interest rates increase the marginal value of assets when $\sigma_2 < 1$ due to this crossing feature.

noting that, although we have not allowed for an annuity market for the effect of uncertainty about time of death, Lemma 1 is not dependent on the presence or not of such a market. The content of Lemma 1 would remain identical if we were to allow for an annuity market similar to that in Blanchard (1985).[38]

An active household's decision problem. Let us now turn to the decision problem of an active household. Its decision problem will incorporate the continuation value of assets in retirement and can be written as:

$$\int_0^\infty e^{-(\delta_1+\rho)t}\left[\frac{c_t^{1-\sigma_1}}{1-\sigma_1}+\delta_1 q V(a_t,\Gamma_t)\right]dt,$$

subject to

$$\dot{a}_t = y_t - c_t \tag{3}$$

with $y_t = w_t + r_t a_t - T_t$, where y_t is disposable income, w_t is labor income and T_t are taxes.

The consumption Euler equation for the active household becomes

$$\frac{\dot{c}_t}{c_t} = \frac{r_t-\rho-\delta_1}{\sigma_1} + \frac{c_t^{\sigma_1}}{\sigma_1}\delta_1 q V_a(a_t,\Gamma_t) \tag{4}$$

Relative to a standard infinitely lived agent Euler equation, this Euler equation incorporates forces associated with both inter-temporal substitution and retirement preoccupations as in Gertler (1999) and Grandmont (1985). The first term in this Euler equation maintains the standard substitution effect of interest rates on consumption. However, this effect now relates to short-term interest rates movements holding the future path of interest rates constant. When both short-term and long-run interest rates move together the net effect is more involved. The additional term in the Euler equation — $\frac{c_t^{\sigma_1}}{\sigma_1}\delta_1 q V_a(a_t,\Gamma_t)$ — represents the incentive to save due to retirement motives and thus is affected by future interest rates. Given this term is always positive, it implies that retirement adds a force towards postpon-

[38] Like Gertler (1999), a key assumption is the absence of a pension system which acts as a perfect insurance market against loss of labor income. The absence of such market implies consumption in retirement depends on the accumulated savings when active.

ing consumption and favoring asset accumulation.[39] The key element for us, and which will allow us to capture features of the data, is that the retirement incentive to save is affected by long run returns to savings. In particular, when interest rates are constant, $r_t = r$, we have seen that $V_{a,r}(a_t, r) < 0$ when $\sigma_2 > 1$. Hence, interest rates have two opposing effects in our set-up when $\sigma_2 > 1$. Low interest rates will favor higher consumption today due to inter-temporal substitution forces, while at the same time, low interest rates are an incentive for greater retirement savings if the low interest rates are viewed as persistent.

To help further highlight implications of this Euler equation, it is helpful to examine the implied long-run asset holdings of the active household when the return of asset a is constant and therefore $\Gamma_t = [\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2} r]^{-1}$. We will denote an active household's steady state asset holding function by $a^{a,ss}(y, r)$. Proposition 1 indicates that $a^{a,ss}(y, r)$ is attractive and describes the key properties of the function $a^{a,ss}(y, r)$.

Proposition 1. *For fixed r, the asset holdings of active households will converge to $a^{a,ss}(y, r)$[40] given by*

$$a^{a,ss}(y, r) = (\delta_1 q)^{\frac{1}{\sigma_2}} \left[\frac{\rho + \delta_2}{\sigma_2} - \frac{1 - \sigma_2}{\sigma_2} r\right]^{-1} [\rho + \delta_1 - r]^{\frac{-1}{\sigma_2}} y^{\frac{\sigma_1}{\sigma_2}}, \tag{5}$$

when r is in the interval defined by $[\frac{\rho+\delta_1-r}{\delta_1 q}(\frac{\rho+\delta_2}{\sigma_2} - \frac{(1-\sigma_2)r}{\sigma_2})^{\sigma_2}]^{\frac{1}{\sigma_1}} > \max[0, r]$.[41]

The long-run asset holdings of active households $a^{a,ss}(y, r)$ are increasing in income. Moreover, if $\sigma_2 \leq 1$, then $a^{a,ss}(y, r)$ are monotonically increasing in asset return r, while if $\sigma_2 > 1$, they are C-shaped in r.[42]

See Appendix D.1 for the proof.

The first property noted in Proposition 1 is straightforward. If an active household has greater income, its target level of asset holding will be higher. This remains true regardless of the degree of inter-temporal substitution. The most important element in Proposition 1 relates to the effects of steady state returns on desired long-run asset holdings. In particular, we see that if $\sigma_2 \leq 1$, then desired long run asset holdings would be monotonically in-

[39]This force is also present in models with warm-glow bequest motives, but in that case it does not depend on interest rates, which is the key feature for our purposes.

[40]When $\sigma_1 = \sigma_2$, this equation implicitly defines the asset-to-income ratio as a function of interest rates.

[41]If r is not in the interval, asset holdings do not converge.

[42]When $\sigma_1 > \sigma_2$, the long-run asset demands exhibit the non-linear Engel curve property emphasized in Mian et al. (2021a), in particular, the desired asset-to-income ratio would be increasing in income holding interest rates constant.

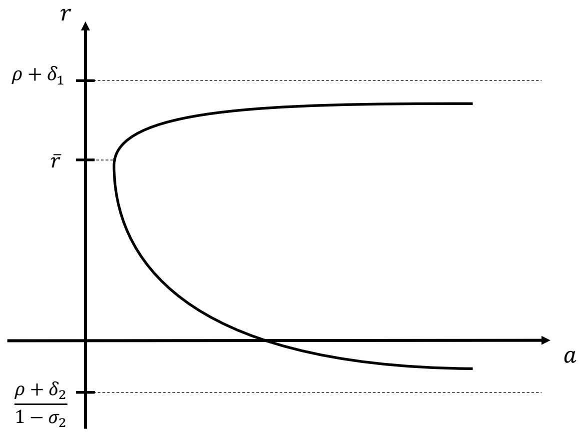


Figure 4
Active households' long-run asset demand

creasing in r because the substitution effect always dominates retirement savings effect.The case that interests us is when $\sigma_2 > 1$. In this case, the effects of returns on long-run asset holdings are non-monotonic. For high levels of returns, desired holdings are increasing in r, while for low returns they are decreasing in r. To understand this effect, recall that interest rates have two effects in this model. At low interest rates, households are encouraged to consume more, and accumulate less, through the standard inter-temporal substitution channel. However, retirement preoccupations play a counterbalancing role. When long-term interest rates are low and $\sigma_2 > 1$, active households have an increased marginal incentive to accumulate assets for retirement needs. What Proposition 1 indicates is that there will be a point of reversal of the effect of steady state r on accumulation incentives. When r is sufficiently high, a marginal increase in steady state r would lead to more accumulation as the positive substitution effect dominates the decreased retirement need effect even if $\sigma_2 > 1$. When interest rates are low then the increased need for retirement income will dominate the inter-temporal substitution effect and favor a greater accumulation of asset when $\sigma_2 > 1$.[43]

The shape of the active household's long-run asset demand $a^{a,ss}(y, r)$ is illustrated in

[43] Our paper has some similarities with the work of Brunnermeier and Koby (2019) on the reversal interest rate. In their work, there is a reversal rate of interest whereby interest rates below the reversal rate become contractionary. Their reversal rate result comes from banking frictions. Our set-up can also be thought as having a reversal rate, which we denote $\bar{r}$. Our reversal rate arises from expected income effects in retirement that drive up households' desired savings while working and therefore depress consumption.

Figure 4 when $\sigma_2 > 1$. Here we see the C-shape of the long-run asset demand keeping income, y, constant. Moreover, we can see that the long-run asset demand (when $\sigma_2 > 1$) is delimited by two levels of r. As r tends to $\rho + \delta_1$, desired asset demands relative to consumption tend to infinity. As r tends to $\frac{\rho+\delta_2}{1-\sigma_2} < 0$, desired asset demand relative to consumption will tend again toward infinity. When $\sigma_2 > 1$, there exists also a threshold or point of inflexion

$$\bar{r} = \left[\frac{\sigma_2(\sigma_2 - 1)(\rho + \delta_1) - (\rho + \delta_2)}{(\sigma_2 - 1)(\sigma_2 + 1)} \right],$$

such that the asset demand of active households is increasing in interest rates when r is above $\bar{r}$ and is decreasing in interest rates when r is below $\bar{r}$. This implies that decreases in interest rate can lead to higher desired assets holding income constant.

Even before we specify the general-equilibrium setting, one can immediately see why the C-shaped property of asset demands by active households may create a situation with multiple steady state interest rates. An economy populated with such households will face a residual asset supply coming from the total asset supply in the economy minus that held by retired households. Even if this residual asset supply faced by active households is monotonic and well-behaved, it is likely to cross the steady state asset demand of active households more than once.

While the objective of the next section is to set up a GE structure where this possibility can be examined explicitly, Figure 5 illustrates three possible equilibrium configurations. In the first panel, we have the case where the asset supply curve faced by active agents is rather inelastic. This creates two equilibrium real rates. The second panel is the case where the asset supply curve is very elastic, it creates only one equilibrium real rate. Finally, the third panel represents the case where the asset supply curve is rather inelastic at high real rates but quite elastic at lower real rates. This can create three equilibrium real rates. The shape of such an asset supply curve will depend, as we shall discuss, on the set of assets available in the economy and the extent to which these assets will exhibit valuation effects.

To simplify the presentation, the remaining sections will focus on the case of interest where $\sigma_1 = \sigma_2 \equiv \sigma > 1$.

4 General equilibrium

We now want to derive the general equilibrium properties of an OLG economy populated with active and retired households with preferences as defined in the previous section. In particular, we want to look at the implications of having active households whose long-run

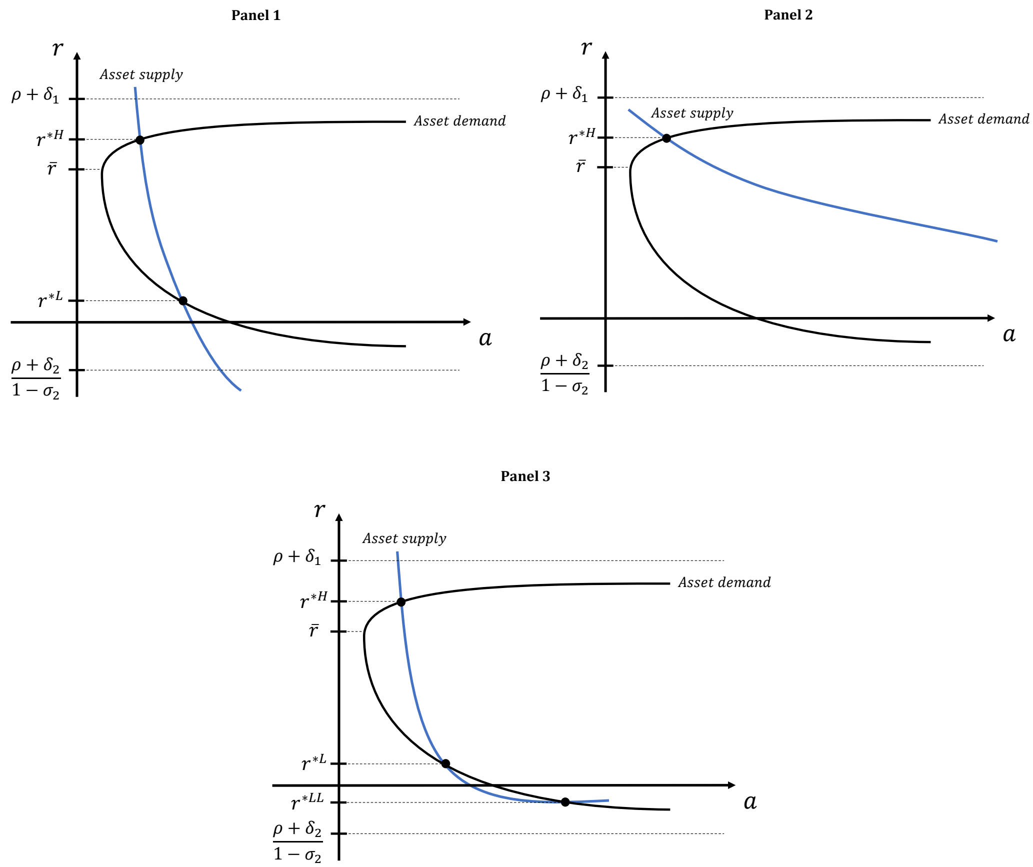


Figure 5
Long-run asset demand and different configurations of the long-run asset supply: multiple steady state interest rates

asset demands are non-monotonic in asset returns when $\sigma > 1$.[44] To begin, we will examine a setting without any nominal constraints. This will allow us to show why real side of this economy is likely to generate more than one steady state real interest rate. Then we add nominal rigidities and examine the effect of monetary policy. In our initial set-up, we will allow for only one asset and this will be a short term government bond. This simplifies the analysis considerably and allows a simpler presentation of how monetary policy can have long-term effects in the presence of two steady state real interest rates. However, the case with only short term bonds is quite restrictive as there can't be asset valuation effects due to changes in interest rates. Accordingly, we will follow up this initial analysis by also presenting the case where there are valuation effects which can lead to more than

[44]Note that we take the empirical observations we presented previously as placing in serious doubt the relevance of the case with $\sigma < 1$ as the observed pattern is not easily reconcilable with asset demands which would be monotonically increasing in real interest rates.

two steady state real interest rates.

In our model economy, we normalize the population to have a measure 1 of households, with the implied fraction $\phi \equiv \frac{\delta_2}{\delta_1 q + \delta_2}$ who are active and the fraction $1 - \phi$ who are retired. When a household dies it is replaced by a new active household.

The government in this one-good economy spends an amount G, has an outstanding real debt in the amount B and levies taxes T_{1t} on active households. The taxes adjust to satisfy the government budget constraint

$$\phi T_{1t} = G + r_t B,$$

where r_t is the real interest rate on government debt. Each active household is endowed with one unit of labor that produces w goods. Total production in this economy is given by output produced with the labor of active workers and therefore is equal ϕw. We will limit attention to cases where B is not so large that it could not be financed by active households. Since $\rho + \delta_1$ is the highest possible interest rate in this economy, we restrict attention to cases where $B(\rho + \delta_1) < \phi w$. Since we have not introduced annuity markets, private agents will generally have positive asset holdings when they die and therefore there will be unintended bequests. We assume that the unintended bequest of a household goes to the newborn household replacing that household. To keep the structure more tractable, we assume that the government ensures — through a tax T_{2t} on active households — that newborn households receiving bequest from retired parents have the same average starting wealth as the newborn households inheriting from active households. Under this assumption, if asset holdings are equal across active households at a point in time, then the system inherits a representative agent structure for active households.[45] The second tax on active households, T_{2t}, is defined by the following budget condition.

$$\delta_1(1-q)\phi a_t + \delta_2(B - \phi a_t) = [\delta_1(1-q)\phi + \delta_2(1-\phi)]\, a_t - \phi T_{2t}.$$

The first term on the left hand side of this equation is the total funds received from accidental bequests.[46] On the right hand side, the first term is the funds needed to give to newborn active households while the second term is the tax levied on all active households to equalize wealth between newborns that inherited from retired and active households. Rearranging the equation, we obtain that $T_{2t} = \delta_2(a_t - B)/\phi$.

[45] Assuming that active households act like a large family as in Gertler et al. (2020) would lead also to maintain the tractability of the representative agent structure.

[46] This equation includes the asset market clearing condition $\phi a_t + a_t^r = B$ implying that the total asset demand of retirees is $a_t^r = B - \phi a_t$.

Definition 1. *An equilibrium for this economy will be composed of a consumption profile and asset allocation profile for the different types of households, a time path of interest rates, and taxes such that (1) given interest rates, taxes, government expenditures and public debt, household consumption and asset allocation profiles maximize households' utility, (2) both the markets for goods and assets clear at each point in time, and (3) the government budget is balanced.*

Let us begin by examining the behavior of total asset demands in this economy in a steady state with constant interest rates and taxes. This demand is comprised of both the long-run asset demand function of active households, $a^{a,ss}(y,r)$, and that of retired households, denoted $a^{r,ss}$.[47] The steady state asset demand function of active households when interest rates are constant is given explicitly in Proposition 1 where it is shown to be C-shaped in r. Since long-run asset demands relative to consumption of active households go to ∞ when either r goes to $\rho+\delta_1$ or $-\frac{\rho+\delta_2}{\sigma-1}$, we can restrict attention to situations where $r \in \left(-\frac{\rho+\delta_2}{\sigma-1}, \rho+\delta_1\right)$ as this is the only feasible range for a steady state equilibrium.

To get the steady state asset demand for retired households, we need to aggregate the asset holdings across the different retirement cohorts. With $r < \rho+\delta_1 \leq \rho+\delta_2$, retired households will be depleting their asset holdings as they age. In particular, this will cause the asset holdings of a retired household who retired τ periods ago with a assets to be given by $ae^{-(\frac{\rho+\delta_2-r}{\sigma})\tau}$.[48] Since in steady state, each retiree starts retirement with the same amount of assets, which is equal to the steady state asset holdings of active households ($a^{a,ss}$), the aggregate asset demand of retirees ($a^{r,ss}$) is given by

$$a^{r,ss} = a^{a,ss}(y,r)(1-\phi)\frac{\delta_2}{\frac{\rho+\delta_2-r}{\sigma}+\delta_2} = a^{a,ss}(y,r)(1-\phi)g(r) \qquad g'(r)>0,$$

where $g(r) \equiv \frac{\delta_2}{\frac{\rho+\delta_2-r}{\sigma}+\delta_2}$.

As a result, total asset demand in the steady state of this economy can be expressed as

$$a^{t,ss}(y,r) = \phi a^{a,ss}(y,r)\left(1+\frac{g(r)(1-\phi)}{\phi}\right).$$

[47]We are focusing on potential steady states where all active households have the same wealth level in the steady state. There may be other types of steady states. However, if wealth levels of active households start from a position of equality, then they always stay equal because of the government tax-transfer scheme.

[48]Note that the consumption of retirees satisfies the relationship $c_t^r = a_t^r \Gamma^{-1}$, where a_t^r is the asset at time t. Hence, asset accumulation dynamics for constant interest rates are given by $\dot{a}_t^r = -\frac{\rho+\delta_2-r}{\sigma}a_r^r$.

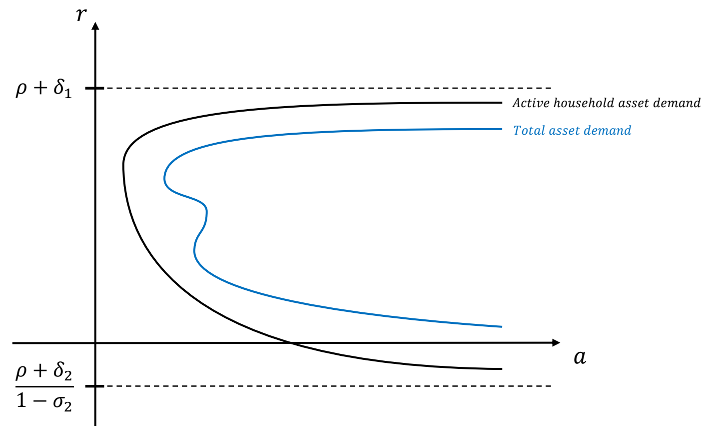


Figure 6
Active households' asset demand and total asset demand

Total asset demand in a steady state is therefore equal to the total asset demand — *i.e.*, $\phi a^{a,ss}(y,r)$ — of active households multiplied by the factor $1 + \frac{g(r)(1-\phi)}{\phi}$. Accordingly, total asset demand will reflect several of the properties of the asset holdings of active households. In particular, this total asset demand will be non-monotonic in r with the additional property that as r goes to either $\rho + \delta_1$ or $-\frac{\rho+\delta_2}{\sigma-1}$, demand will go to infinity. However, even if total asset demand takes this form, it may not always inherit the simple C-shape of the active household's long-run asset demands. An example of a more complex non-monotonic shape for total asset demand that cannot be ruled out is illustrated in Figure 6.

From Figure 6, we can see why such an economy is likely to have more than one steady state values for r. For a given level of total bonds B in the economy, there is likely to be more than one interest rate that clears the asset market. However, this simple argument is not complete as the income of active households, y, is being held fixed in this figure while in this set-up it is endogenous. Proposition 2 nonetheless confirms this line of reasoning.

Proposition 2. *When $\sigma > 1$ and bonds B are in fixed supply, then a unique steady state equilibrium interest rate – a unique r^* – is generically impossible.*[49] *There will either be more than one steady state value for r^* or, if the supply of bonds is sufficiently small, there*

[49]The generic property relates to the amounts of bonds. There can be one value for B where a unique equilibrium can exist if the bond supply happens to satisfy a precise tangency condition. However, such an equilibrium configuration would not be robust to any minor change in the amount of bonds.

will be no equilibrium.

See Appendix D.2 for the proof.

Prudent "perpetual" youth assumption. Much of our analysis could be conducted in the current set-up. However, to allow for an easier presentation of results we will now adopt a very useful simplifying assumption.[50] In the last section we will drop this assumption and come back to the general case to show that this assumption is not driving any of our main insights. To immediately get a sense of why we want to add a simplifying assumption, it is helpful to focus on the long run demand for asset by active households as presented in Figure 4. As we have stressed, this demand is C-shaped when $\sigma > 1$. The equilibrium determination of r^* can then be viewed as depending on the interaction of this C-shaped demand curve for assets with a residual supply curve for assets. The relevant residual supply curve corresponds to the total supply of assets in this economy minus that held by retired agents. Depending on the properties of this residual supply curve, it is obvious from Figure

[50]In the version of the model without nominal rigidities, the equilibrium dynamic system can still be analytically tractable if we are in a situation where the wealth of active households has converged to the same level. Once active households have the same wealth, it will remain that way with active households acting like a representative household. In this case, the equilibrium behavior is described by the following system of three dynamic equations:

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma}\delta_1 q V_a(a_t, \Gamma_t)$$

$$\dot{\Gamma}_t = -1 + \Gamma_t\left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r_t\right]$$

$$\dot{a}_t = w + r_t a_t + \frac{(B - a_t)\delta_2}{\phi} - \frac{G + B r_t}{\phi} - c_t$$

plus the goods market clearing condition

$$\phi c_t = \phi w - G - (B - \phi a_t)\Gamma_t^{-1}$$

where c_t is the consumption of the representative active household and a_t is its asset holdings. However, when we extend the model to include nominal rigidities this dynamic system expands to a 4^{th} and 5^{th} degree system making analytical results very difficult. It is for this reason, we choose to make the additional simplifying assumption of having q go to zero with $\epsilon > 0$. Under this assumption, the dynamic system is reduced, in the absence of nominal rigidities, to the pair of dynamic equations

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma}\delta_1 q^s V_a(B, \Gamma_t)$$

$$\dot{\Gamma}_t = -1 + \Gamma_t\left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r_t\right]$$

plus the goods market clearing condition $c_t = w - G$. This lower dimensional system can be easily extended to allow for nominal rigidities and still be analytically tractable.

4 that many different equilibrium configurations could arise. In particular, as expressed in Proposition 2, there is likely more than one equilibrium value for r^*. In fact, depending on the shape of this residual supply curve, even if it is monotonic, there could be two, three or more equilibrium values for r^*. Our model in its full generality does not rule out any such possibilities. However, analyzing all these possibilities at once can be confusing. The following simplifying assumption will allow us to approach the problem in steps, where we first focus on a case which produces exactly two potential equilibrium values for r^*, then we introduce productive assets and discuss the case of three values. The general case embeds the features emphasized in these special cases but potentially allows for even more equilibria. At this point, we do not see any added insights from the cases with more than two or three potential equilibrium values for r^*. It is for this reason we find the adoption of following simplifying assumption useful.

In particular, consider a modification of the above setting where the probability q of surviving the health shock that moves one to retire has an objective component and a subjective component with the subjective value being denoted q^s while the objective value is still given by q with $q^s = q + \epsilon$ $(\epsilon > 0)$. In this setting, ϵ is governing the extent to which people are over-estimating the probability of needing their retirement savings. Now consider this model as q goes to zero. In this limit, we will have active agents that are saving for retirement but no actual retirees.[51] This simplifies the analysis by removing the need to track the wealth holdings of the retirees. In effect, under this assumption, the steady state demand for assets is now given entirely by the desired wealth holdings of the active population and has a simple C-shape.[52] Hence, steady state equilibrium real interest rates are given by the intersection of the C-shaped asset demand of active households and the exogenous supply of bonds as shown in Figure 7. As can be easily seen on this figure, under this simplifying assumption, the steady state will never be unique. This was also true in the more general case, but was harder to visualize. Moreover, if the supply of asset is sufficiently large, there will always be exactly two steady state values for r, which we will denote r^{*H} and r^{*L} for the high and low real steady state rate respectively. In the continuation, we will assume that B is sufficiently large such that an equilibrium exists as stated below. Lemma 2 indicates some key properties of r^{*H} and r^{*L}.

Going forward we will assume that the quantity of outstanding government bonds

[51] In similarity to the perpetual youth model of Blanchard (1985) and Yaari (1965), this version of our model can be considered as a *"prudent" perpetual youth* model where agents are prudently preparing for retirement even if they never retire. They are young, working and saving for retirement until they die.

[52] In the case without nominal rigidities, the income and consumption levels of the active household become exogenous with $c = y = w - G$.

(B) is sufficiently large to guarantee the existence of an equilibrium, that is, $B > \bar{B} \equiv (\delta_1 q^s)^{1/\sigma}(\rho + \delta_1 - \bar{r})^{-1/\sigma}\left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}\bar{r}\right]$.

Lemma 2. *The low and high natural interest rates r^{*H} and r^{*L} have the following properties: (i) $r^{*H} > \bar{r}$ and $r^{*L} < \bar{r}$; (ii) the high real interest rate (r^{*H}) increases with government bonds B while the low real rate r^{*L} decreases with B; and (iii) r^{*H} increases with the probability of death of retirees δ_2 while r^{*L} falls with δ_2.*

See Appendix D.7 for the proof.

The two steady state real interest rates depicted in Figure 7, r^{*H} and r^{*L}, will continue to play an important role in the presence of nominal rigidities.[53] To foreshadow future results, monetary policy will be shown to potentially affect which of these real interest rates are stable and which are more likely to arise in the long run. However, monetary policy will simultaneously maintain a neutrality property, in that it will not affect the values of the potential long-run real interest rate that can arise. These will always be either r^{*H} or r^{*L}.[54]

4.1 Introducing nominal rigidities that allow for a vertical long-run Phillips curve

In the environment considered up to now, we have not included any nominal rigidities. In this section, we extend the model to allow demand considerations to affect economic activity in the short run while maintaining that in the long run economic activity is entirely determined by the economy's productive capacity. In other words, we extend the model in a way that allows for a Phillips curve which reflects a short-run tradeoff between inflation and activity but not a long-run tradeoff. To this end, we slightly modify the environment and assume that output is produced using labor by a set of competitive firms. The production function is given by $y_t = Al_t$, where productivity $A > 0$ is constant. Goods prices p_t are perfectly flexible and therefore competition between firms will ensure that the price of the output good is equal $\frac{W_t}{A}$, where W_t is now the nominal wage. This implies that real wages

[53]Note that when $r^{*L} < \bar{r}$, the income effects/retirement motives dominate the intertemporal substitution effects associated with interest rate movements. Hence, when B increases, r^{*L} has to fall to provide incentives to demand more assets to clear the long-run asset market. For $r^{*H} > \bar{r}$, the intertemporal substitution effects dominate. Thus, when B goes up, r^{*H} rises to clear the asset market. The intuition is similar for changes in δ_2 where an increased expected retirement duration ($1/\delta_2$) boosts asset holdings.

[54]In terms of stability properties in the absence of nominal rigidities, the high-interest-rate steady state obeys a saddle configuration and therefore locally admits only one equilibrium outcome. While the low-interest-rate steady state has a sink configuration and therefore admits the continuum of rational expectations paths for r that converge to r^{*L}.

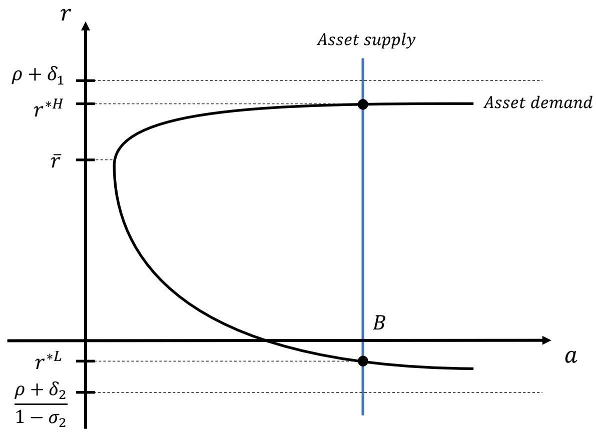


Figure 7
Long-run asset demand and asset supply: multiple steady state real interest rates (r^*)

are always equal to A. We denote $\bar{l}$ and $\bar{y} = A\bar{l}$ the natural rate of employment and output respectively.[55]

The key nominal rigidity we introduce is related to wage determination, where we assume that wage growth increases or decreases depending on whether employment is above or below $\bar{l}$. More specifically, nominal wages W_t are assumed to adjust according to $\frac{\partial\left(\frac{\dot{W}_t}{W_t}\right)}{\partial t} = \kappa'(l_t - \bar{l})$, with $\kappa' > 0$. Since in this model wage inflation is equal to price inflation π_t, the Phillips curve takes the form $\dot{\pi}_t = \kappa'(l_t - \bar{l})$, where $(l_t - \bar{l})$ represents the deviation of employment from full employment $\bar{l}$ and $\kappa' > 0$ governs the relationship between inflation and the employment gap. Expressing this Phillips curve in terms of the output gap leads to

$$\dot{\pi}_t = \kappa(y_t - \bar{y}), \tag{6}$$

where $\kappa = \frac{\kappa'}{A} > 0$ controls the link between inflation and the output gap. Since past wage growth is taken as given, π_t will be treated as a state variable.[56] This formulation of the

[55] The household's budget constraint in real terms is now given by the following, where l_t will be endogenously determined: $c_t + \dot{a}_t = Al_t + r_t a_t - T_t$.

[56] Our assumption of a backward-looking Phillips curve is chosen to keep the presentation simple while simultaneously allowing π_t to be treated as a state variable. It can be shown that our analysis can be generalized to a hybrid Phillips curve of the form $\dot{\pi}_t = \frac{1}{\ln\left(\frac{\beta_1}{1-\beta_1}\right)}\ddot{\pi}_t + \frac{\beta_2}{2\beta_1 - 1}(y_t - \bar{y})$, which is the continuous time analog of a discrete time Phillips curve of the form $\pi_t = (1 - \beta_1)\pi_{t+1} + \beta_1\pi_{t-1} + \beta_2(y_t - \bar{y})$

Phillips curve implies the absence of any long-run tradeoff between inflation and output (or employment).

Since we now allow for variable inflation, we need to distinguish between real and nominal rates of interest. We will denote the nominal rate by i_t with the real rate given by $r_t = i_t - \pi_t$.

The equilibrium dynamics for this economy with nominal wage rigidities (and $q = 0$) is now governed by the following dynamic system

$$\dot{\pi} = \kappa(c_t + G - \bar{y})$$

$$\frac{\dot{c}_t}{c_t} = \frac{i_t - \pi_t - \rho - \delta_1}{\sigma} + \frac{c_t^{\sigma}}{\sigma}\delta_1 q^s V_a(B, \Gamma_t)$$

$$\dot{\Gamma}_t = -1 + \Gamma_t \left[\frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma}(i_t - \pi_t)\right]$$

To complete this model, we need to specify how monetary policy sets the nominal interest rate i_t. Our main focus will be monetary policy that is governed by a simple Taylor rule of the form

$$i_t = \max\left\{0, i^T + \psi(\pi_t - \pi^T)\right\} \qquad \psi > 1, \tag{7}$$

where i^T is a nominal interest rate target, π^T is the central bank's inflation target, $\psi > 1$ satisfies the Taylor principle and the effective lower bound on interest rates is set to 0. However, instead of looking immediately at the implications of this constrained Taylor rule, it is easiest to start with the two embedded sub-cases: (1) $i_t = i^T + \psi(\pi_t - \pi^T)$, that is, disregard the ELB constraint and (2) $i_t = 0$, setting the interest rate at the ELB.

Propositions 3 and 4 highlight how monetary policy affects the stability of the system.[57]

$(0 \leq \beta_1 \leq 1, \beta_2 > 0)$. Note that this alternative specification embeds both forward-looking and backward-looking Phillips curves. When introducing this more general Phillips curve in our analysis, the set of steady states remains the same regardless of the value of β_1, but some stability properties can change if β_1 is sufficiently small. For example, if β_1 is greater than 0.5, then all our main results—including those regarding stability properties and basins of attraction—are maintained under this hybrid Phillips curve specification. None of our results require β_1 to be exactly 1. However, if β_1 is sufficient small, such that the Phillips curve becomes mainly forward looking, certain steady states that our analysis describes as unstable when β_1 is exactly one, can exhibit local indeterminacy as in Benhabib et al. (2001a).

[57]The set-up allows for a more general result: The high real-interest-rate steady state will be stable if nominal policy interest rate setting locally satisfies the Taylor principle, while the low-real-interest-rate steady state will be stable if nominal policy interest rates setting locally does not satisfy the Taylor principle.

In particular, Proposition 3 indicates that if the nominal interest rate setting is unconstrained by an ELB and satisfies the Taylor principle, then the only equilibrium configuration that can be stable is one where the steady state equilibrium real interest rate is equal to r^{*H}. Moreover, if the central bank targets that real natural rate, it will achieve its target for π. Proposition 4 covers the converse case. It shows that if nominal interest rates are set at the ELB, then the only possible configuration for a stable steady state equilibrium is one where the real interest rate equals r^{*L}.

Proposition 3. *If $i_t = i^T + \psi(\pi_t - \pi^T)$ and $\psi > 1$, the economy admits only one stable steady state equilibrium.*[58] *In this equilibrium, the real interest rate equals r^{*H}. If $i^T = r^{*H} + \pi^T$, the central bank attains its inflation target.*

See Appendix D.3 for the proof.

Proposition 4. *If $i_t = 0$, the economy admits only one stable steady state equilibrium. In this equilibrium, the real interest rate equals r^{*L} and $\pi = -r^{*L}$.*

See Appendix D.4 for the proof.

To understand why the stability properties around two different equilibrium real interest rates can depend on the monetary regime, it is helpful to recognize that when inflation is high because consumption is high, endogenous dynamics must favor a reduction in consumption to induce stability. This is actually the case behind both Propositions 3 and 4. However, the underlying mechanisms are quite different. When the system is near the steady state real rate r^{*H} and nominal interest rates increase more than one-to-one with inflation, higher inflation pushes real rates above r^{*H}. With such higher real rates, consumption decreases because the inter-temporal substitution force dominates the retirement motive near r^{*H}. This makes this combination — being near r^{*H} and with real rates rising with inflation — locally stable. In contrast, when the system is near the steady state real rate r^{*L} and i_t is at the ELB, higher inflation pushes down real rates. These lower real rates then depress consumption near r^{*L} because the retirement motive of savings dominates the inter-temporal substitution motive. This makes the alternative combination — being near r^{*L} with real rates falling with inflation — locally stable.

[58] When referring to a stable equilibrium here we are referring to a saddle path stable equilibrium where there are two roots of the system that are positive and one that is negative.

Now we turn to looking at possible equilibrium configurations when the Taylor rule is constrained by the ELB. To give more structure, let us assume that $i^T = r^{*H} + \pi^T$; that is, the central bank targets a real rate (i.e., a natural interest rate) equal to r^{*H}. Given Propositions 3 and 4, one may think that two stable equilibrium configurations would now always be possible with such a rule. However, that is not the case as implied by Proposition 5. Proposition 5 indicates that such a Taylor rule not only has the power to affect the stability properties of different steady state equilibrium real interest rates, it also has the power to affect which actually arise in equilibrium. In particular, if the policy is not very aggressive, that is if $\psi > 1$ is sufficiently close to 1, then only one type of stable equilibrium configuration will arise and that configuration has the real interest rate equal to r^{*H} and inflation on target. In contrast, if the policy is sufficiently aggressive (and π^T is not too small), then two stable equilibrium configurations are possible with two different real interest rates. The low real rate equilibrium is accompanied by a nominal rate at the ELB, while the high real rate equilibrium is accompanied by the nominal rate being at target.

Proposition 5. *When i_t is set according to $i_t = \max\left\{0, r^{*H} + \pi^T + \psi(\pi_t - \pi^T)\right\}$ with $\psi > 1$ and $\pi^T > -r^{*L}$, then there is a cutoff level of monetary tightness $\bar{\psi} \equiv \frac{r^{*H}+\pi^T}{r^{*L}+\pi^T} > 1$, such that the following holds*

- *if $\psi > \bar{\psi}$ (i.e., if monetary policy is sufficiently aggressive), the economy admits two stable steady state equilibrium outcomes; one with the real interest rate equal to r^{*H} and one with the real rate equal to r^{*L}. In the equilibrium with the real interest rate equal to r^{*H}, inflation is on target. In the equilibrium with the real interest rate equal to r^{*L}, inflation is below target and the policy rate i_t is at the ELB.*
- *If $1 < \psi < \bar{\psi}$, then the economy admits only one stable equilibrium. In this equilibrium the real interest rate is equal to r^{*H} and inflation is on target.*

See Appendix D.5 for the proof.

To understand why monetary policy has the power to affect long-run real interest rate outcomes as indicated by Proposition 5, it is useful to recall how a constrained Taylor rule translates inflation into real rates. This is illustrated in Figure 8. As is indicated on the figure, the ELB constraint becomes binding at the inflation level $\pi^{ELB} \equiv \frac{(\psi-1)\pi^T - r^{*H}}{\psi}$. To the right of this binding level of inflation, real rates are increasing with inflation, while to the left, real rates are decreasing with inflation. A higher value of ψ implies that the ELB

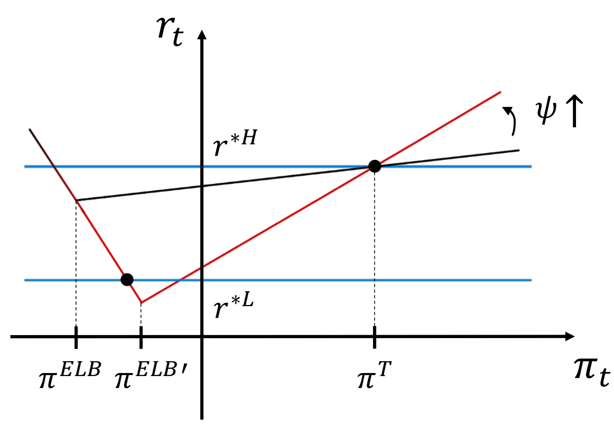


Figure 8
Link between real interest rates and inflation under a Taylor rule constrained by the ELB

constraint will become binding at higher levels of inflation. Therefore, a smaller $\psi > 1$ allows for a smaller range of real interest rates. Accordingly, with a $\psi > 1$ sufficiently close to 1, r^{*L} will not be feasible, while r^{*H} would be feasible. As ψ increases, this allows for a larger range of real rates and generally makes an equilibrium with the real rate at r^{*L} feasible (as long as $\pi^T > -r^{*L}$).[59]

4.2 Illustrating transitional dynamics and how monetary policy can affect basins of attraction

We saw from Propositions 3, 4 and 5 that in the presence of multiple steady state real interest rates r^*, monetary policy can affect which real interest rate may arise in equilibrium and what stability properties it may have. In this section, we illustrate the transitional dynamics associated with the different possible outcomes. In particular, we want to show how aggressive monetary policy can go beyond simply allowing a low real interest rate equilibrium to emerge: it can also affect its basin of attraction and therefore make it more likely to arise the more aggressive policy is (higher ψ).

An illustration of transitional dynamics associated with the case of one stable steady state is represented in Figure 9, while the case with two stable steady states is represented

[59] From this diagram, one can also see that for a given feasible real rate, the system would allow for two associated outcomes: one at the ELB and one above the ELB. Proposition 5 indicates that the stable one will be the one above the ELB when the real rate is r^{*H} and the one at the ELB when the real rate is r^{*L}.

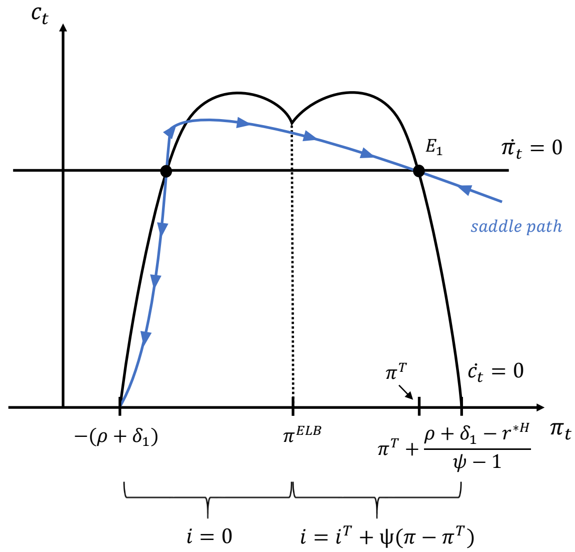


Figure 9
Equilibrium trajectories when monetary policy follows a not too aggressive Taylor rule: one stable steady state

in Figure 10. On this figure, we represent the steady state condition between c and π implied by the $\dot{c}_t = 0$ condition when $\dot{\Gamma}_t = 0$. This is best represented by pieces:

If $\pi < \pi^{ELB}$, the $\dot{c}_t = 0$ curve is given by

$$c = B\left(\delta_1 q^s\right)^{-1/\sigma}\left[\rho+\delta_1+\pi\right]^{1/\sigma}\left[\frac{\rho+\delta_2}{\sigma}+\frac{1-\sigma}{\sigma}\pi\right]$$

and if $\pi \geq \pi^{ELB}$, the $\dot{c}_t = 0$ curve is given by

$$c = B\left(\delta_1 q^s\right)^{-1/\sigma}\left[\rho+\delta_1-(i-\pi)\right]^{1/\sigma}\left[\frac{\rho+\delta_2}{\sigma}-\left(\frac{1-\sigma}{\sigma}\right)(i-\pi)\right]$$

where $i - \pi = r^{*H} + (\psi - 1)(\pi - \pi^T)$.

On this figure, we also depict the steady state condition $\dot{\pi}_t = 0$ which corresponds to $c = A\bar{l} - G$. The crossings between these two curves give us the set of steady states. Finally, on the figure, we plot transitional dynamics in blue which illustrate the stability properties of the steady state. These transitional dynamics should be viewed as a projection of the actual transitional dynamics which are in the three dimensional space $\{c_t, \pi_t, \Gamma_t\}$. Note that the $\dot{c}_t = 0$ curve almost mirrors itself around the cutoff π^{ELB}.

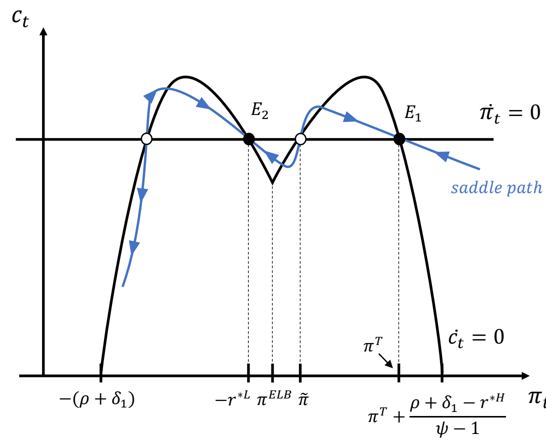


Figure 10
Equilibrium trajectories when monetary policy follows a sufficiently aggressive Taylor rule: two stable steady states

In Figure 9, E_1 is the only stable steady state. E_1 is a high-real-interest-rate (r^{*H}) with inflation on target. There is also a low-inflation steady state in Figure 9, but it is not stable. The nominal interest rate at the unstable steady state in the figure is in the ELB region since the level of inflation arising from that equilibrium point is less than π^{ELB}. This type of configuration, where there is an unstable steady state at the ELB and a stable steady state with $i > 0$, echoes what arises in a standard infinitely lived representative agent environment (see Benhabib et al. (2001a)).[60] In contrast, in Figure 10 we now have two stable steady states. The high-real-rate stable steady state, denoted E_1, remains, but now we also have one low-real-rate (r^{*L}), low-inflation stable steady state denoted E_2. The E_2 steady state is in the ELB region, while the E_1 steady state remains in the region where $i > 0$ and where the Taylor principle is operative. Proposition 5 expresses this possibility. In this setting, given the two stable steady states, the system will exhibit *hysteresis*.[61] If inflation starts above the level $\tilde{\pi} = \pi^T + \frac{r^{*L} - r^{*H}}{\psi - 1}$ denoted on Figure 10, the system will converge to E_1, while if it starts below, it will tend to converge to E_2. In this set-up we can consider the effects of shocks, especially q^s shocks which increase the desire to accumulate more assets for retirement (precautionary) motives. For example, if the economy were

[60] Recall that we are assuming a backward-looking Phillips curve in the main body of the text. When assuming a forward-looking Phillips curve, this equilibrium would exhibit indeterminacy.

[61] In the case where the parameter κ in the Phillips curve is negative, the same two steady states are determinate stable, and the system would jump to one of them instead of exhibiting hysteresis.

to start at E_1, and there was a large temporary rise in q^s, the steady state equilibrium E_1 could temporarily disappear — the reason being that there would then be too much demand for assets relative to supply, which depresses demand. As a result, there would be a contractionary period with deflation. Once the shock reverses itself, the level of inflation would be starting from a lower level. If this new inflation level was below $\tilde{\pi}$, the economy would converge to the long-run equilibrium at E_2 even if it was at equilibrium point E_1 before the temporary shock to q^s.

In this setting, we can highlight the potential role of increasing the aggressiveness of monetary policy—as captured by high values of ψ —in making the low-inflation equilibrium outcome in Figure 10 more likely, that is, making it more likely that the economy converges to a low real interest rate.[62] Recall that we are always assuming that $\psi > 1$, so the Taylor principle is active when not constrained by the ELB. If monetary policy is not too aggressive in the sense of ψ not being much greater than 1, then the equilibrium configuration will take the form we represented in Figure 9.[63] So in this case with monetary policy not too aggressive (but still satisfying the Taylor principle when above the ELB), the economy can only converge to the E_1 equilibrium. This has the desired outcome of supporting inflation close to target. However, as ψ is increased the range of inflation that leads monetary authorities to set i at the ELB increases. A rise in ψ can therefore be seen as changing the equilibrium configuration from that depicted in Figure 9 to that depicted in Figure 10. In fact, as ψ gets very high, the equilibrium dynamics can make the high-real-rate equilibrium fragile. This can be seen in Figure 11. In this figure, we represent equilibria in the (i, π) space as this offers an alternative perspective to discuss the dynamics. The two different real rates are represented in the panels of this figure by lines with slope of one and with a Taylor rule super-imposed. In this space, equilibrium dynamics can be summarized along the π axis, as π is the only state variable and the dynamics are driven by the stability of the different steady states for π. In Panel A of Figure 11, we represent the case where monetary policy is not very aggressive and therefore permits only the high-real-rate equilibrium to arise. In Panel B, monetary policy is more aggressive allowing two stable steady states to arise. One at the high interest rate with inflation on target and one at the low interest rate, where inflation is below target and the ELB is binding. As can also be seen in Figure 11 moving from Panel B to Panel C—that is, when monetary policy reacts more to below

[62] We also examined the effect on equilibrium outcomes of changing the inflation target π^T. Among other results, we find that increasing π^T favors the status quo; that is, we find that the basin of attraction of neither the stable ELB equilibrium nor the non-ELB equilibrium decreases when π^T increases. Hence, if an economy were caught in a low-inflation, low-real-rate trap, increasing π^T would not help exit this trap.

[63] For this equilibrium configuration, we are assuming that $\pi^T > -\bar{r}$ and $\psi > \frac{r^{*H}+\pi^T}{\bar{r}+\pi^T}$. See Appendix D.4.

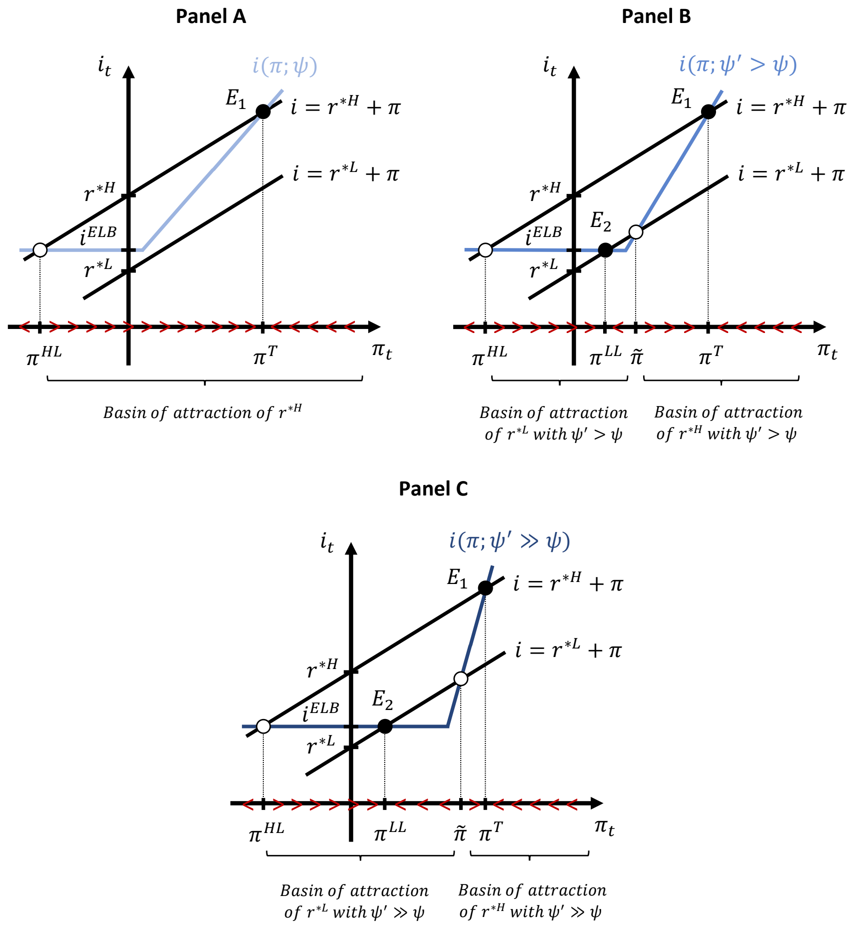
target inflation—leads the range of inflation rates above $\tilde{\pi}$ that support the higher-inflation equilibrium E_1 to become arbitrarily small. This implies that when an economy in Panel C is subjected to shocks, even if it starts at the high-real-rate equilibrium with inflation on target, it is likely to end up at the low-inflation ELB equilibrium. In this sense, a high ψ policy of reducing interest rates aggressively in response to deviation of inflation from target can contribute to the economy ending up at a low steady state real rate of interest. It is worth emphasizing that at the low interest rate equilibrium, inflation is low (possibly negative), but it is nonetheless stable even if the Taylor principle does not hold. Proposition 5 confirms that the existence of the E_2 equilibrium depicted in Figures 10 and 11 actually depends on $\psi > 1$ being sufficiently large. If ψ is not sufficiently large, the configuration depicted in Figure 10 (and in Panels B and C of Figure 11) cannot arise.

4.2.1 Exiting the low-real-rate trap: the effects of inflation shocks and expansionary fiscal policy

When the economy is in a low-real-rate trap, as represented by the equilibrium outcome E_2 shown in Figures 10 and 11, a sufficiently large exogenous shock to inflation could move inflation above the central bank's inflation target.[64] If such a high rate of inflation were to arise, the central bank would increase nominal interest rates aggressively causing real rates to rise also. This would place the economy temporarily in recession in order to reduce inflation. As inflation declines and the employment recovers, interest rates — both real and nominal — gradually decrease. However, the economy would not return to E_2. Instead, it would converge to the steady state E_1 with the high real rate. Hence, when the economy is at E_2 and there is a large inflation shock, this can cause the long-run real interest rate to increase from r^{*L} to r^{*H}.

Fiscal policy can also help create an exit from the low-real-rate steady state, but this exit has non-monotonic properties. An increase in government debt B corresponds to an upward shift in the $\dot{c}_t = 0$ in Figure 10. This implies that the long-run equilibrium point E_2 will move to the right when B is larger, implying higher inflation and lower real interest rates (see Lemma 2). This is expressed in Proposition 6. However, the effect of changes in B on long-run interest rates and long run inflation will be discontinuous. As debt rises, there will come a point where the E_2 equilibrium will cease to exist, as r^{*L} becomes too small to be supported as an equilibrium for given ψ). At that point, the only stable equilibrium will

[64] We are interpreting an inflation shock as an unexpected shock to the production cost of firms, which is passed through to prices. In other words, it is an unexpected shock to the Phillips Curve equation which causes a discrete jump in inflation.



Note: The blue lines in the figure represent a Taylor rule which is constrained by a lower bound, aims for a target level of inflation π^T, and reacts to inflation as governed by the parameter ψ. The black parallel lines reflect different possible real interest rates.

Figure 11
The interaction of a Taylor rule with more than one equilibrium r^*

be E_1. Hence, both the long-run real interest rate and the long run rate of inflation in such an economy can change discretely in response to a large fiscal expansion. A sufficiently large increase in B can create a switch from the long-run equilibrium E_2 to the long-run equilibrium E_1. Fiscal policy in this case, is pushing the economy out of the low-real-rate, low-inflation steady state, but that is coming at the cost of a discrete jump in long-run inflation and r^*.[65]

Proposition 6. *The inflation rate at the ELB stable steady state is increasing in government*

[65] Acharya and Dogra (2022), Eggertsson and Mehrotra (2014), and Mian et al. (2021a) also find that rising public debt favors an escape from the ELB.

debt B, while real interest rates are decreasing. However, when B becomes sufficiently large, the equilibrium at the ELB will cease to exist. At that point, long-run inflation and real interest rates will exhibit a discontinuous jump to higher levels.

See Appendix D.6 for the proof.

5 Extending the Model to Include Productive Assets: Lucas Trees

Up to now, we have been examining the equilibrium determination of long-run real interest rates—and the role of monetary policy—in the presence of only one asset: government bonds. In this section, we enrich the environment by introducing a claim on a productive asset, where the price of the asset increases when interest rates decrease, that is, we introduce valuation effects into the analysis. As we shall see, valuation effects render the analysis more complex but do not overturn our main results regarding both the possibility of multiple r^* and the role of monetary policy in affecting which r^* is most likely to arise. It is for this reason that we left the introduction of valuation effects until now.

To introduce a second asset into our set-up, suppose there is a mass s of Lucas trees that produce a flow f of goods every period.[66] In order to introduce the possibility of something akin to a risk premium on these assets, we will assume that trees die at flow rate $\omega \geq 0$ and that dead trees are continuously replaced with new trees redistributed in a lump sum fashion to active households. In aggregate, trees are not risky, they simply decay at rate ω. A household can now hold a combination of bonds and trees. If we denote by z_t the price of a mass of one of trees at time t, then arbitrage between the two assets will cause z_t to satisfy the following asset pricing relationship

$$\frac{\dot{z}_t}{z_t} = \frac{f}{z_t} - (r_t + \omega),$$

and households will be indifferent between holding bonds or trees. The advantage of allowing for trees to decay is that they permit situations where r can be zero and the price of trees is still finite. The household consumption Euler equation in this case can be re-written as

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \delta_1 q^s \frac{c_t^\sigma}{\sigma} V_a(\Omega_t, \Gamma_t),$$

[66] A Lucas tree set-up is one where we have productive assets but these assets are not themselves expandable. We have also explored the possibility of allowing for reproducible productive assets and have not found it to give novel insights relative to the case analyzed here. For this reason, we chose to focus on the simpler case.

where Ω denotes household wealth which includes both the holdings of bonds and trees. The main effect of introducing this second asset is that it causes the effective supply of assets that must be held by the market, $B + z_t s$, to contain valuation effects. To present this case more easily, it is helpful to recognize that the steady state of the consumption Euler equation can be presented as expressing a household's *desired* ratio of consumption-to-wealth as a function of interest rates. Note that this is simply a reinterpretation of the steady state condition for households' consumption decision in the previous sections. Accordingly, the desired consumption-to-wealth ratio maintains the hump shaped property because it balances inter-temporal substitution effects and retirement incentives. Furthermore, the desired consumption-to-wealth ratio maintains the property that it goes to zero as r goes to either $\rho + \delta_1$ or to $\frac{\rho+\delta_2}{1-\sigma}$. Relative to our analysis with only bonds, household desired wealth holding is unchanged with the introduction of trees. The feature that changes with the introduction of productive assets is the properties of the feasible aggregate long-run consumption-to-wealth ratio. Previously, the feasible long-run consumption-to-wealth ratio was $\frac{A\bar{l}-G}{B}$ and therefore independent of r. The economy's feasible long-run consumption-to-wealth ratios with trees is now given by

$$\frac{c}{\Omega} = \frac{A\bar{l} + sf - G}{B + \frac{sf}{r+\omega}},$$

where the numerator represents full employment output plus the flow of goods from trees less government consumption and the denominator represents the total value of assets in steady state. This is the aggregate consumption-to-wealth ratio that is consistent with full employment and $\dot{z}_t = 0$. This feasible consumption-to-wealth ratio is increasing in r for $r > \omega$, and starts from zero when $r = -\omega$. If $s = 0$, then this feasible consumption-to wealth ratio is independent of interest rates and we are back to our previous analysis where the only possible equilibrium configuration is one where there are two natural interest rates r^*. The introduction of trees increases the possible equilibrium configurations. This is due to it changing the shape of the feasible consumption-to-wealth ratio. In the absence of sticky prices, there now can be at least three equilibrium configurations. These depend on slope of the feasible consumption-to-wealth ratio curve compared to the slope of desired consumption-to-wealth ratio curve.

The dynamic equilibrium equations in the absence of sticky prices are now given by

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \delta_1 q^s \frac{c_t^\sigma}{\sigma} V_a(\Omega_t, \Gamma_t),$$

$$\dot{\Gamma}_t = -1 + \Gamma_t \left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r_t\right]$$

$$\frac{\dot{\Omega}_t}{\Omega_t - B} = \frac{fs}{\Omega_t - B} - (r_t + \omega),$$

where r needs to adjust to satisfy the market clearing condition

$$c_t = A\bar{l} + sf - G.$$

We refer to the desired consumption-to-wealth ratios as the consumption-to-wealth ratios c/Ω that satisfy $\dot{c}_t = \dot{\Omega}_t = \dot{\Gamma}_t = 0$ for different levels of r. The feasible consumption-to-wealth ratios are defined as the set of Ω_t that satisfy $\dot{\Gamma}_t = 0$, $\dot{\Omega}_t = 0$ and the market clearing condition.

In the presence of productive assets, it can still be the case that only two steady state values for r^* are possible. The analysis of the previous section extends directly to such a case, with the only difference being that the consumption-to-wealth ratio is now lower at r^{*L} than at r^{*H}. Given this minor difference, we will not dwell on this case in this section. With productive assets, it is now also possible that there be only a unique equilibrium value of r^* despite the hump shape in desired consumption-to-wealth ratios. This was not possible in the absence of valuation effects. Such a configuration will arise if desired wealth holdings never outpace the valuation effects when r decreases. This case is important as it indicates that even in the presence of C-shaped asset demands, there is not necessarily more than one equilibrium. There can be a unique equilibrium if valuations effects play the right role. Finally, Figure 12 illustrates the case where there are three possible values of r^*. Since this is the novel case, we will focus on it.

The important element to note in Figure 12 is the possibility of three real interest rates (r^*) compatible with full employment. There are two steady states which resemble E_1 and E_2 in terms of how the curves cross, but now a third equilibrium appears. This third equilibrium, which we will denote E_3, has an associated real interest rate which we denoted by r^{*LL}.[67] This steady state equilibrium arises with both very low real interest rates and high households' asset holdings. The households exhibit a very low consumption-to-wealth ratio in this equilibrium. In the absence of Lucas trees, this configuration was not possible as the feasible consumption-to-wealth ratio did not change with r. However, with the Lucas trees, the high demand for assets at r^{*LL} is satisfied by the high valuation of Lucas trees

[67] Note that r^{*LL} may be negative.

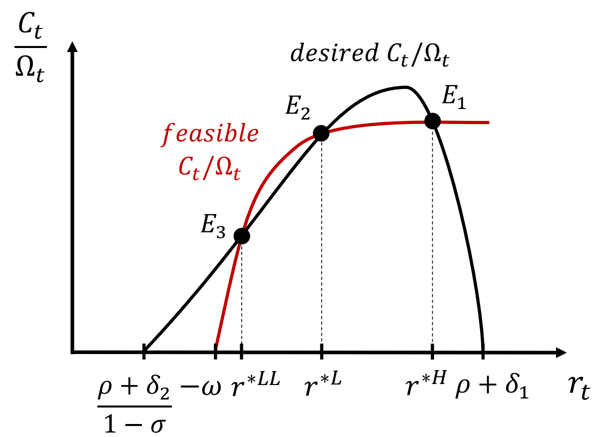


Figure 12
Steady state equilibrium outcomes in the presence of Lucas trees as a function of the size of dividend flow f: three potential steady states

which acts as to endogenously increase the supply of assets.

Assuming that the real side of the economy takes the form as in Figure 12 , we can re-introduce sticky prices and a Taylor rule to look at the joint determination of c_t and π_t as we did before.[68]

There are now two sub-cases to consider. The easy case is when r^{*LL} is small relative to the inflation target π^T in the Taylor rule, that is, when $r^{*LL} < -\pi^T$. In such a case, monetary policy is ruling out the E_3 type equilibrium, and all our previous results again carry over. In particular, if monetary policy is not very aggressive (but still satisfying the Taylor principle), then there can be only one stable steady state equilibrium and that corresponds to the high-real-rate equilibrium E_1. As monetary policy gets more aggressive,

[68]In the presence of Lucas trees, the set of dynamic equations representing the equilibrium with sticky prices and a Taylor rule can be reduced to

$$\frac{\dot{c}_t}{c_t} = \frac{i_t - \pi_t - \rho - \delta_1}{\sigma} + \delta_1 q^s \frac{c_t^\sigma}{\sigma} V_a(\Omega_t, \Gamma_t),$$

$$\dot{\pi}_t = \kappa(c_t + G - A\bar{l} - fs), \qquad \kappa > 0$$

$$\dot{\Gamma}_t = -1 + \Gamma_t \left[\frac{i_t - \pi_t + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma}(i_t - \pi_t) \right]$$

$$\frac{\dot{\Omega}}{\Omega_t - B} = \frac{fs}{\Omega_t - B} - (i_t - \pi_t + \omega).$$

$$i_t = \max\left\{0, r^{*H} + \pi^T + \psi(\pi_t - \pi^T)\right\} \qquad \psi > 1$$

the equivalent of equilibrium E_2 will appear as a stable steady state of the system with nominal frictions.[69] E_2 will again be associated with the nominal interest rate being at the ELB. And as monetary policy becomes gradually more aggressive, the basin of attraction of this E_2 equilibrium will expand while that of E_1 will become small. In this sense, our previous analysis with only bonds extends directly to this case as long as $r^{*LL} < -\pi^T$.

Now if $r^{*LL} > -\pi^T$, then the equilibrium dynamics can get more complex than that presented with only bonds. For example, it can take the form as given in Figure 13. In this case, it is possible to have three stable steady states with different levels of inflation and different real rates. The high-real-rate equilibrium corresponding to E_1 remains. As before, an ELB equilibrium with a low real rate at r^{*L} will also be present when monetary policy is sufficiently aggressive. This is point E_2. But, now we get the possibility of a third equilibrium; this one implements the real rate r^{*LL} and is not in the ELB region. This equilibrium has a low real rate — even lower than that of the E_2 equilibrium — even though the nominal interest rate is positive. The price of the Lucas trees at the E_3 equilibrium, which is given by $z = \frac{f}{\omega+i-\pi}$ in steady state, will be higher in the E_3 equilibrium than in both the E_2 and E_1 steady state equilibria. The inflation at E_3 is given by $\pi = \pi^T - \frac{r^{*H} - r^{*LL}}{\psi - 1}$. With such a configuration, if the economy were to start in the E_1 equilibrium and be subject to a set of negative inflation shocks[70], it would likely go from E_1 to E_3, with a drop in inflation and a rise in asset prices.

Let us now examine how the equilibrium configuration depicted in Figure 13 changes as monetary policy gets more aggressive. This is depicted in Figure 14. Like previously, we return to representing equilibria in the (i, π) space. The three different real rates are represented in the panels of Figure 14 as before by lines with slope of one and with a Taylor rule super-imposed. In this space, equilibrium dynamics can be summarized along the π axis, as π is the only state variable and the dynamics are driven by the stability of the different steady states for π. As can be seen in Figure 14, when policy becomes more aggressive ($\psi > 1$ becomes larger, i.e., moving from Panel A to Panel B), the inflation level at the E_3 steady state equilibrium gets closer and closer to that at the E_1 equilibrium. Hence, with very aggressive monetary policy we can get a situation where the two steady state equilibria E_1 and E_3 are very close together in terms of inflation outcomes, but far apart in real interest rate outcomes. This arises because the nominal rate is much lower at the E_3 equilibrium than at the E_1 equilibrium since monetary policy is very aggressive in cutting rates when inflation is below target. Furthermore, in this case, both E_3 and E_1

[69] The proof of the stability of these steady states is similar to that in Propositions 3-5. It is available upon request.

[70] We are interpreting inflation shocks as shocks that change the initial level of inflation.

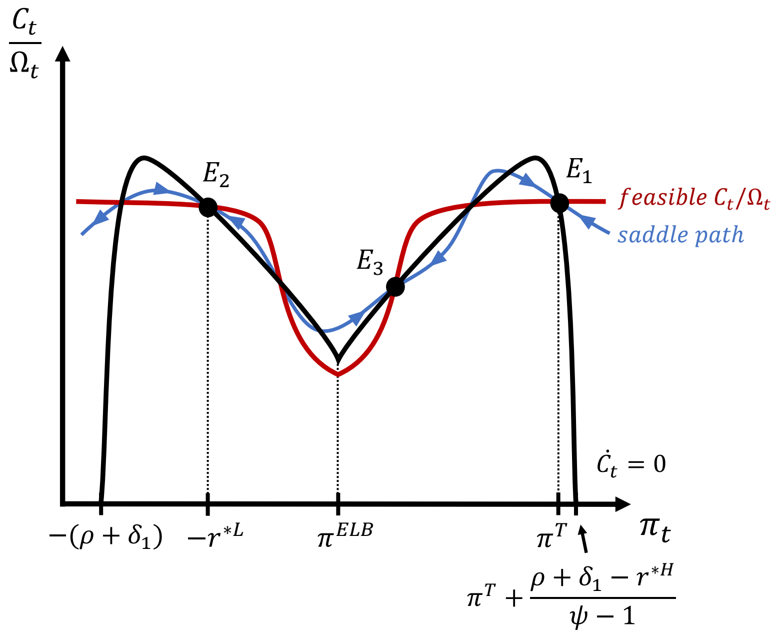


Figure 13
Equilibrium trajectories in the presence of Lucas trees when the inflation target is sufficiently high: three stable steady states

inherit a fragility property. As seen previously, the steady state E_1 becomes fragile with respect to downward shocks to inflation. In contrast, the E_3 equilibrium will be quite robust to downward shocks to inflation as its basin of attraction to its left actually expands as policy gets more aggressive. However, when the policy is very aggressive, the E_3 equilibrium will become fragile to positive shocks to inflation as the relevant basin of attraction to its right can become arbitrarily small.

When the economy is at E_3 or E_1, it could still be pushed to the ELB equilibrium at E_2. This would require a sufficiently large downward shock to inflation. A move from E_3 to E_2 would cause a drop in inflation, but it would also be associated with a fall in asset prices. The mechanism could also work in reverse. If the economy is at either E_3 or E_2 and there were a sufficiently large exogenous positive shock to inflation, then the economy could find itself back to E_1.

In summary, the presence of a productive asset in the form of a Lucas tree enriches our previous analysis but it does not change the basic messages. Because there can be more than one real natural interest rate r^*, monetary policy becomes an important force

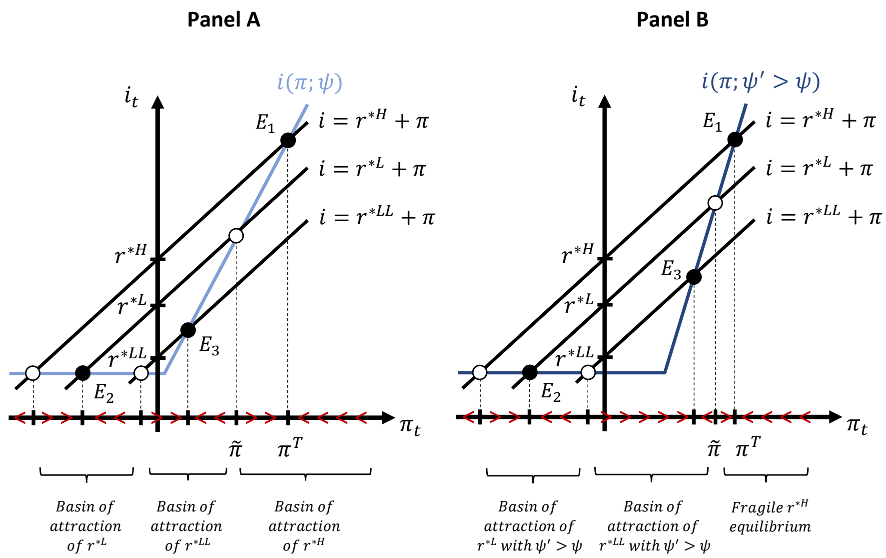
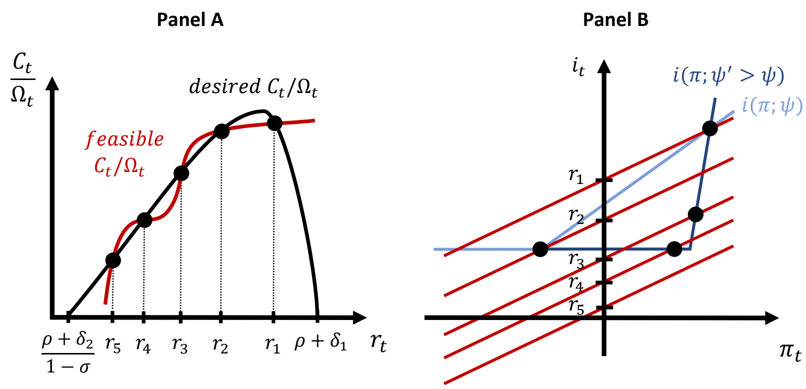


Figure 14
Equilibrium trajectories in the presence of Lucas trees when the inflation target is sufficiently high and monetary policy is very aggressive

in determining long-run real rate outcomes. In particular, the more monetary policy aggressively targets inflation, the more likely it is to cause the high-real-rate equilibrium to be fragile to negative shocks to inflation. This makes the economy likely to converge to a low-real-rate equilibrium. The main additional property that arises with the presence of a Lucas tree is that a lower-real-rate equilibrium does not necessarily happen only at the ELB. It can also arise with nominal interest rates above the ELB and with inflation close to target. Hence this set-up offers an explanation for why economies can get stuck with low real interest rates at either the ELB or above the ELB, where in both cases we would have a high valuation of productive assets.

6 Back to full model

In the previous sections we have been analyzing the monetary policy implications of having C-shaped asset demands by active households, where the C-shape arose due to the competing motives of inter-temporal substitution and retirement. However, we have been conducting most of our analysis under the simplifying assumption that active households were actually the only type of living households. Recall that we assumed toward the end of Section 4 that active households perceived a risk of needing assets to pay for retirement,



Note: The red lines in Panel B represent the steady state Fisher equation $i = r_i + \pi$ for $i = 1, \dots, 5$. The dark blue and light blue lines in the figure represent a Taylor rule which is constrained by a lower bound, aims for a target level of inflation π^T, and reacts to inflation as governed by the parameter ψ.

Figure 15
Multiple equilibrium real interest rates in the general model with both active households and retirees

but that they actually died before needing these funds. This gave rise to a perpetual youth type setup where households always stayed young but nevertheless saved for a possible retirement that never actually happened. In introducing this assumption we claimed that it was not driving our main results. In this Section, we return to the more general case where we remove this assumption and allow active and retired households to co-exist. Our goal in this section is to illustrate why our results carry through to this more general case.

The implications of dropping this assumption (i.e., dropping $q^s \neq q$) can most easily be seen on Panel A of Figure 15. In this figure, we plot the desired long-run consumption-to-wealth ratio of active households against real interest rates. This locus, which is in black, is now familiar and it is not changed with the drop of the assumption. The red line represents the feasible long-run consumption-to-wealth ratio of active households, where now the feasible outcome includes the fact that retired individuals are both consuming resources and holding assets. This figure is abstracting from sticky prices and we are allowing for both bonds and productive assets to be present.[71] This figure is very similar to that we

[71] The equilibrium behavior is now described by the following system of four dynamic equations:

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma}\delta_1 q V_a(\Omega_t, \Gamma_t)$$

$$\dot{\Gamma}_t = -1 + \Gamma_t\left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}r_t\right]$$

presented in Section 5 where we introduced productive assets. However, there is one main difference which relates to the precise properties of the feasibility locus. Previously, this feasibility locus was monotonically increasing in r and concave. However, in the more general model this feasibility locus can be less well behaved, leading it to potentially cross the locus of desired consumption-to-wealth ratio of active households several times. To be clear, in the more general case, there may still only be two or three crossings as before, but we can't rule out more crossings. Hence, as shown in the figure, we could, for example, have five crossings.

In Panel B, we translate this five equilibrium example in Panel A into its implications in terms of feasible stable steady state equilibria in the presence of sticky prices and an ELB constrained monetary policy. The five parallel red lines in Panel B represent the five potential real interest rates from Panel A, while the two different blue lines represent two different monetary policy rules, one being more aggressive than the other (both reflecting our previous specification of a Taylor rule satisfying the Taylor principle when not constrained by the ELB). The black dots at the intersections of the lines represent stable outcomes. In this more general set-up, we can see the main properties we have previously emphasized. First, the stability around the different real interest rates depends on the local property of monetary policy. The highest rate will be stable if monetary policy locally satisfies the Taylor principle; the next highest real rate will be stable if monetary policy locally does not satisfy the Taylor principle as is the case at the ELB.[72] Any additional potential real rate equilibrium will reflect the same pattern, alternating between being stable under a monetary policy that satisfies the Taylor principle or the inverse. The second feature is how monetary policy affects the set of effective steady state equilibria. If monetary policy is not too aggressive, then the set of steady state equilibria will be more limited. In the figure, the light blue line represents a monetary policy with limited aggressiveness which results in only one equilibrium. The dark blue line reflects a more aggressive monetary policy and results in four possible steady states: two above the ELB and two at the ELB. In this later case,

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$$\dot{\Omega}_t = w + r_t\Omega_t + \frac{(B + z_t s - \Omega_t)\delta_2}{\phi} - \frac{G + Br_t}{\phi} - c_t$$

$$\frac{\dot{z}_t}{z_t} = \frac{f}{z_t} - (r_t + \omega)$$

plus the goods market clearing condition $\phi c_t = \phi w - G - (B + z_t s - \phi\Omega_t)\Gamma_t^{-1}$. This system governs asset holdings across active and retired households. However, it does not give a breakdown of holdings of trees versus bonds, as the two are perfect substitutes in equilibrium. An easy fix to this indeterminacy is to assume that both active households and retirees hold the same fraction of wealth in bonds and trees.

[72]The proof of this statement is available from the authors.

the lowest real rate from Panel A remains unattainable as part of a nominal equilibrium with this particular monetary policy.[73]

The main message to convey from Figure 15 is that much of our previous analysis– which relied on an assumption that eliminated retirees but not retirement savings– provided insights regarding monetary policy that are robust to eliminating the assumption.

7 Conclusion

The idea that monetary policy may have contributed to the secular decline in real interest rates is a popular theme among many financial market participants and economic commentators. However, evaluating this type of claim is difficult without first specifying the mechanisms that could in theory generate such an outcome. Motivated by observations regarding within-group changes in wealth-to-income ratios since the late 1980s, we showed how savings behavior which is influenced by both inter-temporal substitution and retirement motives could support/rationalize such claims. Specifically, we highlighted how such savings behavior can give rise to long-run asset demands that are C-shaped with respect to real interest rates and favor multiple steady state equilibrium real rates. Moreover, we showed that in such an environment, an aggressive inflation targeting can render the high-real-rate equilibrium fragile and favor the convergence to a low-real-rate trap. However, the resulting low-real-rate trap is not insurmountable. In particular, we found that the economy may return to a high-real-rate equilibrium if it is subjected to either a large exogenous increase in inflation or to a large increase in public debt. Since both these forces are currently at play, it raises the possibility that the future could involve a much higher real rate.

[73]One aspect that is more complicated to present in this general case is the basin of attraction of the different steady states because the state space is larger.

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Appendix

A Data

For the main analysis we use four waves of the US Survey of Consumer Finances for 1989, 1992, 2016 and 2019. The 1989 and 2019 SCFs are used for the wealth-to-income ratio decomposition into between- and within-group components, while the 1989-1992 and 2016-2019 SCFs are used in the construction of saving rates corresponding to the beginning (1989) and the end (2019) of our period of interest for the joint analysis of changes in wealth-to-income ratios and saving rates. We further supplement the findings using SCF micro-data alone with the results that combine SCF with household-level aggregates reported in the US Flow of Funds Accounts and the National Income and Product Accounts.

Household wealth in the SCF is defined to include all assets of households (both real and financial) net of their liabilities. On the one hand, household non-financial (real) assets include primary and other residential real estate, non-residential real estate equity, as well as equity holdings in privately held businesses (both corporate and non-corporate) and other non-financial assets. Financial assets, on the other hand, include fixed-income assets, e.g. bonds, deposits, as well as mutual fund holdings, and directly and indirectly held stocks, and other financial assets. The split into fixed-income vs. equity components also covers defined contribution pensions of US households. While SCF collects information about the types of pensions households are entitled to (account or traditional pensions), the estimates of the wealth in defined benefit plans are not directly available. Given the importance of these plans in household pension wealth, we use estimates from Sabelhaus and Volz (2020) to construct a measure of wealth in SCF that includes defined benefit pensions, and use aggregate shares from detailed FFA pension accounts to split them into fixed-income vs. equity components, similar to defined contribution account pensions. Unlike other papers, we also do not exclude vehicles as a measure of consumer durables from household wealth in the SCF, given its importance for less wealthy households, which makes our measure of saving closer to the concept used by the Flow of Funds Accounts. On the liability side, we include both mortgage and non-mortgage household debt obligations.

When combining SCF with household-level aggregates from the Flow of Funds Accounts, we follow the literature in consistently defining detailed asset and liability classes in SCF and aggregate data, and then creating a larger number of asset/liability classes (see, for example, Mian et al. (2021b)), for which group ownership shares can be defined. The same grouping into a larger number of asset/liability classes is also useful for the construction of saving rates in raw SCF data, given that pure inflation factors from Mian et al. (2021b) are defined for the same asset and liability classes. We then construct each group's share in the total value of each asset/liability category and distribute FFA aggregates between groups using these shares. Each group's net worth is summed up using the values for each component. On the income side, we follow a similar approach by aggregating each group's income from its components, e.g., wages, business income, interest and dividend income, etc., which, in particular, allows us to be consistent with the balance sheet composition of households, at least on the asset side and the incomes generated by these assets. Similar to the assets/liabilities we do adjustments to the income components reported in SCF to make them consistent with their aggregate counterparts. See Feiveson and Sabelhaus (2019) for the discussion of the comparison between different components of wealth/income reported in FFA/NIPA and SCF.

When reporting results, we prefer using the SCF-based results given that they allow us to construct consistent wealth-to-income ratios and saving rates (in particular, adjusting for net bequests, which can only be constructed in SCF) from the same data source. However, we also show that our wealth-to-income ratio decomposition results are largely unchanged when we use scaled SCF (aggregate) estimates, consistent with the literature. The scaled results in the aggregate do provide a better fit with the saving rates obtained from NIPA/FFA, which is why together with the main results for correlations between group-wise changes in wealth-to-income ratios and changes in saving rates using both raw and scaled SCF data, we provide additional evidence using scaled data as well.

Other data we use for the empirical analysis include pure price inflation factors from Mian et al. (2020), whose replication package provides them until 2016. We extend the series until 2019 using their methodology for different asset categories.[74] Since Mian et al. (2020) measures of wealth and saving do not include consumer durables, we also use an additional factor for consumer durables, and test the results for robustness to its different values.

B Robustness results for the wealth-to-income ratio change decomposition

B.1 Shift-share decomposition: alternative groupings

In this section, we present robustness results associated with using a different number of income-age groups (in Table B1) and using 2019 as a base-year (in Table B2) for the decomposition results.

Table B1
Shift Share Decomposition of the Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019: Robustness to number of age-income groups

Grouping	Total Change	Within, %	Between, %
10 inc gr x 6 age gr	2.82	59.4	40.6
12 inc gr x 6 age gr	2.82	54.9	45.1
15 inc gr x 5 age gr	2.82	51.8	48.2

The 10 income groups are defined as follows: 0-20, 20-40, 40-60, 60-80, 80-120, 120-160, 160-200, 200-250, 250-500, 500+ (000, in 2019 $); while in the 12 income groups the top group is also split into the following additional groups: 500-750, 750-1250, 1250+ (000, in 2019 $). The 15 income groups further split the top 1250+ bracket into 1250-1750, 1750-3000, 3000-15000, and 15000+ (000, in 2019 $). The six age groups split the 65+ age category into 65-74 and 75+ years.

In the first panel of Table B2 for comparison with Auclert et al. (2021) we present results for 12 age groups; in the second panel of the table we report results for different combinations of age and income groups.

[74]For the pure inflation factors on the liability side, however, we are unable to extend the series, and use the last available data point from 2016 for the additional years of interest.

Table B2

Shift Share Decomposition of the Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019: Robustness to base year of income/wealth profiles

Definition	Total Change	Within (%)	Between (%)
	12 Age groups		
1989 base	2.82	65.1	34.9
2019 base	2.82	52.1	47.9
	30 Income-age groups		
1989 base	2.82	61.6	38.4
2019 base	2.82	42.9	57.1
	60 Income-age groups		
1989 base	2.82	59.4	40.6
2019 base	2.82	45.5	54.5
	72 Income-age groups		
1989 base	2.82	54.9	45.1
2019 base	2.82	46.2	53.8
	75 Income-age groups		
1989 base	2.82	51.8	48.2
2019 base	2.82	42.8	57.2

B.2 Regression-based decomposition approach

As the alternative approach to the simple shift-share decomposition presented in the main text, we use the 1989 cross section to estimate a wealth holding function, which we denote by $F_{89}(age, y)$, where as previously *age* represents the age of the household head and y represents real income of a household. Function F can take different forms. In this section, we focus on the polynomial function F in income and age.[75] Then, for each household in the 2019 cross section, we use estimated function $F_{89}(age, y)$ to create a predicted wealth holding, which we denote by $\hat{w}_{19}$. These predicted wealth levels allow us to create a predicted wealth-to-income ratio in 2019 by adding up $\hat{w}_{19}$ across households, and by dividing it by the aggregate income in 2019 (denoted $\left(\frac{\hat{w}}{y}\right)_{19}$). By using the same prediction function for the wealth in 2019, as in 1989, the predicted ratio reflects only the changes in the proportions of different groups in the population. Accordingly, the fraction of the change in the wealth-to-income ratio explained by the within component can be expressed as

[75] We have run our predictive regressions using polynomials of order 3, 4, and 5. Polynomial function of order 5 delivers the best prediction. In Appendix B.3 we show that these results are also similar to using a regression with a set of dummy variables for income and age groups, which we refer to as a step-function regression approach.

$$1 - \left[\frac{\left(\frac{\hat{w}}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89}}{\left(\frac{w}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89}} \right]. \tag{B1}$$

In Table B3, we also report the results of this exercise using our two measures of wealth, which both include defined benefit pensions, but differ in terms of the inclusion of the primary housing wealth. Using a fifth order polynomial in income and age to build predicted wealth, we find that the between component accounts for between 40 and 42 percent of the change in the aggregate wealth-to-income ratio, leaving the within component again accounting for slightly under 60 percent of the rise. While these findings still support an important role of changes in demographics and income inequality in explaining movements in the wealth-to-income ratio, they indicate that an even greater share is due to changes in wealth holdings keeping income and age constant.

Table B3
Total Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019 and the Fraction of the Change due to Within and Between Effects: Decomposition Based on Regression

Definition	Total Change	Within (%)	Between (%)
Wealth (baseline)	2.819	59.8	40.2
Wealth less housing	2.649	57.2	42.8

B.3 Step-function regression decomposition approach and relationship with the shift-share analysis

The regression approach to estimating the between component using a function F with a set of age and income group dummies produces the first term in the decomposition below:

$$\left(\frac{w}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89} = \sum_i \left[\frac{\bar{w}_{i,89} N_{i,19}}{y_{19}} - \frac{\bar{w}_{i,89} N_{i,89}}{y_{89}} \right] + \sum_i \left(\frac{N_{i,19}}{y_{19}} \right) [\bar{w}_{i,19} - \bar{w}_{i,89}] \tag{B2}$$

Table B4 reports the shares of within and between components using this decomposition. These shares are very similar to those obtained using both a baseline shift-share decomposition and a regression approach using continuous age and income variables. In fact, the decomposition in (B3) can be written in a manner that makes it easy to compare with our baseline shift-share decomposition:

$$\left(\frac{w}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89} = \sum_i \frac{\bar{w}_{i,89}}{\bar{y}_{i,89}} \left[\Theta \frac{y_{i,19}}{y_{19}} - \frac{y_{i,89}}{y_{89}}\right] + \sum_i \left(\frac{N_{i,19}\bar{y}_{i,89}}{y_{19}}\right) \left[\frac{\bar{y}_{i,19}}{\bar{y}_{i,89}} \frac{\bar{w}_{i,19}}{\bar{y}_{i,19}} - \frac{\bar{w}_{i,89}}{\bar{y}_{i,89}}\right] \qquad \text{(B3)}$$

where $\Theta_i \equiv \frac{N_{i,19}\bar{y}_{i,89}}{N_{i,19}\bar{y}_{i,19}}$. If the average within group income $\bar{y}_i$ doesn't change much over time, then Θ_i will be close to 1, making the two decompositions very close.

Table B4

Total Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019 and the Fraction of the Change due to Within and Between Effects: Decomposition Using Step-function Regression Approach

Definition	Total Change	Within (%)	Between (%)
Wealth plus DB	2.819	64.8	35.2
Wealth plus DB less housing	2.649	63.7	36.3

Note: DB refers to the value of defined benefit pension schemes. The decomposition is done for 30 groups which are the product of 5 age groups and 6 income groups. The age groups are: 18-34, 34-35, 35-44, 45-54, 54-64, 65+ and the income groups (in thousands) are: 0-20, 20-40, 40-60, 60-80, 80-120, 120+.

C Asset-demand interest-rate link holding income constant in standard models

Consider a household facing an inter-temporal problem of the form

$$\int_0^\infty e^{-(\delta+\rho)t} \left[\frac{c_t^{1-\sigma_c}}{1-\sigma_c} + \Lambda \frac{a^{1-\sigma_a}}{1-\sigma_a}\right] dt, \qquad \sigma_c, \sigma_c > 0, \Lambda \geq 0.$$

where c_t is consumption, ρ is the discount rate, δ is a death rate, a_t is asset holdings, and $\Lambda\frac{a^{1-\sigma_a}}{1-\sigma_a}$ represents any potential additional gains associated with holding asset. Kumhof et al. (2015), Mian et al. (2021a), Michaillat and Saez (2018), De Nardi (2004), and Straub (2019) also allow assets to directly affect utility.

The budget constraint facing the household is given by:

$$\dot{a}_t = y_t - c_t$$

where income y_t is total income and is given by $y_t = w_t + r_t a_t$ with r_t being the return on asset a_t and w_t being non-asset income.

Then the steady state asset demand for this problem is given by

$$a^{ss} = \Lambda^{\frac{1}{\sigma_a}} \left[\rho + \delta - r\right]^{\frac{-1}{\sigma_a}} y^{\frac{\sigma_c}{\sigma_a}}.$$

The effect of a change in r on asset a^{ss} keeping income y constant is given by

$$\frac{\partial a^{ss}}{\partial r} = \frac{1}{\sigma_a}\Lambda^{\frac{1}{\sigma_a}}\left[\rho+\delta-r\right]^{\frac{-1}{\sigma_a}-1} y^{\frac{\sigma_c}{\sigma_a}} = \frac{a^{ss}}{\sigma_a}\left[\rho+\delta-r\right]^{-1}$$

which is always positive as long as $r \leq \rho+\delta$ and goes to infinity as r goes to $\rho+\delta$. For $r > \rho+\delta$, the solution to this problem is not well-defined as asset demands go to infinity creating the absence of a steady state. So signing $\frac{\partial a^{ss}}{\partial r}$ only applies when $r \leq \rho+\delta$. Hence, no member of this class of problems appear well suited to explain the observation that households with similar incomes want to hold more assets in a low interest rate environment than in a higher interest rate environment. While this discussion is in partial equilibrium, note that this difficulty remains in a general equilibrium setting where the fall in asset returns is endogenous – due to a change in the composition of the population – and creates valuation effects.

D Proofs of Propositions and Lemmas

D.1 Proof of Proposition 1

We first prove that asset holdings of active households converge to the long-run asset holdings $a^{a,ss}(y,r)$ and then characterize the properties of $a^{a,ss}(y,r)$.

Convergence of active households' asset holdings to $a^{a,ss}(y,r)$. Let's recall the dynamics of the optimization problem

$$\dot{c}_t = \left(\frac{r_t-\rho-\delta_1}{\sigma_1}\right)c_t + \frac{c_t^{\sigma_1+1}}{\sigma_1}\delta_1 q a_t^{-\sigma_2}\Gamma_t^{\sigma_2},$$

$$\dot{a}_t = r_t a_t + w_t - T_t - c_t,$$

$$\dot{\Gamma}_t = -1 + \Gamma_t\left[\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2}r_t\right].$$

Linearizing this system around the steady state ($\dot{c}_t = 0$, $\dot{a}_t = 0$, and $\dot{\Gamma}_t = 0$) with $r_t = r$ leads to the dynamic system:

$$\begin{pmatrix}\dot{\hat{c}}_t\\ \dot{\hat{a}}_t\\ \dot{\hat{\Gamma}}_t\end{pmatrix} = \underbrace{\begin{bmatrix}\rho+\delta_1-r & -\frac{\sigma_2}{\sigma_1}\frac{c}{a}(\rho+\delta_1-r) & \frac{\sigma_2}{\sigma_1}\frac{c}{\Gamma}(\rho+\delta_1-r)\\ -1 & r & 0\\ 0 & 0 & \frac{\rho+\delta_2}{\sigma_2}-\frac{1-\sigma_2}{\sigma_2}r\end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}}\begin{pmatrix}\hat{c}_t\\ \hat{a}_t\\ \hat{\Gamma}_t\end{pmatrix},$$

where $\hat{x}_t \equiv x_t - x$ means the deviation of a variable x_t from its steady state x, and $\rho+\delta_1-r = \delta_1 q c^{\sigma_1} a^{-\sigma_2}\Gamma^{\sigma_2}$.

The determinant of the 3x3 Jacobian J is given by

$$det(J) = (\rho+\delta_1-r)\left(\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2}r\right)\left(r - \frac{\sigma_2}{\sigma_1}\frac{c}{a}\right)$$

If $r < \frac{\sigma_2}{\sigma_1}\frac{c}{a}$, then $det(J) < 0$, implying that the steady state is saddle stable since $det(J) = \lambda_1\lambda_2\lambda_3$ and the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ have opposite signs.

Combining $\rho + \delta_1 - r = \delta_1 q c^{\sigma_1} a^{-\sigma_2} \Gamma^{\sigma_2}$ and $\Gamma^{-1} = \frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2} r$ leads to

$$\frac{c}{a} = \left[\left(\frac{\rho + \delta_1 - r}{\delta_1 q}\right)\left(\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2} r\right)^{\sigma_2}\right]^{\frac{1}{\sigma_1}} a^{1-\frac{\sigma_2}{\sigma_1}}.$$

Note that this equation also defines the implicit the long-run asset holdings $a^{a,ss}(y, r)$ where the disposable income y equal c.

Therefore, the convergence condition toward $a^{a,ss}(y, r)$ is

$$r < \left[\left(\frac{\rho + \delta_1 - r}{\delta_1 q}\right)\left(\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2} r\right)^{\sigma_2}\right]^{\frac{1}{\sigma_1}} \left(\frac{\sigma_2}{\sigma_1}\right) a^{1-\frac{\sigma_2}{\sigma_1}}.$$

This represents a necessary condition. A sufficient condition is

$$\max\{r, 0\} < \left[\left(\frac{\rho + \delta_1 - r}{\delta_1 q}\right)\left(\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2} r\right)^{\sigma_2}\right]^{\frac{1}{\sigma_1}},$$

where $\max\{r, 0\}$ guarantees consumption to be non-negative.

Properties of $a^{a,ss}(y, r)$. Recall the steady state asset holdings

$$a^{a,ss}(y, r) = (\delta_1 q)^{\frac{1}{\sigma_2}} \left[\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2} r\right]^{-1} [\rho + \delta_1 - r]^{\frac{-1}{\sigma_2}} y^{\frac{\sigma_1}{\sigma_2}}.$$

Let us take the derivative of $a^{a,ss}$ with respect to income y

$$\frac{da^{a,ss}}{dy} = \frac{\sigma_1}{\sigma_2}(\delta_1 q)^{\frac{1}{\sigma_2}} \left[\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2} r\right]^{-1} [\rho + \delta_1 - r]^{\frac{-1}{\sigma_2}} y^{\frac{\sigma_1}{\sigma_2}-1}.$$

$\frac{da^{a,ss}}{dy} > 0$ since $r \in \left(-\frac{\rho+\delta_2}{\sigma-1}, \rho + \delta_1\right)$. Hence, the long-run asset holdings of active households are increasing in income y.

Taking the derivative of $a^{a,ss}$ with respect to r, we have

$$\frac{da^{a,ss}}{dr} = (\delta_1 q)^{\frac{1}{\sigma_2}} (\rho + \delta_1 - r)^{\frac{-1}{\sigma_2}-1} y^{\frac{\sigma_1}{\sigma_2}} \left(\frac{1}{\rho + \delta_2 + (\sigma_2 - 1)r}\right)\left[1 - \frac{\sigma_2(\sigma_2-1)(\rho+\delta_1-r)}{\rho+\delta_2+(\sigma_2-1)r}\right].$$

If $\sigma_2 \leq 1$, $\frac{da^{a,ss}}{dr} \geq 0$ and hence the steady state asset holdings of active households are increasing in the interest rate.

Now let us assume that $\sigma_2 > 1$. When $r = \bar{r}$, we have $\frac{da^{a,ss}}{dr} = 0$ where

$$\bar{r} \equiv \frac{\sigma_2(\sigma_2-1)(\rho+\delta_1) - (\rho+\delta_2)}{(\sigma_2-1)(\sigma_2+1)}.$$

If $r > \bar{r}$, $\frac{da^{a,ss}}{dr} > 0$. And if $r < \bar{r}$, $\frac{da^{a,ss}}{dr} < 0$. As a result, $a^{a,ss}$ is increasing (decreasing) in the interest rate when r is above (below) $\bar{r}$. Hence, $a^{a,ss}$ is C-shaped in the space (r, a).

Q.E.D.

D.2 Proof of Proposition 2

In the steady state $\dot{c}_t = 0$, $\dot{\Gamma}_t = 0$, and $\dot{a}_t = 0$. Combining $\dot{c}_t = 0$ and $\dot{\Gamma}_t = 0$ we obtain the desired consumption-to-wealth ratio (c/a):

$$\frac{c}{a} = (\delta_1 q)^{-\frac{1}{\sigma}} (\rho + \delta_1 - r)^{\frac{1}{\sigma}} \left[\frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma} r\right] \equiv D(r), \tag{D4}$$

where we denote the desired consumption-to-wealth ratio $D(r)$.

Combing the asset $([\phi + (1 - \phi)]a = B)$ and goods $(\phi c = \phi w - G - (B - \phi a)\Gamma^{-1})$ markets clearing conditions leads to the feasible consumption-to-wealth ratio (c/a)

$$\frac{c}{a} = \frac{y}{B}\left[1 + \frac{(1 - \phi)}{\phi} g(r)\right] - \frac{(1 - \phi)}{\phi} g(r)\Gamma^{-1} \equiv F(r), \tag{D5}$$

where $F(r)$ represents the feasible c/a ratio, $\Gamma = \left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r\right]^{-1}$, $g(r) = \frac{\delta_2 \sigma}{\rho+\delta_2 - r + \sigma\delta_2}$, and $y = \phi w - G$.

The function $D(r)$ has the following properties

- D is hump shape and continuous over the interval $[\frac{\rho+\delta_2}{1-\sigma}, \rho + \delta_1]$. $D(\bar{r}) = 0$, if $r < \bar{r}$, $D' < 0$ and if $r > \bar{r}$, $D' > 0$,
- $D(\rho + \delta_1) = 0$ and $D\left(\frac{\rho+\delta_2}{1-\sigma}\right) = 0$.

Similarly, the function $F(r)$ has these properties

- F is continuous over the interval $[\frac{\rho+\delta_2}{1-\sigma}, \rho + \delta_1]$,
- $F(\rho + \delta_1) > 0$ if $\frac{B}{y} < \frac{1+\frac{(1-\phi)}{\phi} g(\rho+\delta_1)}{\frac{(1-\phi)}{\phi}\Gamma^{-1}}$ and $F\left(\frac{\rho+\delta_2}{1-\sigma}\right) > 0$.

The steady state equilibrium is obtained when the desired and feasible consumption-to-wealth ratios cross, that is, when $D(r) = F(r)$.

Given that $F\left(\frac{\rho+\delta_2}{1-\sigma}\right) > 0 = D\left(\frac{\rho+\delta_2}{1-\sigma}\right)$, if D and F cross once, they must cross at least one more time again since F and D are continuous over the interval $[\frac{\rho+\delta_2}{1-\sigma}, \rho + \delta_1]$, D is hump shape and $F\left(\frac{\rho+\delta_2}{1-\sigma}\right) > 0 = D\left(\frac{\rho+\delta_2}{1-\sigma}\right)$. *Q.E.D.*

D.3 Proof of Proposition 3

Recall that $\pi^{ELB} = \frac{(\psi-1)\pi^T - r^{*H}}{\psi}$ where the ELB constraint is binding when $\pi \leq \pi^{ELB}$ and it is non binding when $\pi > \pi^{ELB}$. π^{ELB} is increasing in ψ. Assume that the ELB constraint is not binding and hence the Taylor rule is given by $i_t = r^{*H} + \pi^T + \psi(\pi_t - \pi^T)$ with $\psi > 1$. Let's also recall the equilibrium dynamics for the economy with nominal wage rigidities is now governed by the following dynamic system

$$\dot{\pi}_t = \kappa(c_t + G - \bar{y})$$

$$\frac{\dot{c}_t}{c_t} = \frac{i_t - \pi_t - \rho - \delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma}\delta_1 q^s V_a(B, \Gamma_t)$$

$$\dot{\Gamma}_t = -1 + \Gamma_t \left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma}(i_t - \pi_t)\right]$$

In the steady state, $\dot{\pi}_t = 0$, $\dot{c}_t = 0$, and $\dot{\Gamma}_t = 0$. The $\dot{\pi}_t = 0$ curve is given by $c = \bar{y} - G$. The $\dot{\Gamma}_t = 0$ curve is $\Gamma = \left[\frac{\rho+\delta_2}{\sigma} + \frac{1-\sigma}{\sigma}\pi\right]^{-1}$, while the $\dot{c}_t = 0$ curve (together with $\dot{\Gamma}_t = 0$) is given by

$$\begin{aligned} c = (\delta_1 q^s)^{-1/\sigma} B \left[\rho + \delta_1 - r^{*H} - (\psi - 1)(\pi - \pi^T)\right]^{1/\sigma} & \\ \left[\frac{\rho + \delta_2}{\sigma} - \left(\frac{1-\sigma}{\sigma}\right)(r^{*H} + (\psi-1)(\pi - \pi^T))\right]. & \end{aligned} \tag{D6}$$

Some properties of the $\dot{c}_t = 0$ curve. We denote the $\dot{c}_t = 0$ curve $F^c(\pi)$. When $\pi > \pi^{ELB}$, $c = 0$ if $\pi = \pi^T + \frac{\rho+\delta_1 - r^{*H}}{\psi - 1} > \pi^T$. The derivative of $F^c(\pi)$ with respect to π is

$$\begin{aligned} F^{c'}(\pi) = -(\delta_1 q^s)^{-\frac{1}{\sigma}} B \left[\rho + \delta_1 - r^{*H} - (\psi-1)(\pi - \pi^T)\right]^{\frac{1}{\sigma}-1} \left(\frac{\psi - 1}{\sigma^2}\right) & \\ \left[\rho + \delta_2 - (1-\sigma)(r^{*H} + (\psi-1)(\pi - \pi^T)) + \sigma(1-\sigma)\left(\rho + \delta_1 - r^{*H} - (\psi - 1)(\pi - \pi^T)\right)\right], & \end{aligned}$$

where $F^{c'}(\pi) = 0$ when $\pi^{opt} = \pi^T + \frac{\bar{r} - r^{*H}}{\psi - 1}$.

If $\pi < \pi^{opt}$, $F^{c'} > 0$ and if $\pi > \pi^{opt}$, $F^{c'} < 0$. Hence, the $\dot{c}_t = 0$ curve is hump shaped in π with the optimal consumption being equal to $F^c(\pi^{opt}) = (\delta_1 q)^{-1/\sigma} B(\rho + \delta_1 - \bar{r})^{1/\sigma}\left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}\bar{r}\right]$. The $\dot{c}_t = 0$ curve (given by F^c) and the $\dot{\pi}_t = 0$ curve are displayed in Figures 9 and 10.

Existence of steady state equilibria. A steady state equilibrium is determined by the intersection of the $\dot{\pi}_t = 0$ and $\dot{c}_t = 0$ curves, that is $F^c(\pi) = \bar{y} - G$.

A necessary condition for an equilibrium to exist is $0 < \bar{y} - G < F^c(\pi^{opt})$ which is satisfied when

$$B > (\delta_1 q^s)^{1/\sigma}(\rho + \delta_1 - \bar{r})^{-1/\sigma}\left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma}\bar{r}\right] \equiv \bar{B}.$$

It is helpful to consider two cases for the discussion of the equilibrium. In the first case, we assume that $F^c(\pi^{ELB}) < \bar{y} - G$ (see the proof of Proposition 5 for the condition). We start with the scenario where $\pi > \pi^{opt}$ and F^c is strictly decreasing in π. Since $0 < \bar{y} - G < F^c(\pi^{opt})$ and F^c is decreasing, there is an equilibrium inflation $\pi_1 = F^{c-1}(\bar{y} - G)$. We denote this equilibrium E_1. We also consider the scenario where $\pi^{ELB} < \pi < \pi^{opt}$ and F^c is strictly increasing in π. Similarly, since F^c is strictly increasing there is a second equilibrium $\tilde{\pi} = F^{c-1}(\bar{y} - G)$ (see Section D.4 for the value of $\tilde{\pi}$). We denote this equilibrium $\tilde{E}_1$. The first case shows that there are two equilibria. In the second case where $F^c(\pi^{ELB}) > \bar{y} - G$, only equilibrium E_1 exists.

Real rate and inflation at equilibrium E_1. Are r^{*H} and π^T the real interest rate and inflation rate at E_1 respectively? To answer this question, first recall that the r^{*H} (in the model without nominal rigidities) is determined by the following equations

$$
\begin{aligned}
&F(r) \equiv (\delta_1 q^s)^{-1/\sigma} B \left[\rho + \delta_1 - r\right]^{1/\sigma} \left[\frac{\rho + \delta_2}{\sigma} - \left(\frac{1-\sigma}{\sigma}\right) r\right] = \bar{y} - G, \\
&r^{*H} = F^{-1}(\bar{y} - G) > \bar{r}, \\
&F'(r) < 0 \quad \text{if} \quad r > \bar{r}.
\end{aligned}
$$

Now note that $F(i - \pi^T) = F^c(\pi^T)$. Hence $i - \pi^T = F^{-1}(\bar{y} - G) = r^{*H}$ and the inflation rate at the equilibrium E_1 is $\pi_1 = \pi^T$. We also need to check whether r^{*H} is higher $\bar{r}$ in the presence of nominal rigidities. We know that at the equilibrium E_1, $\pi^T > \pi^{opt}$ and $i - \pi^T < i - \pi^{opt}$. Using the definition of π^{opt}, we obtain $i - \pi^T = r^{*H} > \bar{r}$. Therefore, at the equilibrium E_1, inflation is at target $\pi = \pi^T$ and as a result the real interest rate is $r^{*H} > \bar{r}$.

Stability. The stability analysis of these two steady states is given by the following system:

$$
\begin{pmatrix} \dot{\hat{\pi}}_t \\ \dot{\hat{c}}_t \\ \dot{\hat{\Gamma}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & \kappa & 0 \\ \frac{(\psi-1)c}{\sigma} & J_{22} & J_{23} \\ -\Gamma(\psi-1)\left(\frac{1-\sigma}{\sigma}\right) & 0 & J_{33} \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{\pi}_t \\ \hat{c}_t \\ \hat{\Gamma}_t \end{pmatrix},
$$

where $\hat{x}_t \equiv x_t - x$ means the deviation of a variable x_t from its steady state x,

$$
J_{22} = (\rho + \delta_1) - (r^{*H} + (\psi - 1)(\pi - \pi^T)),
$$

$$
J_{23} = [(\rho + \delta_1) - (r^{*H} + (\psi - 1)(\pi - \pi^T))]\frac{c}{\Gamma},
$$

and

$$
J_{33} = \frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma}(r^{*H} + (\psi - 1)(\pi - \pi^T)).
$$

The determinant of the 3x3 Jacobian J is given by

$$
\begin{aligned}
det(J) = &-\frac{\kappa(\psi - 1)c}{\sigma^2} \\
&\left[\rho + \delta_2 - (1-\sigma)(1+\sigma)(\psi-1)(\pi - \pi^T) - (1-\sigma)(1+\sigma)r^{*H} + \sigma(1-\sigma)(\rho+\delta_1)\right].
\end{aligned}
$$

If $\pi > \pi^{opt}$, then $det(J) < 0$, implying that the steady state equilibrium E_1 is saddle stable since $det(J) = \lambda_1\lambda_2\lambda_3$ and the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ have opposite signs. If $\pi < \pi^{opt}$, then $det(J) > 0$, meaning that the steady state with $\tilde{\pi}_1 < \pi^{opt}$ is unstable. Hence, only one stable steady state equilibrium exists. *Q.E.D.*

D.4 Proof of Proposition 4

This proof is similar to that of Proposition 3 except that now $\pi < \pi^{ELB}$ and $i_t = 0$. Let's recall the equilibrium dynamics for the economy is governed by the following system

$$\dot{\pi}_t = \kappa(c_t + G - \bar{y}),$$

$$\frac{\dot{c}_t}{c_t} = \frac{-\pi_t - \rho - \delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma}\delta_1 q^s V_a(B, \Gamma_t),$$

$$\dot{\Gamma}_t = -1 + \Gamma_t\left[\frac{\rho + \delta_2}{\sigma} + \frac{1-\sigma}{\sigma}\pi_t\right].$$

In the steady state, $\dot{\pi}_t = 0$, $\dot{c}_t = 0$, and $\dot{\Gamma}_t = 0$. The $\dot{\pi}_t = 0$ curve is given by $c = \bar{y} - G$. The $\dot{\Gamma}_t = 0$ curve is $\Gamma = \left[\frac{\rho+\delta_2}{\sigma} + \frac{1-\sigma}{\sigma}\pi\right]^{-1}$, while the $\dot{c}_t = 0$ curve (together with $\dot{\Gamma}_t = 0$) is

$$c = (\delta_1 q^s)^{-1/\sigma} B\left[\rho + \delta_1 + \pi\right]^{1/\sigma}\left[\frac{\rho + \delta_2}{\sigma} + \left(\frac{1-\sigma}{\sigma}\right)\pi\right] \equiv H^c(\pi). \tag{D7}$$

The $\dot{c}_t = 0$ curve given by H^c and the $\dot{\pi}_t = 0$ are displayed in Figures 9 and 10.

Some properties of $\dot{c}_t = 0$ curve. When $\pi < \pi^{ELB}$, $c = 0$ if $\pi = -(\rho + \delta_1)$. The derivative of $H^c(\pi)$ with respect to π is

$$H^{c'}(\pi) = (\delta_1 q^s)^{-\frac{1}{\sigma}} B\left[\rho + \delta_1 + \pi\right]^{\frac{1}{\sigma}-1}\left(\frac{1}{\sigma^2}\right)\left[\rho + \delta_2 + (1-\sigma)\pi + \sigma(1-\sigma)\left(\rho + \delta_1 + \pi\right)\right].$$

$H^{c'}(\tilde{\pi}^{opt}) = 0$ where $\tilde{\pi}^{opt} = \frac{-(\rho+\delta_2)-\sigma(1-\sigma)(\rho+\delta_1)}{(1-\sigma)(\sigma+1)} = -\bar{r}$.

If $\pi < \tilde{\pi}^{opt}$, $H^{c'} > 0$ and if $\pi > \tilde{\pi}^{opt}$, $H^{c'} < 0$. Hence the $\dot{c}_t = 0$ curve is hump shaped in π with maximal being given by $H^c(\tilde{\pi}^{opt}) = (\delta_1 q^s)^{-1/\sigma} B(\rho + \delta_1 - \bar{r})^{1/\sigma}\left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}\bar{r}\right]$.[76] In this configuration, the condition $\tilde{\pi}^{opt} < \pi^{ELB}$ must hold since the ELB binds. This is satisfied if $\pi^T > -\bar{r}$ and $\psi > \frac{r^{*H}+\pi^T}{\bar{r}+\pi^T}$.

Existence of equilibria. The steady state equilibrium is determined by the intersection of the $\dot{\pi}_t = 0$ and $\dot{c}_t = 0$ curves, that is $H^c(\pi) = \bar{y} - G$. A necessary condition for an equilibrium to exist is $0 < \bar{y} - G < H^c(\tilde{\pi}^{opt})$ which is satisfied when $B > \bar{B}$.

It is useful to consider two cases for the discussion of the equilibrium. In the first case, we assume that $H^c(\pi^{ELB}) < \bar{y} - G$ (see the proof of Proposition 5 for the condition). We start with the scenario where $\pi > \tilde{\pi}^{opt}$ and H^c is strictly decreasing in π. Since $0 < \bar{y} - G < H^c(\tilde{\pi}^{opt})$ and

[76] Note that $H^c(\tilde{\pi}^{opt}) = F^c(\pi^{opt})$.

H^c is decreasing, there is an equilibrium inflation $\pi_2 = H^{c-1}(\bar{y} - G)$. We denote this equilibrium E_2. We also consider the scenario where $\pi^{ELB} < \pi < \tilde{\pi}^{opt}$ and H^c is strictly increasing in π. Similarly, since H^c is strictly increasing there is a second equilibrium $\pi_2^{'} = H^{c-1}(\bar{y} - G)$. We denote this equilibrium $\tilde{E}_2$. The first case shows that there are two equilibria. In the second case where $H^c(\pi^{ELB}) > \bar{y} - G$, only equilibrium $\tilde{E}_2$ exists.

Real rate and inflation at equilibrium E_2. We now show that at the equilibrium E_2, inflation $\pi = -r^{*L}$ where the r^{*L} is the low real interest rate. To do so, first recall that the r^{*L} (in the model without nominal rigidities) is determined by the following equations

$$
\begin{aligned}
&F(r) \equiv (\delta_1 q^s)^{-1/\sigma} B\,[\rho + \delta_1 - r]^{1/\sigma} \left[\frac{\rho + \delta_2}{\sigma} - \left(\frac{1-\sigma}{\sigma}\right) r\right] = \bar{y} - G,\\
&r^{*L} = F^{-1}(\bar{y} - G) < \bar{r},\\
&F'(r) > 0 \quad \text{if} \quad r < \bar{r}.
\end{aligned}
$$

Now note that $F(-\pi) = H^c(\pi)$. Hence, in equilibrium $-\pi = F^{-1}(\bar{y} - G) = r^{*L}$. Since $\pi < \tilde{\pi}^{opt} = -\bar{r}$ we also have that $-\pi < \bar{r}$. As a result, in equilibrium E_2, inflation $\pi = -r^{*L}$.

Stability. The stability analysis of these two steady states is given by the following system:

$$
\begin{pmatrix} \dot{\hat{\pi}}_t \\ \dot{\hat{c}}_t \\ \dot{\hat{\Gamma}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & \kappa & 0 \\ -\frac{c}{\sigma} & \rho + \delta_1 + \pi & (\rho + \delta_1 + \pi)\frac{c}{\Gamma} \\ \Gamma\left(\frac{1-\sigma}{\sigma}\right) & 0 & \frac{\rho+\delta_2}{\sigma} + \frac{1-\sigma}{\sigma}\pi \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{\pi}_t \\ \hat{c}_t \\ \hat{\Gamma}_t \end{pmatrix},
$$

where $\hat{x}_t \equiv x_t - x$ means the deviation of a variable x_t from its steady state x. The determinant of the 3x3 Jacobian J is given by $det(J) = \frac{\kappa c}{\sigma^2}\left[(1-\sigma)(1+\sigma)\pi + \rho + \delta_2 + \sigma(1-\sigma)(\rho + \delta_1)\right]$.

If $\pi > \tilde{\pi}^{opt}$, then $det(J) < 0$, implying that the steady state equilibrium E_2 is saddle stable since $det(J) = \lambda_1\lambda_2\lambda_3$ and the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ have opposite signs. If $\pi < \tilde{\pi}^{opt}$, then $det(J) > 0$, meaning that the steady state with $\tilde{\pi}_2 < \tilde{\pi}^{opt}$ is unstable. Hence only the steady state equilibrium E_2 is stable.

Note to find $\tilde{\pi}$. At $\tilde{\pi}$, we must have $F^c(\tilde{\pi}) = H^c(-r^{*L})$. Rearranging this equation leads to[77]

$$
\tilde{\pi} = \pi^T + \frac{r^{*L} - r^{*H}}{\psi - 1}, \text{ where } \tilde{\pi} < \pi^{opt} = \pi^T + (\bar{r} - r^{*H})/(\psi - 1).
$$

Q.E.D.

D.5 Proof of Proposition 5

This proof builds on the proofs of Propositions 3 and 4. We start by noting that at π^{ELB}, $F^c(\pi^{ELB}) = H^c(\pi^{ELB})$ where F^c and H^c are given by equations (D6) and (D7) respectively.[78]

[77] Note that $\tilde{\pi}$ also solves the following equation in π: $r^{*L} + \pi = r^{*H} + \pi^T + \psi(\pi - \pi^T)$.

[78] Note that at the π^{ELB}, the real interest rates are identical: $-\pi^{ELB} = r^{*H} + (\psi - 1)(\pi^{ELB} - \pi^T)$.

Recall that $\pi^{ELB} = \frac{(\psi-1)\pi^T - r^{*H}}{\psi}$ and is increasing in ψ.

From the proofs of Propositions 3 and 4, we know that the steady state equilibrium E_1 is always stable. For the second stable steady state equilibrium E_2 to exist, we must have $H^c(\pi^{ELB}) < \bar{y} - G$. Since $\pi^{ELB} > \tilde{\pi}^{opt} = -\bar{r}$, H^c is decreasing which implies that $\pi^{ELB} > H^{c-1}(\bar{y} - G)$. Using the definition of π^{ELB}, we obtain $\psi > \frac{r^{*H} + \pi^T}{\pi^T - H^{c-1}(\bar{y} - G)}$. Knowing from the proof of Proposition 4 that $H^{c-1}(\bar{y} - G) = -r^{*L}$ implies that $\psi > \frac{r^{*H} + \pi^T}{r^{*L} + \pi^T}$.

Hence there exists a threshold[79] $\bar{\psi} \equiv \frac{r^{*H} + \pi^T}{r^{*L} + \pi^T} > 1$ such that if $\psi > \bar{\psi}$, there are two stable steady state equilibria given by E_1 and E_2. If $\psi < \bar{\psi}$ only equilibrium E_1 exists. *Q.E.D.*

D.6 Proof of Proposition 6

The proof is similar to the proof of Proposition 5. Recall that the inflation rate at the ELB stable steady state (E_2) is $-r^{*L}$. Since $r^{*L} < \bar{r}$ we have $\frac{dr^{*L}}{dB} < 0$. Hence, the inflation rate $-r^{*L}$ at the equilibrium is increasing in government debt B.

First, note that $H^c(\pi; B) \equiv \hat{F}(B)$ increases with B for any inflation rate π. It is also important to note that $\frac{\partial \hat{F}(B)}{\partial B} = \frac{\partial H^c(\pi^{ELB}; B)}{\partial B} + \frac{\partial H^c(\pi^{ELB}; B)}{\partial \pi^{ELB}} \frac{\partial \pi^{ELB}}{\partial B} > 0$ since $\pi^{ELB} > \tilde{\pi}^{opt}$, $\frac{\partial H^c}{\partial \pi^{ELB}} < 0$, $\frac{\partial \pi^{ELB}}{\partial B} < 0$, and $\frac{\partial r^{*H}}{\partial B} > 0$. Therefore, $\hat{F}(B)$ is strictly increasing in B.

For the ELB equilibrium (E_2) to cease to exist, the following relationship must hold:

$$H^c(\pi^{ELB}; B) \equiv \hat{F}(B) > \bar{y} - G.$$

This implies that $B > \hat{F}^{-1}(\bar{y} - G) \equiv B^{cutoff}$. Consequently, when $B > B^{cutoff}$, the ELB equilibrium ceases to exist. If $B < \bar{B}$, $\lim_{B \to B^{cutoff}}(\pi) = \pi^{ELB}$. At the cutoff B^{cutoff}, there is a discontinuity and the stable ELB equilibrium disappears. *Q.E.D.*

D.7 Proof of Lemma 2

Recall the Euler equation in the steady state which includes the asset and goods market clearing conditions $\rho + \delta_1 - r = \delta_1 q c^\sigma B^{-\sigma} \left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r\right]^{-\sigma}$.

The derivative of this equation with respect to B is $\frac{dr}{dB} = \frac{-\sigma B^{-1}(\rho + \delta_1 - r)(\rho + \delta_2 - (1-\sigma)r)}{(1-\sigma)(1+\sigma)r + \sigma(\sigma-1)(\rho+\delta_1) - (\rho+\delta_2)}$. The numerator is negative, so the sign of $\frac{dr}{dB}$ depends on the denominator. Hence, we have:

$$\frac{dr^{*H}}{dB} > 0 \quad \text{if} \quad r^{*H} > \bar{r} \quad \text{and} \quad \frac{dr^{*L}}{dB} < 0 \quad \text{if} \quad r^{*L} < \bar{r}.$$

Second, we show how r^{*L} and r^{*H} change with δ_2: $\frac{dr}{d\delta_2} = \frac{-\sigma(\rho + \delta_1 - r)}{(1-\sigma)(1+\sigma)r + \sigma(\sigma-1)(\rho+\delta_1) - (\rho+\delta_2)}$.

Similarly $\frac{dr^{*H}}{d\delta_2} > 0 \quad \text{if} \quad r^{*H} > \bar{r} \quad \text{and} \quad \frac{dr^{*L}}{d\delta_2} < 0 \quad \text{if} \quad r^{*L} < \bar{r}$.

Q.E.D.

[79] Note that $\psi > \frac{r^{*H} + \pi^T}{r^{*L} + \pi^T} > \frac{r^{*H} + \pi^T}{\bar{r} + \pi^T} > 1$ as $r^{*L} < \bar{r} < r^{*H}$. This shows that if $\psi > \bar{\psi}$, both $H^c(\pi^{ELB}) > \bar{y} - G$ and $\pi < \pi^{ELB}$.