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Testing Collusion and Cooperation in Binary Choice Games

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Abstract

This paper studies the testable implication of players' collusive or cooperative behaviours in a binary choice game with complete information. In this paper, these behaviours are defined as players coordinating their actions to maximize the weighted sum of their payoffs. I show that this collusive model is observationally equivalent to an equilibrium model that imposes two restrictions. The first restriction is on each player's strategic effect and the second one requires a particular equilibrium selection mechanism. Under the equilibrium condition, these joint restrictions are simple to test using tools in the literature on empirical games. This test, as suggested by the observational equivalence result, is the same as testing collusive and cooperative behaviours. I illustrate the implementation of this test by revisiting the entry game between Walmart and Kmart studied by Jia (2008). Under the equilibrium condition, Jia's original estimates are consistent with the first restriction on the strategic effects, serving as a warning sign of potential collusion. This paper tests and rejects the second restriction on the equilibrium selection mechanism. Thus, the empirical evidence suggests that Walmart and Kmart did not collude on their entry decisions.

Topics: Econometric and statistical methods; Market structure and pricing
JEL codes: C57, L13

Résumé

Cette étude s'intéresse à l'implication vérifiable des comportements collusoires ou coopératifs des joueurs dans un jeu de choix binaire en situation d'information parfaite. Pour les besoins de l'étude, ces comportements sont définis comme une décision des joueurs de coordonner leurs actions afin de maximiser la somme pondérée de leurs gains. Je montre que ce modèle de collusion donne lieu aux mêmes observations qu'un modèle d'équilibre imposant deux restrictions : l'une concernant l'effet stratégique de chaque joueur, et l'autre exigeant un mécanisme particulier de sélection de l'équilibre. Sous la condition d'équilibre, ces restrictions conjointes sont faciles à tester à l'aide d'outils mis au point dans les travaux sur les jeux empiriques. Ce test, comme l'indique le résultat de l'équivalence observationnelle, revient à tester les comportements collusoires et coopératifs. J'illustre la mise en œuvre du test en réexaminant le jeu d'entrée entre Walmart et Kmart étudié par Jia (2008). Sous la condition d'équilibre, les estimations initiales de Jia concordent avec la première restriction sur l'effet stratégique, ce qui constitue un signe de possibilité de collusion. Dans la présente étude, je teste et rejette la seconde restriction sur le mécanisme de sélection de l'équilibre. Les résultats empiriques semblent donc indiquer que Walmart et Kmart ne se sont pas concertés sur leurs décisions d'entrée.

Sujets : Méthodes économétriques et statistiques; Structure de marché et fixation des prix
Codes JEL : C57, L13

1 Introduction

Collusion—if it exists—can undermine market efficiency, reduce consumer welfare, and hamper economic growth. Consequently, the empirical testing and detection of collusion remain a central focus in economics, particularly in the field of industrial organization. Over the past three decades, economists have developed powerful econometric tools to test collusion in quantity or price decisions[1] and in the context of auctions.[2] These decisions are naturally modeled as *continuous choice*.

In addition to the aforementioned scenarios, firms also make other crucial decisions such as entry and product choice, which are inherently modeled as *discrete choice*. As documented by Sullivan (2020), Bourreau et al. (2021), and De Leverano (2023), collusion can also occur in these discrete decision dimensions. However, compared to research on continuous choice, there is a notable scarcity of econometric tools to test collusion in discrete choice, with the exception of the work by Aradillas-Lopez and Kosenkova (2023). The objective of this paper is to address this gap by deriving a testable implication of collusive or cooperative behaviors in the context of complete information binary choice games with $N \geq 2$ players.

To illustrate the key results of this paper, consider for simplicity a two-firm entry game. This paper focuses on testing the null hypothesis of collusive behaviors, where each firm can coordinate their entry decisions to maximize the (weighted) sum of both firms' profits. I show that this model of collusive behaviors is *observationally equivalent* to a model of equilibrium behaviors with two specific restrictions. In the corresponding equilibrium model, the competitive / strategic effects are constrained to be homogeneous across firms, known as the *equal strategic effect restriction*. Furthermore, when multiple equilibria exist, firms are restricted to selecting the most efficient one, referred to as the *equilibrium selection restriction*. In other words, consider games with any payoff structure and suppose that firms are actually colluding. However, instead of a collusive model, researchers estimate a dataset consisting of firms' choice under the equilibrium condition. Then asymptotically, the estimated structural functions will satisfy two conditions—one regarding the strategic effects and the other regarding the equilibrium selection mechanism.

The aforementioned equivalence result allows the testing of a collusive model to be transformed into a

[1] Important examples include Bresnahan (1982), Genesove and Mullin (1998), Berry and Haile (2014), Miller and Weinberg (2017), and Duarte et al. (2023).

[2] Important examples include McAfee and McMillan (1992), Porter and Zona (1993), Pesendorfer (2000), Bajari and Ye (2003), and Kawai and Nakabayashi (2022).

joint testing of the strategic effects and the equilibrium selection restrictions within an equilibrium model. This joint test can be easily implemented using well-established tools that estimate discrete choice games with complete information (Bajari et al., 2010; Kline, 2015, 2016; Kashaev and Salcedo, 2021). See also Aradillas-Lopez (2020) for a recent survey. To be more precise, suppose that researchers reject the joint restrictions on the strategic effects and equilibrium selection mechanism within an equilibrium framework. This rejection directly suggests the rejection of collusive behaviors, as implied by the equivalence result.

While the observational equivalence has the power to reject collusive behaviors, it does have a limitation in the detection of collusion. Specifically, if the joint restrictions on the strategic effects and equilibrium selection mechanism are satisfied in a dataset, the equivalence result implies that researchers cannot differentiate between the equilibrium model with the joint restrictions and the collusive behavior. However, this limitation can be mitigated when researchers are confident that players are heterogeneous. Under this scenario, the joint restrictions, particularly the equal strategic effect restriction, are extremely unlikely to hold. Therefore, if the joint restrictions are satisfied in an equilibrium model, it could be interpreted as strong evidence of collusion, as the only plausible explanation for their satisfaction is that firms have colluded on their entry decisions. In other words, when researchers believe that players are sufficiently heterogeneous, the satisfaction of the joint restrictions under an equilibrium model serves as compelling evidence of collusion. At last, even in absence of prior assumptions about player heterogeneity, the non-rejection of the joint restrictions suggests at least a possibility of collusion in the corresponding industry. This non-rejection then serves as a warning sign, indicating that researchers may want to collect more information in such an industry to further investigate firms' behaviors.

To highlight the practical value of the key results in this paper, I revisit the empirical study by Jia (2008) that examines the entry decisions of Wal-Mart and Kmart. In this study, Jia estimates a static entry game played by these two chains across geographical markets in the US, under the equilibrium condition. The estimated parameters provide insights on the competitive structure between Wal-Mart and Kmart and their impact on local discount stores. Notably, Jia's findings reveal that, under various equilibrium selection mechanisms, there is no evidence to reject the null hypothesis that Wal-Mart and Kmart share identical competitive effects. As suggested by the observational equivalence result, this restriction on equal strategic effect indicates a warning sign of potential collusion. However, Jia (2008) does not explicitly test the restriction on the equilibrium selection mechanism. Therefore, based on her estimates alone, we cannot conclusively determine whether these two chains exhibit collusive entry behaviors or not. To investigate this issue, this paper performs a test on the joint restrictions of the equal strategic effect and the mechanism that selects the most efficient equilibrium. The results strongly reject the joint restrictions, indicating that collusive entry behaviors are not a concern in this industry.

To the best of my knowledge, Aradillas-Lopez and Kosenkova (2023) is the only existing paper that also examines the testable implication of collusion in discrete choice games when researchers have access to choice data. Specifically, their results are based on two assumptions. First, they assume that the payoff functions are *exchangeable* across players when players share same realizations of their control variables. In a two-firm entry game, this exchangeability requires each firm to share identical distribution of the monopoly profits, thereby ruling out important features such as fixed effect. The second assumption in Aradillas-Lopez and Kosenkova (2023) pertains to collusive behaviors, where players jointly maximize a *symmetric* objective function. The symmetric assumption suggests that the joint objective function is independent of the order or the identity of the players. In contrast, this paper allows for any payoff structure and does not require the exchangeability. Furthermore, I consider both symmetric and *asymmetric* objective functions. Allowing for asymmetry is relevant in scenarios where firms have different bargaining power in dividing their collusive profits. Another notable difference is that Aradillas-Lopez and Kosenkova (2023) derive a testable implication on the joint distribution of players' decisions. They further construct a novel *nonparametric* method to test the implication. In contrast, this paper derives the collusive implication on structural functions (e.g., strategic effects and equilibrium selection mechanism) if researchers estimate the data under an equilibrium framework. Even though these structural functions can be nonparametrically identified (Kline, 2016; Kashaev and Salcedo, 2021), they are difficult to be recovered in an actual dataset (Khan and Nekipelov, 2018). Therefore, practical application of the method in this paper often requires parametric assumptions. Consequently, in summary, the method in this paper is preferable when researchers are more concerned about exchangeability and symmetric restrictions but are less concerned about the specific parametric family. On the other hand, if researchers prioritize other considerations such as nonparametric structural functions, the method derived in Aradillas-Lopez and Kosenkova (2023) would be preferred. Consequently, this paper complements the analysis by Aradillas-Lopez and Kosenkova (2023).[3]

[3]Another advantage of Aradillas-Lopez and Kosenkova (2023) is that their results extend to scenarios where players have

In addition to the entry decision, binary choice games with complete information have been estimated in various other contexts, including the intra-household labour supply decision (Kooreman, 1994) and social interactions among teenagers (Soetevent and Kooreman, 2007). In these examples, the testable implication derived in this paper could be seen as a test to discern whether individuals are engaged in a non-cooperative or cooperative game.

The rest of this paper is organized as follows. Section 2 presents an empirical model of a two-player binary choice game. It also describes the collusive / cooperative behavior that maximizes a symmetric joint objective function. The testable implication of such a behavior is derived in Section 3. Section 4 extends the results to games with $N \geq 2$ players and asymmetric objective functions. Section 5 illustrates the method by revisiting the study by Jia (2008), and Section 6 concludes.

2 Empirical Model

For simplicity, this section considers a two-player binary choice game, and Section 4 extends the results to games with $N \geq 2$ players. Two players in this game are represented by players 1 and 2. Letter i indexes an arbitrary player and $-i$ denotes the other player. Each player i simultaneously chooses an action—denoted by a_i—from their binary choice set $\{0,1\}$. Given an action profile or realized outcome $\mathbf{a} = (a_1, a_2)$, player i's payoff function $u_i(\mathbf{a}, \mathbf{x}_i, \varepsilon_i)$ is defined by Equation (1):

$$u_i(\mathbf{a}, \mathbf{x}_i, \varepsilon_i) = \begin{cases} 0, \text{ if } a_i = 0 \\ \pi_i(\mathbf{x}_i) + \delta_i(\mathbf{x}_i) \cdot \mathbb{1}(a_{-i} = 1) + \varepsilon_i, \text{ if } a_i = 1 \end{cases} . \tag{1}$$

As is standard in the discrete-choice literature, Equation (1) normalizes player i's payoff of action $a_i = 0$ to zero.[4] When player i chooses action $a_i = 1$, their payoff is influenced by the choice of the other player. In particular, if player $-i$ chooses $a_{-i} = 0$, player i's payoff is given by $\pi_i(\mathbf{x}_i)$, which is referred to as the *base return*. However, if player $-i$ deviates and chooses $a_{-i} = 1$, it will have an impact on player i's payoff. Such an impact is captured by the term $\delta_i(\mathbf{x}_i)$ and it is known as the *strategic effect*. In an entry game,

more than two actions. In contrast, this paper focuses exclusively on binary choice games.

[4]It is important to note that this normalization does not affect the key results of this paper. Specifically, suppose that the payoff of action $a_i = 0$ is modeled as $u_i(a_i = 0, a_{-i}, \mathbf{x}_i, \varepsilon_i) = \tilde{u}_i(a_{-i}, \mathbf{x}_i)$, where $\tilde{u}_i(\cdot)$ is a nonparametric function of its arguments. Under this general payoff function, it can be shown that the observational equivalence result in this paper still holds.

term $\delta_i(\mathbf{x}_i)$ corresponds to the competitive effect on firm i's profit caused by firm $-i$'s entry. Importantly, both $\pi_i(\mathbf{x}_i)$ and $\delta_i(\mathbf{x}_i)$ are considered as structural functions and can take any form with dependence on control variables $\mathbf{x}_i$. Vector $\mathbf{x}_i$ and $\mathbf{x}_{-i}$ may include variables that are common to all players (e.g., market characteristics) as well as individual-specific variables (e.g., firm characteristics). Notably, while $\pi_i(\mathbf{x}_i)$ and $\delta_i(\mathbf{x}_i)$ appear to be additively separable, the payoff function is actually nonparametrically specified without any restrictions.[5] Lastly, the variable $\varepsilon_i \in \mathbb{R}$ represents the unobserved component of the payoff function to researchers.

Table 1 presents the aforementioned game in its normal form. I consider an environment of complete information. Specifically, all model primitives in Table 1 are common knowledge between both players. In contrast, researchers have a dataset consisting of each player's action $\mathbf{a} = (a_1, a_2)$ and control variables $\mathbf{x} = (\mathbf{x}_1', \mathbf{x}_2')'$. However, researchers do not observe the latent variables $\boldsymbol{\epsilon} = (\varepsilon_1, \varepsilon_2)'$.

Table 1: Payoff Matrix of the 2×2 Game

	$a_2 = 0$	$a_2 = 1$
$a_1 = 0$	$0\ ,\ 0$	$0\ ,\ \pi_2(\mathbf{x}_2) + \varepsilon_2$
$a_1 = 1$	$\pi_1(\mathbf{x}_1) + \varepsilon_1\ ,\ 0$	$\pi_1(\mathbf{x}_1) + \delta_1(\mathbf{x}_1) + \varepsilon_1\ ,\ \pi_2(\mathbf{x}_2) + \delta_2(\mathbf{x}_2) + \varepsilon_2$

To illustrate players' collusive and cooperative behaviors, Equation (2) defines a joint objective function $V(\mathbf{a}, \mathbf{x}, \boldsymbol{\epsilon})$ as the sum of both players' payoffs:

$$V(\mathbf{a}, \mathbf{x}, \boldsymbol{\epsilon}) = u_1(\mathbf{a}, \mathbf{x}_1, \varepsilon_1) + u_2(\mathbf{a}, \mathbf{x}_2, \varepsilon_2). \tag{2}$$

The joint objective function $V(\cdot)$ weights each player's payoff equally. It could capture the scenario when players have equal bargaining power upon collusion. Section 4 further considers asymmetric objective functions that allow for heterogeneous bargaining power.

This paper aims to test the null hypothesis of collusive and cooperative behaviors. These behaviors are summarized by Definition 1.

Definition 1 (**Collusion & Cooperation**). *Players coordinate their actions to maximize their joint objective function $V(\mathbf{a}, \mathbf{x}, \boldsymbol{\epsilon})$. In other words, let (a_1^*, a_2^*) denote the observed action profile; then $(a_1^*, a_2^*) =$*

[5] To clarify, consider that $u_i(a_i = 1, a_{-i}, \mathbf{x}_i, \varepsilon_i) = \tilde{u}_i(a_{-i}, \mathbf{x}_i) + \varepsilon_i$, where $\tilde{u}_i(\cdot)$ is a nonparametric function of its arguments without any restrictions. This general payoff specification can be redefined as $\pi_i(\mathbf{x}_i) = \tilde{u}_i(a_{-i} = 0, \mathbf{x}_i)$ and $\delta_i(\mathbf{x}_i) = \tilde{u}_i(a_{-i} = 1, \mathbf{x}_i) - \tilde{u}_i(a_{-i} = 0, \mathbf{x}_i)$ without loss of generality.

$\text{argmax}_{\mathbf{a}} V(\mathbf{a}, \mathbf{x}, \boldsymbol{\epsilon})$.

This paper considers the regular case where each player incurs a non-zero strategic effect on the other player (i.e., $\delta_i \neq 0 \ \forall i$). Furthermore, I assume that ε_i is continuously distributed. This condition of continuity is summarized by Assumption 1. It implies that—with probability 1—there exists a unique action profile that maximizes the joint objective function $V(\cdot)$. Consequently, it is sufficient to consider only scenarios in which one action profile or one action is strictly preferred to another, without having to deal with cases of indifference.

Assumption 1. *Let $F(\boldsymbol{\epsilon})$ denote the cumulative distribution function of the vector $\boldsymbol{\epsilon} = (\varepsilon_1, \varepsilon_2)' \in \mathbb{R}^2$; then $F(\boldsymbol{\epsilon})$ is absolutely continuous with respect to the Lebesgue measure over its domain.*

Suppose that two players are involved in the game described by Table 1 and they behave according to Definition 1; this scenario is referred to as the **collusive / cooperative model**. Before proceeding to the testable implication of such a model, it is worth noting some important properties. First, given Equation (2), the collusive / cooperative model always achieves Pareto efficiency.

To further illustrate other properties, consider an example of entry games in different geographic markets (i.e., $\delta_1 < 0,\ \delta_2 < 0$). In markets with relatively low demand, firms' monopoly profits (i.e., $\pi_i(\cdot) + \varepsilon_i$) are negative. Therefore, neither of the firms will enter, and the collusive model has the same prediction as the Nash Equilibrium (NE). In markets with intermediate demand, only the firm with the highest monopoly profit enters. Interestingly, consider one of these markets where only one firm—say firm i—enters. The other firm $-i$ will not enter even when it could earn a positive profit upon entry. This can happen when firm $-i$ imposes a strong competitive effect on i that outweighs firm $-i$'s potential entry profit. Notably, this prediction is in sharp contrast to NE, as NE predicts the entry of firm $-i$ under such a scenario. This result also has the flavor of market division, as surveyed by Asker and Nocke (2021). Lastly, in markets with extremely high demand, both firms will enter. This is because the market is highly profitable, making entry attractive despite each firm internalizing its impact on the other.

The collusive entry behaviors in the above example can also be interpreted as follows: suppose that firm 1 and firm 2 are commonly owned by the same company. Instead of making decentralized choices, the central company makes entry decisions for each firm, taking into account the interactive / strategic effects between the two firms. Under this scenario, the centralized decision made by the company is identical to the collusive entry profile chosen by the two firms. This type of common ownership has been studied by Backus et al. (2021).

3 Testable Implication of Collusion and Cooperation

With a slight abuse of notation, let $\tilde{\mathbf{x}}$ denote the common variables in vector $\mathbf{x}_1$ and $\mathbf{x}_2$. The rest of this paper considers the case where player i's strategic effect δ_i depends only on $\tilde{\mathbf{x}}$; for instance $\delta_i(\mathbf{x}_i) = \delta_i(\tilde{\mathbf{x}})$. Note that player i's base return π_i is still allowed to depend on the entire vector $\mathbf{x}_i$ as $\pi_i(\mathbf{x}_i)$. It can be shown that the observational equivalence result in this paper also holds when $\delta_i(\mathbf{x}_i)$ depends on the entire vector $\mathbf{x}_i$. However, exploitation of such an equivalence result in practice requires identification of the structural functions. Consequently, I follow the literature that identifies complete information games (Tamer, 2003; Kline, 2015, 2016; Kashaev and Salcedo, 2021) and assume the existence of a player-specific variable that only affects player i's base return but has no impact on the strategic effect and the payoff of the other player.[6]

This paper assumes that researchers have access to a dataset consisting of players' chosen action profile $\mathbf{a} = (a_1, a_2)'$ and control variables $\mathbf{x} = (\mathbf{x}_1', \mathbf{x}_2')'$. With this dataset, researchers can consistently estimate the conditional probability of each action profile, denoted as $Pr(\mathbf{a}|\mathbf{x})$. Based on this knowledge, the paper proceeds to derive the testable implications of the collusive / cooperative model.

To simplify the presentation, this section first focuses on games with strategic substitutes (i.e., $\delta_i < 0\ \forall i$) and then studies games with strategic complements (i.e., $\delta_i > 0\ \forall i$). Additionally, an extension in Subsection 4.2 considers games with potentially more players and does not impose any sign restrictions on the strategic effects. Importantly, in this extension, researchers are not required to know the signs of these strategic effects.

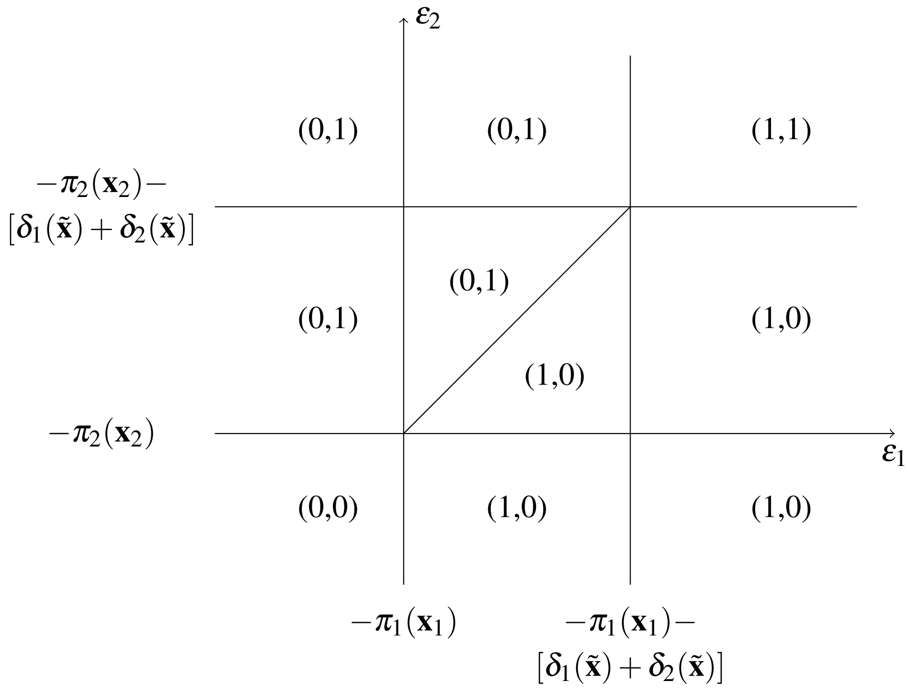
3.1 Games with Strategic Substitutes

Recall that $\mathbf{a}^* = (a_1^*, a_2^*)$ denotes the predicted action profile under the collusive / cooperative model. Assumption 1 implies that such an action profile is strictly preferred with probability 1. Therefore, through

[6] To be precise, the identification allows all player-specific variables but one to enter $\delta_i(\cdot)$. This excluded variable from $\delta_i(\cdot)$ then enters $\pi_i(\cdot)$ linearly. This paper does not consider this general structure for notation simplicity.

the relationship $V(\mathbf{a}^*, \mathbf{x}, \boldsymbol{\epsilon}) > V(\mathbf{a}, \mathbf{x}, \boldsymbol{\epsilon})\ \forall \mathbf{a} \neq \mathbf{a}^*$, one can derive the conditions on $\pi_i(\cdot)$, $\delta_i(\cdot)$, and ε_i such that $\mathbf{a}^*$ is chosen under the collusive / cooperative model. Following Bresnahan and Reiss (1991) and Tamer (2003), these conditions can be represented by a graph in the space of $\boldsymbol{\epsilon}$. Figure 1 presents such a graph. A complete derivation of these conditions in this graph is provided in the Appendix.

Figure 1: Prediction under the Collusive / Cooperative Model in Games with Strategic Substitutes



Compared to the predictions of NE as presented by Bresnahan and Reiss (1991) and Tamer (2003), the collusive predictions by Figure 1 exhibit two salient differences. First, in the collusive model, the coordinates of the right vertical line and the top horizontal line are represented as $-\pi_i - (\delta_i + \delta_{-i})$, whereas in the equilibrium model, they are in the form of $-\pi_i - \delta_i$ in the equilibrium model. Second, under NE, the middle rectangle represents the region with multiple equilibria. However, under collusion, this region is divided equally by a 45-degree line, as depicted in Figure 1. The multiplicity disappears because players could coordinate to choose the generically unique action profile that maximizes the total payoff.

The above discussion has an intuitive implication. Specifically, Figure 1 also represents the predictions of NE for a different game under a particular equilibrium selection mechanism. To illustrate this point, let us consider another game with a different payoff than the one presented in Section 2. In this new game with *perturbed payoffs*, each player's base return still remains as $\pi_i(\mathbf{x})$. However, player *i internalizes* their impact on the other player. Therefore, each player's strategic effect in this perturbed game changes from δ_i to $\tilde{\delta}_i(\tilde{\mathbf{x}}) = \delta_1(\tilde{\mathbf{x}}) + \delta_2(\tilde{\mathbf{x}})$. Due to this internalization, both players' perturbed strategic effects are **identical**. Table 2 presents this perturbed game in its normal form.

Table 2: Payoff Matrix of the 2×2 Perturbed Game

	$a_2 = 0$	$a_2 = 1$
$a_1 = 0$	$0\ ,\ 0$	$0\ ,\ \pi_2(\mathbf{x}_2) + \varepsilon_2$
$a_1 = 1$	$\pi_1(\mathbf{x}_1) + \varepsilon_1\ ,\ 0$	$\pi_1(\mathbf{x}_1) + \delta_1(\tilde{\mathbf{x}}) + \delta_2(\tilde{\mathbf{x}}) + \varepsilon_1\ ,\ \pi_2(\mathbf{x}_2) + \delta_1(\tilde{\mathbf{x}}) + \delta_2(\tilde{\mathbf{x}}) + \varepsilon_2$

Suppose that players play a pure strategy NE in this perturbed game. Note that there exists a region with two pure strategy NEs: $(1,0)$ and $(0,1)$. Within this region, players can coordinate and select the *most efficient* equilibrium; for instance, the equilibrium with the highest total payoff (i.e., the sum of two players' payoffs). Notably, Figure 1 also captures the conditions for predicting the pure strategy NE of the perturbed game represented in Table 2, under the assumption that players select the most efficient equilibrium when multiple equilibria exist. A complete derivation of this point is provided in the Appendix.

In summary, this paper has thus far considered two models. The first one is the collusive / cooperative model described in Section 2, which focuses on a game with a general payoff structure as shown by Table 1. The second model focuses on a perturbed game with equal strategic effects as presented by Table 2. In this second model, players are assumed to choose the pure strategy predicted by NE. Moreover, when multiple equilibria exist, players coordinate on the most efficient one. Importantly, both models share the same predictions of players' behaviors, as represented by Figure 1. Therefore, these two models are observationally equivalent. This key result is summarized by Proposition 1.

Proposition 1. *Suppose that $\delta_i(\cdot) < 0\ \forall i$ and Assumption 1 holds; then the collusive / cooperative model described in Section 2 leads to the same conditional probability $Pr(\mathbf{a}|\mathbf{x})$ as the model that satisfies the following two restrictions:*

(a) Players play the pure strategy NE in the game as presented by Table 2. Note that in this game, two players share an identical strategic effect.

(b) When multiple equilibria exist, players select the most efficient equilibrium.

Consider a scenario where two firms are actually colluding; however, researchers estimate the corresponding dataset under the equilibrium condition. Proposition 1 implies that the estimated results must indicate equal strategic effects and the mechanism selecting the most efficient equilibrium. Consequently, in any empirical study of complete information binary choice games under the equilibrium framework, if the estimates are consistent with these joint restrictions, it could serve as a warning sign of potential collusion. In such cases, researchers may need to collect more information to further investigate players' behaviors. Moreover, as described in the Introduction, if researchers believe that players are sufficiently heterogeneous so that the restriction of equal strategic effect is extremely unlikely to hold, then the satisfaction of the above joint restrictions could be a strong evidence of collusion.

In contrast, suppose that—under the equilibrium framework—the estimated results are inconsistent with the joint restrictions of equal strategic effects and the efficient equilibrium selection mechanism. Proposition 1 then implies the rejection of the collusive / cooperative model. In other words, the observational equivalence result has the power to reject the collusion. Moreover, testing the joint restrictions is straightforward using the existing tools in the literature (Bajari et al., 2010; Kline, 2015, 2016; Kashaev and Salcedo, 2021). The empirical application in Section 5 describes the testing procedure in more detail. Importantly, by transforming the test of collusion to the test of joint restrictions under the equilibrium condition, the test becomes a by-product of many empirical studies that maintain the equilibrium restriction, and therefore can be implemented with little additional cost.

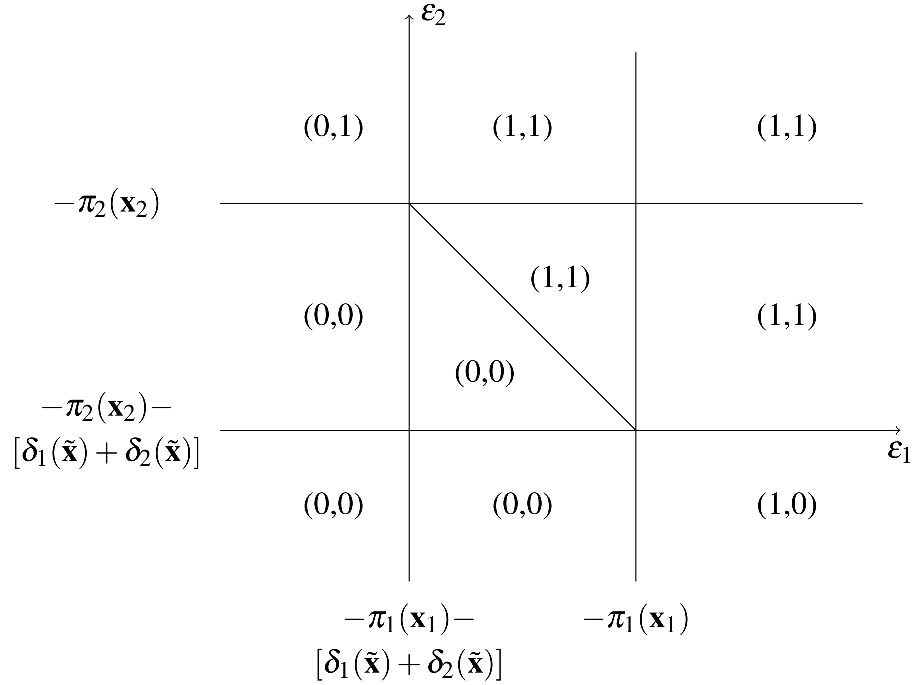
3.2 Games with Strategic Complements

When each player's strategic effect $\delta_i(\cdot)$ is positive—following a similar argument as in the above subsection—one could derive the predictions of the collusive / cooperative model. Figure 2 presents such predictions. A complete derivation of this figure is provided in the Appendix.

Similarly to Subsection 3.1, let us depart from the original game represented by Table 1 and consider its perturbed version as represented by Table 2. Recall that, in this perturbed game, each player internalizes their impact on the other player; therefore, they share an identical perturbed strategic effect as $\tilde{\delta}_i = \delta_1 + \delta_2$.

In contrast to Subsection 3.2 that focuses on games with strategic substitutes, in this subsection, strategies in the perturbed game are complements. Consequently, the set of equilibria would change and we shall impose another equilibrium selection mechanism. To be precise, suppose that players are playing a pure strategy NE in the game represented by Table 2. Moreover, since $\delta_i > 0$, there exists a region with two pure strategy NEs: $(0,0)$ and $(1,1)$. Instead of selecting the efficient one as in Subsection 3.1, this

Figure 2: Prediction Under the Collusive / Cooperative Model in Games with Strategic Complements



subsection considers that players coordinate to choose the *risk dominant* equilibrium (Harsanyi and Selten, 1988) when multiple equilibria exist.

In summary, in addition to the collusive / cooperative model, this subsection constructs a second model. The second one focuses on the pure strategy NE in the perturbed game and imposes a mechanism that selects the risk dominant equilibrium. Importantly, it can be shown that Figure 2 also presents the predictions of this second model. A complete derivation of this point is provided in the Appendix. Therefore, the second model shares the same model predictions as the collusive / cooperative model. Consequently, these two models are observationally equivalent. This result is summarized by the following Proposition 2.

Proposition 2. *Suppose that* $\delta_i(\cdot) > 0$ $\forall i$ *and Assumption 1 holds; the collusive / cooperative model described in Section 2 leads to the same conditional probability* $Pr(\mathbf{a}|\mathbf{x})$ *as the model that satisfies the following two restrictions:*

(a) Players play the pure strategy NE in the game as presented by Table 2. Note that in this game, two players share an identical strategic effect.

(b) When multiple equilibria exist, players select the risk dominant equilibrium.

4 Extension

This section extends the main results in Section 3 in two important directions. These extensions allow for (1) asymmetric joint objective functions and (2) games with more than two players.

4.1 Asymmetric Joint Objective Function

For brevity, this subsection focuses on games with strategic substitutes (i.e., $\delta_i < 0\ \forall i$). The results for games with strategic complements (i.e., $\delta_i > 0\ \forall i$) follow the same logic and are omitted. In particular, this subsection introduces the asymmetric joint objective function by defining the objective $V(\mathbf{a},\mathbf{x},\boldsymbol{\epsilon})$ as the weighted sum of two players' payoffs:

$$V(\mathbf{a},\mathbf{x},\boldsymbol{\epsilon}) = u_1(\mathbf{a},\mathbf{x}_1,\varepsilon_1) + \kappa \cdot u_2(\mathbf{a},\mathbf{x}_2,\varepsilon_2), \tag{3}$$

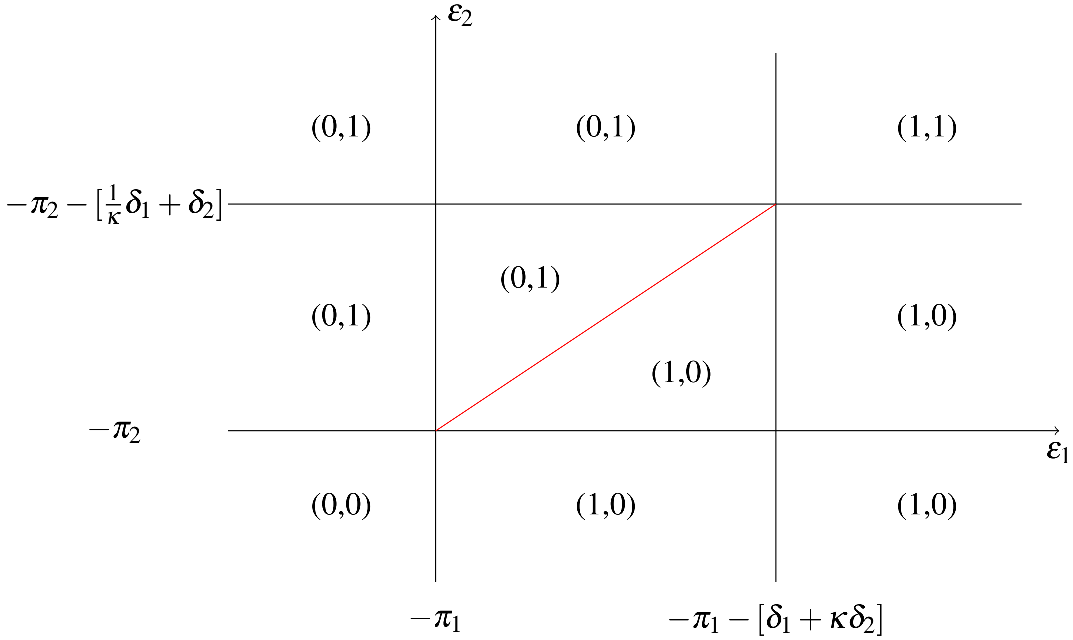
where $\kappa \in (0,\infty)$ represents the weight on player 2 and player 1's weight is normalized to 1. This asymmetric joint objective function nests the symmetric function as defined by Equation (2) as a special case when $\kappa = 1$. Moreover, when $\kappa \neq 1$, it can be interpreted as two players having different bargaining power upon collusion. Specifically, player 1's payoff becomes more important in the joint objective function as $\kappa \to 0$, indicating a stronger bargaining power for player 1. Conversely, player 2's bargaining power becomes stronger as κ increases.

This subsection considers the collusive / cooperative behaviors where players could coordinate their actions to maximize their joint objective function $V(\mathbf{a},\mathbf{x},\boldsymbol{\epsilon})$ defined by Equation (3). Specifically, let (a_1^*,a_2^*) denote the observed action profile, then $(a_1^*,a_2^*) = \text{argmax}_{\mathbf{a}} V(\mathbf{a},\mathbf{x},\boldsymbol{\epsilon}) = \text{argmax}_{\mathbf{a}} \left[u_1(\cdot) + \kappa \cdot u_2(\cdot)\right]$.

By applying similar reasoning as in Section 3, one could derive the conditions for predicting each action profile under the collusive / cooperative model with an asymmetric joint objective function. Figure 3 presents these conditions. A complete derivation of this figure is left to the Appendix.

Due to the asymmetric parameter κ, Figure 3 has three important changes as compared to Figure 1. First, the coordinate of the right vertical line shifts from $-\pi_1 - (\delta_1 + \delta_2)$ to $-\pi_1 - (\delta_1 + \kappa\delta_2)$. Second, the coordinate of the top horizontal line changes from $-\delta_2 - (\delta_1 + \delta_2)$ to $-\pi_2 - (\frac{1}{\kappa}\delta_1 + \delta)$. Finally, the red line in the middle rectangle represents the division between the profile $(0,1)$ and $(1,0)$. This line now has

Figure 3: Prediction Under the Collusive / Cooperative Model with an Asymmetric Objective Function



a slope of $\frac{1}{\kappa}$ instead of 1 as in Figure 1.

Similarly to Section 3, let us consider another game with perturbed payoffs. However, due to the presence of an asymmetric joint objective function, each player now internalizes their impact on the other player, taking into account their relative weight in the joint objective function. Specifically, let $\tilde{\delta}_1 = \delta_1 + \kappa\delta_2$ and $\tilde{\delta}_2 = \frac{1}{\kappa}\delta_1 + \delta_2$ be the perturbed strategic effects of players 1 and 2, respectively. Notably, this internalization leads to the relationship $\frac{\tilde{\delta}_2}{\tilde{\delta}_1} = \frac{1}{\kappa}$. Table 3 presents this perturbed game with an asymmetric joint objective function in its normal form.

Table 3: Payoff Matrix of the 2×2 Perturbed Game with an Asymmetric Objective Function

	$a_2 = 0$	$a_2 = 1$
$a_1 = 0$	$0\ ,\ 0$	$0\ ,\ \pi_2(\mathbf{x}_2) + \varepsilon_2$
$a_1 = 1$	$\pi_1(\mathbf{x}_1) + \varepsilon_1\ ,\ 0$	$\pi_1(\mathbf{x}_1) + \delta_1(\tilde{\mathbf{x}}_1) + \kappa\delta_2(\tilde{\mathbf{x}}) + \varepsilon_1\ ,\ \pi_2(\mathbf{x}_2) + \frac{1}{\kappa}\delta_1(\tilde{\mathbf{x}}) + \delta_2(\tilde{\mathbf{x}}) + \varepsilon_2$

Suppose that players play a pure strategy NE in the game presented by Table 3. Furthermore, there exists a region where $(1,0)$ and $(0,1)$ are the two possible pure strategy NEs. In this region, instead of choosing the most efficient equilibrium, players select the one that maximizes the sum of their weighted payoffs, with player 2's weight being κ. Remarkably, the Appendix shows that Figure 3 also represents the prediction of this equilibrium model in the perturbed game presented by Table 3, along with the aforementioned equilibrium selection mechanism. This feature leads to the following observational equivalence result, as shown by Proposition 3.

Proposition 3. *Suppose that $\delta_i(\cdot) < 0 \ \forall i$ and Assumption 1 holds, then the collusive / cooperative model with an asymmetric objective function defined by Equation* (3) *leads to the same conditional probability $Pr(\mathbf{a}|\mathbf{x})$ as the model that satisfies the following two restrictions:*

(a) Players play the pure strategy NE in the game as presented by Table 3. Note that in this game, the ratio of two players' perturbed strategic effects is $\frac{1}{\kappa}$; that is, $\frac{\tilde{\delta}_2(\cdot)}{\tilde{\delta}_1(\cdot)} = \frac{1}{\kappa}$.

(b) When multiple equilibria exist, players select the equilibrium that is represented by the form: $(\mathbb{1}[\pi_1(\mathbf{x}_1)+\varepsilon_1 > \kappa\pi_2(\mathbf{x}_2)+\kappa\varepsilon_2], \mathbb{1}[\pi_1(\mathbf{x}_1)+\varepsilon_1 < \kappa\pi_2(\mathbf{x}_2)+\kappa\varepsilon_2])$.

Under the equilibrium framework, conditions (a) and (b) in Proposition 3 are testable and the joint restrictions of these two conditions are equivalent to testing the collusive / cooperative model with an asymmetric objective function. For instance, if the perturbed strategic effect is linearly specified as $\delta_i(\tilde{\mathbf{x}}) = \beta_i'\tilde{\mathbf{x}}$ with $\tilde{\mathbf{x}}$ consisting of multiple variables, then condition (a) implies the testable restriction $\frac{\beta_{1,k}}{\beta_{2,k}} = \frac{\beta_{1,k'}}{\beta_{2,k'}} \ \forall k \neq k'$, where $\beta_{i,k}$ is player i's coefficient on variable x_k. In some other specifications, suppose that $\tilde{\delta}_i$ is modeled as a single unknown parameter. Recall that—under the equilibrium selection mechanism by condition (b)—the slope of the line that divides the profiles $(1,0)$ and $(0,1)$ (i.e., the red line in Figure 3) is $\frac{1}{\kappa}$. This slope then equals the ratio of strategic effects $\frac{\tilde{\delta}_2}{\tilde{\delta}_1}$. It is a testable restriction, and the empirical application in Section 5 explains how to test it in practice.

4.2 Games with $N \geq 2$ Players

This subsection extends the main results in this paper to games with potentially more than two players. The game consists of $N \geq 2$ players. Letter i indexes an arbitrary player and $-i$ denotes all players other than i. Moreover, let j represent one of the players other than i. This paper considers the following payoff function for player i:

$$u_i(\mathbf{a}, \mathbf{x}_i, \varepsilon_i) = \begin{cases} 0, \text{ if } a_i = 0 \\ \pi_i(\mathbf{x}_i) + \sum_{j\neq i} \delta_{i,j}(\tilde{\mathbf{x}}) \cdot a_j + \varepsilon_i, \text{ if } a_i = 1 \end{cases} . \tag{4}$$

The function $\delta_{i,j}(\tilde{\mathbf{x}})$ represents the strategic effect that player j incurs on the payoff of player i. This paper assumes that these strategic effects are additively separable and allows them to be potentially heterogeneous across players. In addition, it is assumed that these strategic effects always exist (i.e., $\delta_{i,j} \neq \ \forall i, j$) and they could be nonparametrically specified. Importantly, I allow the strategic effects to have different signs; for instance $sign(\delta_{i,j})$ does not necessarily equal $sign(\delta_{i',j'})$. Moreover, researchers do not need to know the signs of these strategic effects. Clearly, this game with potentially more players and the general payoff structure nests the 2×2 game described in Section 2.

The collusive / cooperative model assumes that players could coordinate their actions to maximize the sum of their payoffs. This joint objective function is defined by:

$$V(\mathbf{a}, \mathbf{x}, \boldsymbol{\epsilon}) = \sum_{i=1}^{N} u_i(\mathbf{a}, \mathbf{x}_i, \boldsymbol{\epsilon}). \tag{5}$$

To be precise, let $\mathbf{a}^* = (a_1^*, \cdots, a_N^*)'$ be the predicted outcome of the collusive / cooperative model; we then have $\mathbf{a}^* = \text{argmax}_{\mathbf{a}} V(\mathbf{a}, \mathbf{x}, \boldsymbol{\epsilon})$. Again, we assume that the cumulative distribution function $F(\boldsymbol{\epsilon})$ of $\boldsymbol{\epsilon} = (\varepsilon_1, \cdots, \varepsilon_N)'$ is absolutely continuous over its domain. It ensures that—generically—there is a unique action profile that maximizes the joint objective function.

Similarly to Section 3, when player i internalizes their impact on player j, the perturbed strategic effect of player i is then $\tilde{\delta}_{i,j} = \delta_{i,j} + \delta_{j,i}$. Notably, due to this internalization, the strategic effects are identical across any pair of two players; for instance, $\tilde{\delta}_{i,j} = \tilde{\delta}_{j,i} \ \forall i, j$. Equation (6) describes each player's payoff function in this perturbed game.

$$\tilde{u}_i(\mathbf{a}, \mathbf{x}_i, \varepsilon_i) = \begin{cases} 0, \text{ if } a_i = 0 \\ \pi_i(\mathbf{x}_i) + \sum_{j \neq i} [\underbrace{\delta_{i,j}(\tilde{\mathbf{x}}) + \delta_{j,i}(\tilde{\mathbf{x}})}_{= \tilde{\delta}_{i,j}(\tilde{\mathbf{x}})}] \cdot a_j + \varepsilon_i, \text{ if } a_i = 1 \end{cases} . \tag{6}$$

Proposition 4 establishes that the collusive / cooperative model for a game with $N \geq 2$ players is observationally equivalent to a model that assumes pure strategy NE in the above perturbed game with a particular equilibrium selection mechanism. This result can be used to test collusion in games with more than two players.

Proposition 4. *Suppose that the cumulative distribution function $F(\boldsymbol{\epsilon})$ of $\boldsymbol{\epsilon} = (\varepsilon_1, \cdots, \varepsilon_N)'$ is absolutely continuous with respect to the Lebesgue measure over its domain; then the collusive / cooperative model with the payoff function described by Equation* (4) *leads to the same conditional probability $Pr(\mathbf{a}|\mathbf{x})$ as the model that satisfies the following two restrictions:*

(a) Players play the pure strategy NE in the game as described by Equation (6). *Note that in this perturbed game, the strategic effects are identical across any pair of two players; for instance, $\tilde{\delta}_{i,j} = \tilde{\delta}_{j,i} = \delta_{i,j} + \delta_{j,i}\ \forall i, j$.*

(b) When multiple equilibria exist, players select the equilibrium that maximizes the function $V'(\mathbf{a}, \mathbf{x}, \boldsymbol{\epsilon}) = \sum_{i=1}^{N} a_i[\pi_i(\mathbf{x}_i) + \frac{1}{2}\sum_{j \neq i} \tilde{\delta}_{i,j}(\tilde{\mathbf{x}}) \cdot a_j + \varepsilon_i]$.

Proof. See the Appendix. □

The proof of Proposition 4 also indicates that, generically, the perturbed game defined by Equation (6) has at least one pure strategy NE. In addition, when there are only $N = 2$ players, the equilibrium selection mechanism described in condition (b) reduces to selecting the most efficient equilibrium in games with strategic substitutes and the risk dominant equilibrium in games with strategic complements. Consequently, Proposition 4 nests Propositions 1 and 2 as special cases.

5 Empirical Application

This section revisits the empirical study by Jia (2008). She estimates an entry game between Wal-Mart and Kmart, with chain / network effects. Her empirical model provides valuable insights into how the competitive structure between these two retail chains, Wal-Mart and Kmart, influence the local retail landscape. Specifically, Jia's estimates reveal that the entry of a single chain store would lead to approximately 50% of local discount stores unprofitable.

In more detail, Jia (2008) collects a cross-sectional dataset, consisting of 2,065 geographic markets across the United States. For each market, she observes the entry decisions by both Wal-Mart and Kmart, along with market-level and chain-level characteristics. This paper focuses on her dataset collected for the year 1997. In Table 4, the first three columns display Jia's estimates of the strategic effects for both chains, under three equilibrium selection mechanisms that she considers. As a robustness check, the fourth column presents the results without imposing any equilibrium selection mechanism. Instead, it employs the technique introduced by Bresnahan and Reiss (1991), defining observed outcome as the number of chains in each market. In the subsequent analysis, subscript $_K$ indexes Kmart and subscript $_W$ represents Wal-Mart.

Table 4: Estimated Strategic Effects in Jia (2008)

	Equilibrium Selection Mechanism			
	Favors Kmart	Favors Wal-Mart	Regional Advantage	No Mechanism Is Imposed
δ_W	-0.68^{***}	-0.74^{**}	-0.59^{***}	-0.73^{*}
	(0.26)	(0.34)	(0.19)	(0.42)
δ_K	-0.74^{***}	-0.77^{***}	-0.59^{***}	-1.00^{***}
	(0.19)	(0.25)	(0.14)	(0.29)
p-value of the test $\delta_W = \delta_K$	0.85	0.94	1.00	0.57
Observations			2,065	

Notes: *, **, and *** represent significance at 10%, 5%, and 1% levels, respectively. Standard errors are in parentheses. When multiple equilibria exist, the equilibrium selection mechanisms are as follows: "Favors Kmart" selects the equilibrium most profitable for Kmart, "Favors Wal-Mart" chooses the one most profitable for Wal-Mart, and "Regional Advantage" selects the most profitable one for Kmart in the Midwest region and the most profitable one for Wal-Mart in the South. The calculation of p-value for the first three columns assume that the estimates of δ_W and δ_K are uncorrelated, since this correlation has not been reported by Jia (2008).

Table 4 shows that the estimates of δ_W and δ_K are extremely similar in their magnitudes. The difference of these two parameters is highly insignificant from zero. Importantly, this result is robust to different equilibrium selection mechanisms. As Proposition 1 suggests, these estimates of almost identical strategic effects under the equilibrium framework could be a warning sign of potential collusion.

To further investigate firms' conduct, this section utilizes the key results of this paper. In addition to identical strategic effects, if Wal-Mart and Kmart are colluding on their entry decisions, the estimates under the equilibrium condition must also be consistent with a mechanism that selects the most efficient equilibrium. However, none of the four columns in Table 4 considers this specific mechanism. Therefore, to test the potential collusive entry behaviors, this paper exploits Proposition 1. In particular, under an equilibrium framework, I aim to test whether the estimates satisfy the joint restrictions of equal strategic effects and the mechanism that selects the most efficient equilibrium.

Following Jia (2008), I consider a linear specification for each firm's profit / payoff function. In particular, for each $i = W,\ K$, the profit of entry (i.e., $a_i = 1$) is:

$$\pi_i(\mathbf{x}_i) = \beta_i' \mathbf{x}_i,$$
$$\delta_i(\tilde{\mathbf{x}}) = \delta_i. \tag{7}$$

Firm i's base return depends linearly on $\mathbf{x}_i$. As in Jia (2008), $\mathbf{x}_i$ includes both market demand shifters and firm i's characteristics. The market shifters consist of three variables: (1) population, (2) retail sales per capita, and (3) percentage of urban population. The firm-specific characteristics for Wal-Mart's profit function include a dummy variable representing the South region and the market distance to Wal-Mart's headquarters in Bentonville, Arkansas. For Kmart's profit function, the firm-specific characteristics include a dummy variable representing the Midwest region, where its headquarters are originally located. In line with Jia (2008), each firm's strategic effect is specified to be a single unknown parameter δ_i that is independent of control variables $\mathbf{x}$.

This section assumes that the latent variables $\epsilon = (\varepsilon_W, \varepsilon_K)$ follow a joint normal distribution with mean zero and variances normalized to one. Equation (8) displays such a distribution:

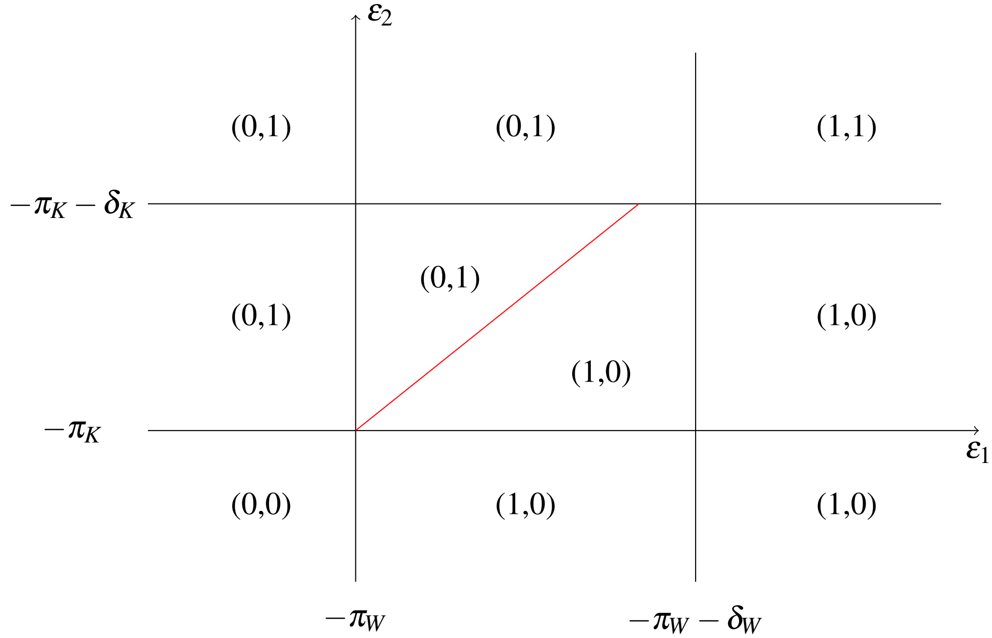
$$\begin{pmatrix} \varepsilon_W \\ \varepsilon_K \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right], \tag{8}$$

where ρ represents the correlation between ε_W and ε_K. To simplify computation, instead of estimating this correlation parameter, this section sets the value of ρ to 0.5. This value is based on the estimates reported by Jia (2008), where ρ ranges from 0.44 to 0.53 across different specifications (see her Table V). Importantly, the estimates obtained in this paper are robust to any value of ρ within this range. Note that the parameter ρ in this paper corresponds to $\sqrt{1-\rho^2}$ in Jia (2008).

This analysis assumes that each market is independent. In particular, consider any two markets m and m', then $\epsilon_m \perp\!\!\!\perp \epsilon_{m'}$. Moreover, the decision of firm i in market m, denoted as $a_{i,m}$, has no impact on any firm's profit in market m'.

The observational equivalence results suggest that the estimates under the assumption of pure strategy NE can be used to test the collusive / cooperative model. In the aforementioned entry game between Wal-Mart and Kmart, the predictions of pure strategy NE are represented by the following Figure 4.

Figure 4: Prediction of Pure Strategy NE in the Entry Game by Wal-Mart & Kmart



As shown in Figure 4, in each of the eight peripheral regions, there exists a unique pure strategy NE. In contrast, the middle rectangle (i.e., $-\pi_W < \varepsilon_W < -\pi_W - \delta_W$ and $-\pi_K < \varepsilon_K < -\pi_K - \delta_K$) consists of two equilibria: $(0,1)$ and $(1,0)$. In this region, I consider the following equilibrium selection mechanism:

$$\text{Wal-Mart Enters} \;\Leftrightarrow\; \alpha \cdot (\pi_W + \varepsilon_W) > \pi_K + \varepsilon_K, \tag{9}$$

where the parameter α is an unknown parameter. It represents the slope of the red line in Figure 4 that divides the two equilibria: $(0,1)$ and $(1,0)$. Specifically, when $\alpha \to 0$, the mechanism chooses the equilibrium that is most profitable for Kmart. In contrast, it selects the most profitable one for Wal-Mart as $\alpha \to \infty$. At last, when $\alpha = \frac{\delta_K}{\delta_W}$, the red line in Figure 4 will be the diagonal line of the middle rectangle. This red line will divide $(0,1)$ and $(1,0)$ equally.

Let $\theta = (\beta_W', \beta_K', \delta_W, \delta_K, \alpha)'$ be the vector of all unknown parameters. Given the equilibrium selection mechanism by Equation (9), the model is complete in the sense that—for any fixed value of θ—there is a unique expression for the conditional probability $Pr(\mathbf{a}|\mathbf{x};\theta)$. Consequently, the unknown parameters can be estimated by maximizing the following log-likelihood function:

$$LL = \max_{\theta} \sum_{m=1}^{M} \log[Pr(\mathbf{a}_m|\mathbf{x}_m;\theta)]. \tag{10}$$

Proposition 1 states that if Wal-Mart and Kmart collude under the symmetric joint objective function, then the estimates under the equilibrium condition would satisfy two restrictions: (i) equal strategic effects (i.e., $\delta_W = \delta_K$) and (ii) efficient equilibrium selection mechanism (i.e., $\alpha = 1$). One could estimate an equilibrium model with these two restrictions by maximizing the following constrained log-likelihood function:

$$LL^{Sym-Collusion} = \max_{\theta} \sum_{m=1}^{M} \log[Pr(\mathbf{a}_m|\mathbf{x}_m;\theta)],\ s.t.\ \delta_W = \delta_K \ and\ \alpha = 1. \tag{11}$$

Similarly, if Wal-Mart and Kmart collude with an asymmetric joint objective function, Proposition 3 proves that the equilibrium estimates would satisfy the restriction that $\alpha = \frac{\delta_K}{\delta_W}$. Consequently, this restricted model could be also estimated by MLE:

$$LL^{Asy-Collusion} = \max_{\theta} \sum_{m=1}^{M} \log[Pr(\mathbf{a}_m|\mathbf{x}_m;\theta)],\ s.t.\ \alpha = \frac{\delta_K}{\delta_W}. \tag{12}$$

Intuitively, we can test the null hypothesis of collusive entry behaviors using the standard likelihood ratio test. In particular, the collusion with the symmetric joint objective function is tested by the statistic $2(LL - LL^{Sym-Collusion})$ with a degree of freedom of 2. Similarly, the collusion with an asymmetric joint objective is tested by the statistic $2(LL - LL^{Asym-Collusion})$ with a degree of freedom of 1.

Table 5 presents the estimation results for each firm's strategic effect and the test of collusive entry behaviors. These results indicate strong rejection of collusion under both the symmetric joint objective function (with the restrictions $\delta_W = \delta_K$ and $\alpha = 1$) and the asymmetric one (with the restriction $\alpha = \frac{\delta_K}{\delta_W}$). Consequently, there is no evidence supporting the collusive behavior between Wal-Mart and Kmart in their entry game.

Notably, as shown by the last column in Table 5, the restriction of equal strategic effects across firms cannot be rejected and exhibits a high p-value. However, the estimate of parameter α is zero, consistent with a mechanism that always selects the most profitable equilibrium for Kmart when multiple equilibria exist (i.e., "Favors Kmart"). This mechanism significantly differs from the one that selects the most efficient equilibrium. It is this estimated result that rejects the collusive entry behaviors, under both symmetric and asymmetric joint objective functions. Interestingly, Jia (2008) also chooses the mechanism "Favors Kmart" as the preferred or benchmark specification in her empirical study. The estimated result for α in this paper provides an empirical support for Jia's choice.

Table 5: Estimated Strategic Effects and the Test of Collusion

	Restrictions of Symmetric Collusion: $\delta_W = \delta_K$ and $\alpha = 1$	Restrictions of Asymmetric Collusion: $\alpha = \frac{\delta_K}{\delta_W}$	Unrestricted Model
δ_W	-0.90^{***} (0.07)	-0.99^{***} (0.19)	-0.92^{***} (0.09)
δ_K	-0.90^{***} (0.07)	-0.81^{***} (0.22)	-0.74^{***} (0.19)
α	1 By Restriction	0.82 By Restriction	0.00 (0.62)
log-likelihood	-1270.1	-1270.0	-1265.6
Test of Equal Strategic Effects	*n.a.*	p-value = 0.614	p-value = 0.651
Test of Symmetric Collusion		p-value = 0.003	
Test of Asymmetric Collusion		p-value = 0.011	

In summary, Section 3 establishes that a collusive model is observationally equivalent to an equilibrium model with two specific restrictions on the strategic effects and the equilibrium selection mechanism. The estimation results presented in Table 5 are in line with the first restriction on the strategic effects but strongly reject the second restriction on the equilibrium selection mechanism. Consequently, these results reject the collusive entry behaviors. Interestingly, the estimation results also suggest a high degree of homogeneity in the profit functions of Wal-Mart and Kmart. This homogeneity is supported not only by the non-rejection of equal strategic effects but also by the quantitatively very similar estimates of their base returns, as shown by Table 6.

6 Conclusion

This paper focuses on binary choice games with complete information and studies the testable implication of the collusive / cooperative behaviors. Suppose that players actually collude by coordinating their actions to maximize the weighted sum of every player's payoff. However, instead of imposing this collusive model, researchers simply estimate the dataset under the equilibrium condition. This paper shows that the estimated results will satisfy two restrictions: one on the strategic effects and the other on the equilibrium selection mechanism. In other words, when the estimates under the equilibrium condition are consistent with the joint restrictions mentioned above, it could be a warning sign of potential collusion. On the other hand, if researchers reject the joint restrictions, it automatically rejects the collusive / cooperative

Table 6: Estimation Results of Base Return

	Wal-Mart	Kmart
log(*Population*)	1.89^{***}	1.74^{***}
	(0.09)	(0.13)
log(*Retail Sales Per Capita*)	1.59^{***}	1.72^{***}
	(0.12)	(0.14)
Percentage of Urban Population	1.33^{***}	1.35^{***}
	(0.18)	(0.21)
log(*Distance to Benton*)	-1.06^{***}	
	(0.06)	
South	0.62^{***}	
	(0.08)	
Midwest		0.30^{***}
		(0.09)
Constant	-12.86^{***}	-21.29^{***}
	(1.07)	(1.39)

behaviors.

The test of the aforementioned joint restrictions is straightforward to implement with the existing tools that estimate complete information discrete games. This paper illustrates the implementation of such a test by revisiting the entry game between Wal-Mart and Kmart studied by Jia (2008). Under the condition of pure strategy NE, the estimation results are consistent with the first restriction on the strategic effects, but strongly reject the second restriction on the equilibrium selection mechanism. Consequently, there is no empirical evidence supporting that Wal-Mart and Kmart collude on their entry decisions.

This paper focuses on a static framework. However, many crucial decisions, such as entry and investment, involve dynamic interactions. Investigation of the empirical implication of collusion in dynamic discrete choice games is an important area for future research.

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Online Appendix

Derivation of the Conditions in Figure 1

For notation simplicity, suppress $\mathbf{x}$ and $\boldsymbol{\epsilon}$ in the joint objective function so that $V(a_1,a_2,\mathbf{x},\varepsilon)$ reduces to $V(a_1,a_2)$. The following items describe the conditions under which each action profile is chosen within the collusive / cooperative model.

- Case 1: $(a_1^*,a_2^*)=(0,0)$ is chosen
  - $V(0,0)>V(1,0) \;\Leftrightarrow\; 0>\pi_1(\mathbf{x}_1)+\varepsilon_1 \;\Leftrightarrow\; \varepsilon_1<-\pi_1(\mathbf{x}_1)$
  - $V(0,0)>V(0,1) \;\Leftrightarrow\; 0>\pi_2(\mathbf{x}_2)+\varepsilon_2 \;\Leftrightarrow\; \varepsilon_2<-\pi_2(\mathbf{x}_2)$
  - $V(0,0)>V(1,1) \;\Leftrightarrow 0>\pi_1(\mathbf{x}_1)+\delta_1(\tilde{\mathbf{x}})+\varepsilon_1+\pi_2(\mathbf{x}_2)+\delta_2(\tilde{\mathbf{x}})+\varepsilon_2$: Note that since $\delta_i<0\;\forall i$, this condition is implied by the previous two conditions. Therefore, it is a redundant condition.
- Case 2: $(a_1^*,a_2^*)=(1,0)$ is chosen
  - $V(1,0)>V(0,0) \;\Leftrightarrow\; \pi_1(\mathbf{x}_1)+\varepsilon_1>0 \;\Leftrightarrow\; \varepsilon_1>-\pi_1(\mathbf{x}_1)$
  - $V(1,0)>V(0,1) \;\Leftrightarrow\; \pi_1(\mathbf{x}_1)+\varepsilon_1>\pi_2(\mathbf{x}_2)+\varepsilon_2$
  - $V(1,0)>V(1,1) \;\Leftrightarrow\; \pi_1(\mathbf{x}_1)+\varepsilon_1>\pi_1(\mathbf{x}_1)+\delta_1(\tilde{\mathbf{x}})+\varepsilon_1+\pi_2(\mathbf{x}_2)+\delta_2(\tilde{\mathbf{x}})+\varepsilon_2$
    $\Leftrightarrow\; \varepsilon_2<-\pi_2(\mathbf{x}_2)-\delta_1(\tilde{\mathbf{x}})-\delta_2(\tilde{\mathbf{x}})$
- Case 3: $(a_1^*,a_2^*)=(0,1)$ is chosen
  - $V(0,1)>V(0,0) \;\Leftrightarrow\; \pi_2(\mathbf{x}_2)+\varepsilon_2>0 \;\Leftrightarrow\; \varepsilon_2>-\pi_2(\mathbf{x}_2)$
  - $V(0,1)>V(1,0) \;\Leftrightarrow\; \pi_2(\mathbf{x}_2)+\varepsilon_2>\pi_1(\mathbf{x}_1)+\varepsilon_1$
  - $V(0,1)>V(1,1) \;\Leftrightarrow\; \pi_2(\mathbf{x}_2)+\varepsilon_2>\pi_1(\mathbf{x}_1)+\delta_1(\tilde{\mathbf{x}})+\varepsilon_1+\pi_2(\mathbf{x}_2)+\delta_2(\tilde{\mathbf{x}})+\varepsilon_2$
    $\Leftrightarrow\; \varepsilon_1<-\pi_1(\mathbf{x}_1)-\delta_1(\tilde{\mathbf{x}})-\delta_2(\tilde{\mathbf{x}})$
- Case 4: $(a_1^*,a_2^*)=(1,1)$ is chosen
  - $V(1,1)>V(1,0) \;\Leftrightarrow\; \pi_1(\mathbf{x}_1)+\delta_1(\tilde{\mathbf{x}})+\varepsilon_1+\pi_2(\mathbf{x}_2)+\delta_2(\tilde{\mathbf{x}})+\varepsilon_2>\pi_1(\mathbf{x}_1)+\varepsilon_1$
    $\Leftrightarrow\; \varepsilon_2>-\pi_2(\mathbf{x}_2)-\delta_1(\tilde{\mathbf{x}})-\delta_2(\tilde{\mathbf{x}})$

- $V(1,1) > V(0,1) \Leftrightarrow \pi_1(\mathbf{x}_1) + \delta_1(\tilde{\mathbf{x}}) + \varepsilon_1 + \pi_2(\mathbf{x}_2) + \delta_2(\tilde{\mathbf{x}}) + \varepsilon_2 > \pi_2(\mathbf{x}_2) + \varepsilon_2$
  $\Leftrightarrow \varepsilon_1 > -\pi_1(\mathbf{x}_1) - \delta_1(\tilde{\mathbf{x}}) - \delta_2(\tilde{\mathbf{x}})$
- $V(1,1) > V(0,0) \Leftrightarrow \pi_1(\mathbf{x}_1) + \delta_1(\tilde{\mathbf{x}}) + \varepsilon_1 + \pi_2(\mathbf{x}_2) + \delta_2(\tilde{\mathbf{x}}) + \varepsilon_2 > 0$: Note that since $\delta_i < 0 \; \forall i$, this condition is implied by the previous two conditions. Therefore, it is a redundant condition.

Plotting all of the above conditions in the space of $(\varepsilon_1, \varepsilon_2)$ would directly imply Figure 1.

Derivation of the Pure Strategy NE of the Perturbed Game by Table 2 with Strategic Substitutes and under the Efficient Equilibrium Selection Mechanism

Recall that in this perturbed game, $\tilde{\delta}_1(\tilde{\mathbf{x}}) = \tilde{\delta}_2(\tilde{\mathbf{x}}) = \delta_1(\tilde{\mathbf{x}}) + \delta_2(\tilde{\mathbf{x}})$. Since $\tilde{\delta}_i < 0 \; \forall i$, it is clear that when $\pi_i(\mathbf{x}_i) + \varepsilon_i < 0$, player i's strict dominant strategy is $a_i = 0$. On the other hand, when $\pi_i(\mathbf{x}_i) + \tilde{\delta}_i(\tilde{\mathbf{x}}) + \varepsilon_i > 0$, player i's strict dominant strategy is $a_i = 1$. The regions for these strict dominant strategies directly imply the following four results:

- when $\varepsilon_i < -\pi_i(\mathbf{x}_i) \; \forall i$, action profile $(0,0)$ is the unique NE
- when $\varepsilon_i > -\pi_i(\mathbf{x}_i) - \tilde{\delta}_i(\tilde{\mathbf{x}}) \; \forall i$, action profile $(1,1)$ is the unique NE
- when $\varepsilon_1 < -\pi_1(\mathbf{x}_1)$ and $\varepsilon_2 > -\pi_2(\mathbf{x}_2)$ or $-\pi_1(\mathbf{x}_1) < \varepsilon_1 < -\pi_1(\mathbf{x}_1) - \tilde{\delta}_1(\tilde{\mathbf{x}})$ and $\varepsilon_2 > -\pi_2(\mathbf{x}_2) - \tilde{\delta}_2(\tilde{\mathbf{x}})$, action profile $(0,1)$ is the unique NE
- when $\varepsilon_1 > -\pi_1(\mathbf{x}_1) - \tilde{\delta}_1(\tilde{\mathbf{x}})$ and $\varepsilon_2 < -\pi_2(\mathbf{x}_2) - \tilde{\delta}_2(\tilde{\mathbf{x}})$ or $-\pi_1(\mathbf{x}_1) < \varepsilon_1 < -\pi_1(\mathbf{x}_1) - \tilde{\delta}_1(\tilde{\mathbf{x}})$ and $\varepsilon_2 < -\pi_2(\mathbf{x}_2)$, action profile $(1,0)$ is the unique NE.

The only remaining case is when $-\pi_1(\mathbf{x}_1) < \varepsilon_1 < -\pi_1(\mathbf{x}_1) - \tilde{\delta}_1(\tilde{\mathbf{x}})$ and $-\pi_2(\mathbf{x}_2) < \varepsilon_2 < -\pi_2(\mathbf{x}_2) - \tilde{\delta}_2(\tilde{\mathbf{x}})$. In this case, there are two pure strategy NEs, where only one of the players chooses action 1 while the other player chooses action 0. Given the equilibrium selection mechanism that chooses the most efficient one, player 1 will choose action 1 if and only if $\pi_1(\mathbf{x}_1) + \varepsilon_1 > \pi_2(\mathbf{x}_2) + \varepsilon_2$.

In summary, in a model that considers the pure strategy NE with the mechanism that selects the most efficient equilibrium, the model prediction can be summarized by the following four scenarios:

- Case 1: $(0,0)$ is chosen
  - $\varepsilon_i < -\pi_i(\mathbf{x}_i) \; \forall i$

- Case 2: $(1,0)$ is chosen
  - $\varepsilon_1 > -\pi_1(\mathbf{x}_1)$
  - $\varepsilon_2 < -\pi_2(\mathbf{x}_2) - \tilde{\delta}_2(\tilde{\mathbf{x}})$
  - $\pi_1(\mathbf{x}_1) + \varepsilon_1 > \pi_2(\mathbf{x}_2) + \varepsilon_2$
- Case 3: $(0,1)$ is chosen
  - $\varepsilon_2 > -\pi_2(\mathbf{x}_2)$
  - $\varepsilon_1 < -\pi_1(\mathbf{x}_1) - \tilde{\delta}_1(\tilde{\mathbf{x}})$
  - $\pi_2(\mathbf{x}_2) + \varepsilon_2 > \pi_1(\mathbf{x}_1) + \varepsilon_1$
- Case 4: $(1,1)$ is chosen
  - $\varepsilon_i > -\pi_i(\mathbf{x}_i) - \tilde{\delta}_i(\tilde{\mathbf{x}})\ \forall i$

All of the above conditions are identical to the ones under the collusive / cooperative model in games with strategic substitutes. These conditions also lead to Figure 1.

Derivation of the Conditions in Figure 2

Similarly to the derivation of Figure 1, the following items describe the conditions under which each action profile is chosen within the collusive / cooperative model.

- Case 1: $(a_1^*, a_2^*) = (0,0)$ is chosen
  - $V(0,0) > V(1,0) \Leftrightarrow 0 > \pi_1(\mathbf{x}_1) + \varepsilon_1 \Leftrightarrow \varepsilon_1 < -\pi_1(\mathbf{x}_1)$
  - $V(0,0) > V(0,1) \Leftrightarrow 0 > \pi_2(\mathbf{x}_2) + \varepsilon_2 \Leftrightarrow \varepsilon_2 < -\pi_2(\mathbf{x}_2)$
  - $V(0,0) > V(1,1) \Leftrightarrow 0 > \pi_1(\mathbf{x}_1) + \delta_1(\tilde{\mathbf{x}}) + \varepsilon_1 + \pi_2(\mathbf{x}_2) + \delta_2(\tilde{\mathbf{x}}) + \varepsilon_2$
- Case 2: $(a_1^*, a_2^*) = (1,0)$ is chosen
  - $V(1,0) > V(0,0) \Leftrightarrow \pi_1(\mathbf{x}_1) + \varepsilon_1 > 0 \Leftrightarrow \varepsilon_1 > -\pi_1(\mathbf{x}_1)$
  - $V(1,0) > V(0,1) \Leftrightarrow \pi_1(\mathbf{x}_1) + \varepsilon_1 > \pi_2(\mathbf{x}_2) + \varepsilon_2$: Note that since $\delta_i > 0\ \forall i$, this condition is implied by the two other conditions under Case 2. Therefore, it is a redundant condition.

- $V(1,0) > V(1,1) \Leftrightarrow \pi_1(\mathbf{x}_1)+\varepsilon_1 > \pi_1(\mathbf{x}_1)+\delta_1(\tilde{\mathbf{x}})+\varepsilon_1+\pi_2(\mathbf{x}_2)+\delta_2(\tilde{\mathbf{x}})+\varepsilon_2$
  $\Rightarrow \varepsilon_2 < -\pi_2(\mathbf{x}_2)-\delta_1(\tilde{\mathbf{x}})-\delta_2(\tilde{\mathbf{x}})$

- Case 3: $(a_1^*, a_2^*) = (0,1)$ is chosen
  - $V(0,1) > V(0,0) \Leftrightarrow \pi_2(\mathbf{x}_2)+\varepsilon_2 > 0 \Rightarrow \varepsilon_2 > -\pi_2(\mathbf{x}_2)$
  - $V(0,1) > V(1,0) \Leftrightarrow \pi_2(\mathbf{x}_2)+\varepsilon_2 > \pi_1(\mathbf{x}_1)+\varepsilon_1$: Note that since $\delta_i > 0\ \forall i$, this condition is implied by the two other conditions under Case 3. Therefore, it is a redundant condition.
  - $V(0,1) > V(1,1) \Leftrightarrow \pi_2(\mathbf{x}_2)+\varepsilon_2 > \pi_1(\mathbf{x}_1)+\delta_1(\tilde{\mathbf{x}})+\varepsilon_1+\pi_2(\mathbf{x}_2)+\delta_2(\tilde{\mathbf{x}})+\varepsilon_2$
    $\Leftrightarrow \varepsilon_1 < -\pi_1(\mathbf{x}_1)-\delta_1(\tilde{\mathbf{x}})-\delta_2(\tilde{\mathbf{x}})$

- Case 4: $(a_1^*, a_2^*) = (1,1)$ is chosen
  - $V(1,1) > V(1,0) \Leftrightarrow \pi_1(\mathbf{x}_1)+\delta_1(\tilde{\mathbf{x}})+\varepsilon_1+\pi_2(\mathbf{x}_2)+\delta_2(\tilde{\mathbf{x}})+\varepsilon_2 > \pi_1(\mathbf{x}_1)+\varepsilon_1$
    $\Leftrightarrow \varepsilon_2 > -\pi_2(\mathbf{x}_2)-\delta_1(\tilde{\mathbf{x}})-\delta_2(\tilde{\mathbf{x}})$
  - $V(1,1) > V(0,1) \Rightarrow \pi_1(\mathbf{x}_1)+\delta_1(\tilde{\mathbf{x}})+\varepsilon_1+\pi_2(\mathbf{x}_2)+\delta_2(\tilde{\mathbf{x}})+\varepsilon_2 > \pi_2(\mathbf{x}_2)+\varepsilon_2$
    $\Leftrightarrow \varepsilon_1 > -\pi_1(\mathbf{x}_1)-\delta_1(\tilde{\mathbf{x}})-\delta_2(\tilde{\mathbf{x}})$
  - $V(1,1) > V(0,0) \Leftrightarrow \pi_1(\mathbf{x}_1)+\delta_1(\tilde{\mathbf{x}})+\varepsilon_1+\pi_2(\mathbf{x}_2)+\delta_2(\tilde{\mathbf{x}})+\varepsilon_2 > 0$

Plotting all of the above conditions in the space of $(\varepsilon_1, \varepsilon_2)$ would directly imply Figure 2.

Derivation of the Pure Strategy NE of the Perturbed Game by Table 2 with Strategic Complements and Under the Mechanism that Selects the Risk Dominant Equilibrium

Recall that in this perturbed game, $\tilde{\delta}_1(\tilde{\mathbf{x}}) = \tilde{\delta}_2(\tilde{\mathbf{x}}) = \delta_1(\tilde{\mathbf{x}})+\delta_2(\tilde{\mathbf{x}})$. Since $\tilde{\delta}_i > 0\ \forall i$, it is clear that when $\pi_i(\mathbf{x}_i)+\varepsilon_i > 0$, player i's strict dominant strategy is $a_i = 1$. On the other hand, when $\pi_i(\mathbf{x}_i)+\tilde{\delta}_i(\tilde{\mathbf{x}})+\varepsilon_i < 0$, player i's strict dominant strategy is $a_i = 0$. The regions for these strict dominant strategies directly imply the following four results:

- when $\varepsilon_1 > -\pi_1(\mathbf{x}_1)$ and $\varepsilon_2 > -\pi_2(\mathbf{x}_1)-\tilde{\delta}_2(\tilde{\mathbf{x}})$ or $-\pi_1(\mathbf{x}_1)-\tilde{\delta}_1(\tilde{\mathbf{x}}) < \varepsilon_1 < -\pi_1(\mathbf{x}_1)$ and $\varepsilon_2 > -\pi_2(\mathbf{x}_2)$, action profile $(1,1)$ is the unique NE
- when $\varepsilon_1 < -\pi_1(\mathbf{x}_1)-\tilde{\delta}_1(\tilde{\mathbf{x}})$ and $\varepsilon_2 < -\pi_2(\mathbf{x}_1)$ or $-\pi_1(\mathbf{x}_1)-\tilde{\delta}_1(\tilde{\mathbf{x}}) < \varepsilon_1 < -\pi_1(\mathbf{x}_1)$ and $\varepsilon_2 < -\pi_2(\mathbf{x}_2)-\tilde{\delta}_2(\tilde{\mathbf{x}})$, action profile $(0,0)$ is the unique NE

- when $\varepsilon_1 < -\pi_1(\mathbf{x}_1) - \tilde{\delta}_1(\tilde{\mathbf{x}})$ and $\varepsilon_2 > -\pi_2(\mathbf{x}_2)$, action profile $(0,1)$ is the unique NE
- when $\varepsilon_1 > -\pi_1(\mathbf{x}_1)$ and $\varepsilon_2 < -\pi_2(\mathbf{x}_2) - \tilde{\delta}_2(\tilde{\mathbf{x}})$, action profile $(1,0)$ is the unique NE.

The only remaining case is when $-\pi_1(\mathbf{x}_1) - \tilde{\delta}_1(\tilde{\mathbf{x}}) < \varepsilon_1 < -\pi_1(\mathbf{x}_1)$ and $-\pi_2(\mathbf{x}_2) - \tilde{\delta}_2(\tilde{\mathbf{x}}) < \varepsilon_2 < -\pi_2(\mathbf{x}_2)$. In this case, there are two pure strategy NEs, where either both players choose action 1 or both players choose action 0. As defined by Harsanyi and Selten (1988), the profile $(1,1)$ is the risk dominant equilibrium if and only if $[0 - \pi_1 - \tilde{\delta}_1 - \varepsilon_1] \cdot [0 - \pi_2 - \tilde{\delta}_2 - \varepsilon_2] > [\pi_1 + \varepsilon_1] \cdot [\pi_2 + \varepsilon_2]$. Simple algebra simplifies this inequality as $\pi_1(\mathbf{x}_1) + \delta_1(\tilde{\mathbf{x}}) + \varepsilon_1 + \pi_2(\mathbf{x}_2) + \delta_2(\tilde{\mathbf{x}}) + \varepsilon_2 > 0$.

In summary, in a model that considers the pure strategy NE with the mechanism that selects the risk dominant equilibrium, the model prediction can be summarized by the following four scenarios:

- Case 1: $(0,0)$ is chosen
  - $\varepsilon_1 < -\pi_1(\mathbf{x}_1)$
  - $\varepsilon_2 < -\pi_2(\mathbf{x}_2)$
  - $\pi_1(\mathbf{x}_1) + \delta_1(\tilde{\mathbf{x}}) + \varepsilon_1 + \pi_2(\mathbf{x}_2) + \delta_2(\tilde{\mathbf{x}}) + \varepsilon_2 < 0$
- Case 2: $(1,0)$ is chosen
  - $\varepsilon_1 > -\pi_1(\mathbf{x}_1)$
  - $\varepsilon_2 < -\pi_2(\mathbf{x}_2) - \tilde{\delta}_2(\tilde{\mathbf{x}})$
- Case 3: $(0,1)$ is chosen
  - $\varepsilon_2 > -\pi_2(\mathbf{x}_2)$
  - $\varepsilon_1 < -\pi_1(\mathbf{x}_1) - \tilde{\delta}_1(\tilde{\mathbf{x}})$
- Case 4: $(1,1)$ is chosen
  - $\varepsilon_1 > -\pi_1(\mathbf{x}_1) - \tilde{\delta}_1(\tilde{\mathbf{x}})$
  - $\varepsilon_2 > -\pi_2(\mathbf{x}_2) - \tilde{\delta}_2(\tilde{\mathbf{x}})$
  - $\pi_1(\mathbf{x}_1) + \delta_1(\tilde{\mathbf{x}}) + \varepsilon_1 + \pi_2(\mathbf{x}_2) + \delta_2(\tilde{\mathbf{x}}) + \varepsilon_2 > 0$

All of the above conditions are identical to the ones under the collusive / cooperative model in games with strategic complements. These conditions also lead to Figure 2.

Derivation of the Conditions in Figure 3

The following items describe the conditions under which each action profile is chosen within the collusive / cooperative model under an asymmetric joint objective function.

- Case 1: $(a_1^*, a_2^*) = (0,0)$ is chosen
  - $V(0,0) > V(1,0) \Leftrightarrow 0 > \pi_1(\mathbf{x}_1) + \varepsilon_1 \Leftrightarrow \varepsilon_1 < -\pi_1(\mathbf{x}_1)$
  - $V(0,0) > V(0,1) \Leftrightarrow 0 > \kappa\pi_2(\mathbf{x}_2) + \kappa\varepsilon_2 \Leftrightarrow \varepsilon_2 < -\pi_2(\mathbf{x}_2)$
  - $V(0,0) > V(1,1) \Leftrightarrow 0 > \pi_1(\mathbf{x}_1) + \delta_1(\tilde{\mathbf{x}}) + \varepsilon_1 + \kappa\pi_2(\mathbf{x}_2) + \kappa\delta_2(\tilde{\mathbf{x}}) + \kappa\varepsilon_2$: Note that since $\delta_i < 0 \ \forall i$, this condition is implied by the previous two conditions. Therefore, it is a redundant condition.
- Case 2: $(a_1^*, a_2^*) = (1,0)$ is chosen
  - $V(1,0) > V(0,0) \Leftrightarrow \pi_1(\mathbf{x}_1) + \varepsilon_1 > 0 \Leftrightarrow \varepsilon_1 > -\pi_1(\mathbf{x}_1)$
  - $V(1,0) > V(0,1) \Leftrightarrow \pi_1(\mathbf{x}_1) + \varepsilon_1 > \kappa\pi_2(\mathbf{x}_2) + \kappa\varepsilon_2$
  - $V(1,0) > V(1,1) \Leftrightarrow \pi_1(\mathbf{x}_1) + \varepsilon_1 > \pi_1(\mathbf{x}_1) + \delta_1(\tilde{\mathbf{x}}) + \varepsilon_1 + \kappa\pi_2(\mathbf{x}_2) + \kappa\delta_2(\tilde{\mathbf{x}}) + \kappa\varepsilon_2$ $\Leftrightarrow \varepsilon_2 < -\pi_2(\mathbf{x}_2) - [\frac{1}{\kappa}\delta_1(\tilde{\mathbf{x}}) + \delta_2(\tilde{\mathbf{x}})]$
- Case 3: $(a_1^*, a_2^*) = (0,1)$ is chosen
  - $V(0,1) > V(0,0) \Leftrightarrow \kappa\pi_2(\mathbf{x}_2) + \kappa\varepsilon_2 > 0 \Leftrightarrow \varepsilon_2 > -\pi_2(\mathbf{x}_2)$
  - $V(0,1) > V(1,0) \Leftrightarrow \kappa\pi_2(\mathbf{x}_2) + \kappa\varepsilon_2 > \pi_1(\mathbf{x}_1) + \varepsilon_1$
  - $V(0,1) > V(1,1) \Leftrightarrow \kappa\pi_2(\mathbf{x}_2) + \kappa\varepsilon_2 > \pi_1(\mathbf{x}_1) + \delta_1(\tilde{\mathbf{x}}) + \varepsilon_1 + \kappa\pi_2(\mathbf{x}_2) + \kappa\delta_2(\tilde{\mathbf{x}}) + \kappa\varepsilon_2$ $\Leftrightarrow \varepsilon_1 < -\pi_1(\mathbf{x}_1) - \delta_1(\tilde{\mathbf{x}}) - \kappa\delta_2(\tilde{\mathbf{x}})$
- Case 4: $(a_1^*, a_2^*) = (1,1)$ is chosen
  - $V(1,1) > V(1,0) \Leftrightarrow \pi_1(\mathbf{x}_1) + \delta_1(\tilde{\mathbf{x}}) + \varepsilon_1 + \kappa\pi_2(\mathbf{x}_2) + \kappa\delta_2(\tilde{\mathbf{x}}) + \kappa\varepsilon_2 > \pi_1(\mathbf{x}_1) + \varepsilon_1$ $\Leftrightarrow \varepsilon_2 > -\pi_2(\mathbf{x}_2) - [\frac{1}{\kappa}\delta_1(\tilde{\mathbf{x}}) + \delta_2(\tilde{\mathbf{x}})]$

- $V(1,1) > V(0,1) \Leftrightarrow \pi_1(\mathbf{x}_1)+\delta_1(\tilde{\mathbf{x}})+\varepsilon_1+\kappa\pi_2(\mathbf{x}_2)+\kappa\delta_2(\tilde{\mathbf{x}})+\kappa\varepsilon_2 > \kappa\pi_2(\mathbf{x}_2)+\kappa\varepsilon_2$
  $\varepsilon_1 > -\pi_1(\mathbf{x}_1)-\delta_1(\tilde{\mathbf{x}})-\kappa\delta_2(\tilde{\mathbf{x}})$
- $V(1,1) > V(0,0) \Leftrightarrow \pi_1(\mathbf{x}_1)+\delta_1(\tilde{\mathbf{x}})+\varepsilon_1+\kappa\pi_2(\mathbf{x}_2)+\kappa\delta_2(\tilde{\mathbf{x}})+\kappa\varepsilon_2 > 0$: Note that since $\delta_i < 0\ \forall i$, this condition is implied by the previous two conditions. Therefore, it is a redundant condition.

Plotting all above conditions in the space of $(\varepsilon_1,\varepsilon_2)$ would directly imply Figure 3.

Derivation of the Pure Strategy NE Prediction of the Game Presented by Table 3

Recall that $\tilde{\delta}_1(\tilde{\mathbf{x}}) = \delta_1(\tilde{\mathbf{x}})+\kappa\delta_2(\tilde{\mathbf{x}})$ and $\tilde{\delta}_2(\tilde{\mathbf{x}}) = \frac{1}{\kappa}\delta_1(\tilde{\mathbf{x}})+\delta_2(\tilde{\mathbf{x}})$. In the perturbed game presented by Table 3, since $\tilde{\delta}_i < 0\ \forall i$, it is clear that when $\pi_i(\mathbf{x}_i)+\varepsilon_i < 0$, player i's strict dominant strategy is $a_i = 0$. On the other hand, when $\pi_i(\mathbf{x}_i)+\tilde{\delta}_i(\tilde{\mathbf{x}})+\varepsilon_i > 0$, player i's strict dominant strategy is $a_i = 1$. The regions for these strict dominant strategies directly imply the following four results:

- when $\varepsilon_i < -\pi_i(\mathbf{x}_i)\ \forall i$, action profile $(0,0)$ is the unique NE
- when $\varepsilon_i > -\pi_i(\mathbf{x}_i)-\tilde{\delta}_i(\tilde{\mathbf{x}})\ \forall i$, action profile $(1,1)$ is the unique NE
- when $\varepsilon_1 < -\pi_1(\mathbf{x}_1)$ and $\varepsilon_2 > -\pi_2(\mathbf{x}_2)$ or $-\pi_1(\mathbf{x}_1) < \varepsilon_1 < -\pi_1(\mathbf{x}_1)-\tilde{\delta}_1(\tilde{\mathbf{x}})$ and $\varepsilon_2 > -\pi_2(\mathbf{x}_2)-\tilde{\delta}_2(\tilde{\mathbf{x}})$, action profile $(0,1)$ is the unique NE
- when $\varepsilon_1 > -\pi_1(\mathbf{x}_1)-\tilde{\delta}_1(\tilde{\mathbf{x}})$ and $\varepsilon_2 < -\pi_2(\mathbf{x}_2)-\tilde{\delta}_2(\tilde{\mathbf{x}})$ or $-\pi_1(\mathbf{x}_1) < \varepsilon_1 < -\pi_1(\mathbf{x}_1)-\tilde{\delta}_1(\tilde{\mathbf{x}})$ and $\varepsilon_2 < -\pi_2(\mathbf{x}_2)$, action profile $(1,0)$ is the unique NE.

The only remaining case is when $-\pi_1(\mathbf{x}_1) < \varepsilon_1 < -\pi_1(\mathbf{x}_1)-\tilde{\delta}_1(\tilde{\mathbf{x}})$ and $-\pi_2(\mathbf{x}_2) < \varepsilon_2 < -\pi_2(\mathbf{x}_2)-\tilde{\delta}_2(\tilde{\mathbf{x}})$. In this case, there are two pure strategy NEs, where only one of the players chooses action 1 while the other player chooses action 0. Given the equilibrium selection mechanism that chooses the one that maximizes the sum of weighted payoffs, player 1 will choose action 1 if and only if $\pi_1(\mathbf{x}_1)+\varepsilon_1 > \kappa\pi_2(\mathbf{x}_2)+\kappa\varepsilon_2$. This action profile takes the form: $(\mathbb{1}[\pi_1(\mathbf{x}_1)+\varepsilon_1 > \kappa\pi_2(\mathbf{x}_2)+\kappa\varepsilon_2], \mathbb{1}[\pi_1(\mathbf{x}_1)+\varepsilon_1 < \kappa\pi_2(\mathbf{x}_2)+\kappa\varepsilon_2])$.

In summary, in a model that considers the pure strategy NE with the mechanism that selects the equilibrium with the highest weighted sum of payoffs, the model predictions can be summarized by the following four scenarios:

- Case 1: $(0,0)$ is chosen
  - $\varepsilon_i < -\pi_i(\mathbf{x}_i)\ \forall i$
- Case 2: $(1,0)$ is chosen
  - $\varepsilon_1 > -\pi_1(\mathbf{x}_1)$
  - $\varepsilon_2 < -\pi_2(\mathbf{x}_2) - \tilde{\delta}_2(\tilde{\mathbf{x}})$
  - $\pi_1(\mathbf{x}_1) + \varepsilon_1 > \kappa\pi_2(\mathbf{x}_2) + \kappa\varepsilon_2$
- Case 3: $(0,1)$ is chosen
  - $\varepsilon_2 > -\pi_2(\mathbf{x}_2)$
  - $\varepsilon_1 < -\pi_1(\mathbf{x}_1) - \tilde{\delta}_1(\tilde{\mathbf{x}})$
  - $\kappa\pi_2(\mathbf{x}_2) + \kappa\varepsilon_2 > \pi_1(\mathbf{x}_1) + \varepsilon_1$
- Case 4: $(1,1)$ is chosen
  - $\varepsilon_i > -\pi_i(\mathbf{x}_i) - \tilde{\delta}_i(\tilde{\mathbf{x}})\ \forall i$

All of the above conditions are identical to the ones under the collusive / cooperative model with an asymmetric objective function. These conditions also lead to Figure 3.

Proof of Proposition 4: Let $\mathbf{a}^* = (a_i^*, \mathbf{a}_{-i}^*)$ denote the predicted action profile of the collusive / cooperative model. First, I will prove that $\mathbf{a}^*$ is a pure strategy NE of the perturbed game by Equation (6). Furthermore, within the set of pure strategy NEs, $\mathbf{a}^*$ is the one that maximizes the function $V'(\mathbf{a}, \mathbf{x}, \boldsymbol{\epsilon})$.

For any player i, consider another action profile $\mathbf{a}'$ that differs from $\mathbf{a}^*$ only by player i's action; for instance, $\mathbf{a}' = (a_i', \mathbf{a}_{-i}^*)$ where $a_i' \neq a_i^*$. Since $\mathbf{a}^*$ is the predicted outcome of the collusive / cooperative model, it implies that $V(\mathbf{a}^*) > V(\mathbf{a}')$. This inequality implies the following condition:

$$\pi_i(\mathbf{x}_i) + \sum_{j \neq i} \underbrace{[\delta_{i,j}(\tilde{\mathbf{x}}) + \delta_{j,i}(\tilde{\mathbf{x}})]}_{=\tilde{\delta}_{i,j}(\tilde{\mathbf{x}})} \cdot a_j^* + \varepsilon_i > 0, \text{ if } a_i^* = 1 \tag{13}$$

$$\pi_i(\mathbf{x}_i) + \sum_{j \neq i} [\delta_{i,j}(\tilde{\mathbf{x}}) + \delta_{j,i}(\tilde{\mathbf{x}})] \cdot a_j^* + \varepsilon_i < 0, \text{ if } a_i^* = 0.$$

Equation (13) essentially represents the no deviation condition for player i in the perturbed game, given all other players choose the profile $\mathbf{a}^*_{-i}$. Since this condition holds for every player i, it suggests that $\mathbf{a}^*$ is a pure strategy NE of the perturbed game.

In addition, note that $V'(\mathbf{a},\mathbf{x},\boldsymbol{\epsilon}) = V(\mathbf{a},\mathbf{x},\boldsymbol{\epsilon})$ as proved by the following equation

$$\begin{aligned}
V'(\mathbf{a},\mathbf{x},\boldsymbol{\epsilon}) &= \sum_{i=1}^{N} a_i[\pi_i(\mathbf{x}_i) + \frac{1}{2}\sum_{j\neq i}\tilde{\delta}_{i,j}(\tilde{\mathbf{x}})\cdot a_j + \varepsilon_i] \\
&= \sum_{i=1}^{N} a_i[\pi_i(\mathbf{x}_i) + \sum_{j<i}\tilde{\delta}_{i,j}(\tilde{\mathbf{x}})\cdot a_j + \varepsilon_i] \quad \textit{since } \tilde{\delta}_{i,j} = \tilde{\delta}_{j,i} \\
&= \sum_{i=1}^{N} a_i[\pi_i(\mathbf{x}_i) + \sum_{j<i}[\delta_{i,j}(\tilde{\mathbf{x}}) + \delta_{j,i}(\tilde{\mathbf{x}})]\cdot a_j + \varepsilon_i] \\
&= \sum_{i=1}^{N} a_i[\pi_i(\mathbf{x}_i) + \sum_{j\neq i}\delta_{i,j}(\tilde{\mathbf{x}})\cdot a_j + \varepsilon_i] \\
&= V(\mathbf{a},\mathbf{x},\boldsymbol{\epsilon}).
\end{aligned}$$

Since $\mathbf{a}^*$ maximizes $V(\cdot)$, it must be the pure strategy NE that maximizes $V'(\cdot)$.

Conversely, within the set of pure strategy NEs in the perturbed game, suppose that $\mathbf{a}^*$ is the one that maximizes $V'(\cdot)$. Next, I will prove that $\mathbf{a}^*$ also maximizes $V(\cdot)$. Suppose that, on the contrary, the collusive / cooperative model predicts the action profile $\mathbf{a}' \neq \mathbf{a}^*$. By the above arguments, it must be the case that $\mathbf{a}'$ is the pure strategy NE that maximizes $V'(\cdot)$. This contradicts the prior condition that $\mathbf{a}^*$ is such an equilibrium. Therefore, $\mathbf{a}^*$ must be the maximizer of $V(\mathbf{a},\mathbf{x},\boldsymbol{\epsilon})$. This completes the proof.

□