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ELECTRICAL NOISE THEORY

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CONTENTS

PREFACE

The purpose of this booklet is to introduce electrical noise theory. It is primarily intended for workers in the electrical noise measurement field. To restrict the topic to an introduction, only certain traditional cases are treated.

The booklet is divided into three chapters and an Appendix. The first chapter treats the fundamental concepts and definitions of the two ideal kinds of noise-random and impulsive. The definition of impulse is not yet rigourously defined in the standards, to the author's knowledge; hence a definition appears that gives the impulse as a fundamental physical quantity - the charge. The second chapter discusses reception by an ideal receiver. The AM receiver is particularly important; its solution provides the basis for interpreting meter readings of the ideal RIFI meter discussed in Chapter Three. The third chapter includes solutions giving theoretical meter readings for noises as well as giving the bandwidth ratios of the gaussian bandpass. Finally, the third chapter discusses the uncertainty principle and its uses in RIFI meter applications.

The booklet was based on a literature survey of both theoretical and experimental work. It was written over the summer of 1970 at the Telecommunications Engineering Lab. of the D.O.C., in Ottawa. The author wishes to thank the Lab. staff for the help and encouragement they gave.

Further research is required into the more modern techniques of electrical noise measurement. Current work appears to favour the identification of noises by using multi-detector instruments to measure probability density functions. Information theory may be required to specify acceptable noise levels, and to optimize monitoring. Also, engineers and scientists co-operating at the international level can organize monitoring stations over the earth's surface to furnish planetary noise indexes. Conserving the electromagnetic spectrum as a world communications resource is truly one of the most important challenges to electrical engineering in the dawn of the Second Industrial Revolution.

Paul Irwin

Ottawa, 1970

CHAPTER ONE

NOISE

1.1. Random Noise.

1.1.1. Gaussian or White Noise.

Random noise is often called gaussian noise because it follows a so-called gaussian or normal probability distribution. Probability distributions are necessary to describe the behaviour of random variables such as noise voltages and currents.

To see how probability distributions do this, consider a random variable, v, and let $P(v_o)$ be the probability that v is less than v_o. This is usually written as

$$Pr(v \leq v_o) = P(v_o)$$

Next, consider the probability that v lies within some range, say from v_1 to v_2. A little thought will show that this is the probability v is less than the larger value v_2, minus the probability v is less than v_1. In notation:

$$Pr(v_1 \leq v \leq v_2) = Pr(v \leq v_2) - Pr(v \leq v_1)$$

or, using function notation, this is

$$P(v_2) - P(v_1)$$

The probability that v lies in a narrow range, v_1 to v_1+dv_1 is therefore

$$P(v_1 + dv_1) - P(v_1)$$

by substituting $v_1 + dv_1$ for v_2 above. This quantity is the differential of P(v) at v_1, therefore it may be written as dP(v). A function called the probability density function may now be defined as

$$p(v) = \frac{dP(v)}{dv}$$

where dP(v) = p(v)dv is the probability the random variable is between v and v + dv. Now the probability v lies between v_1 and v_2 may be written as

$$P(v_2) - P(v_1) = \int_{v_1}^{v_2} p(v)dv$$

The probability v has any value is 1, giving

$$P(\infty) = \int_{-\infty}^{\infty} p(v)dv = 1$$

For any particular value, v_o, $P(v_o)$ may be found from p(v) by

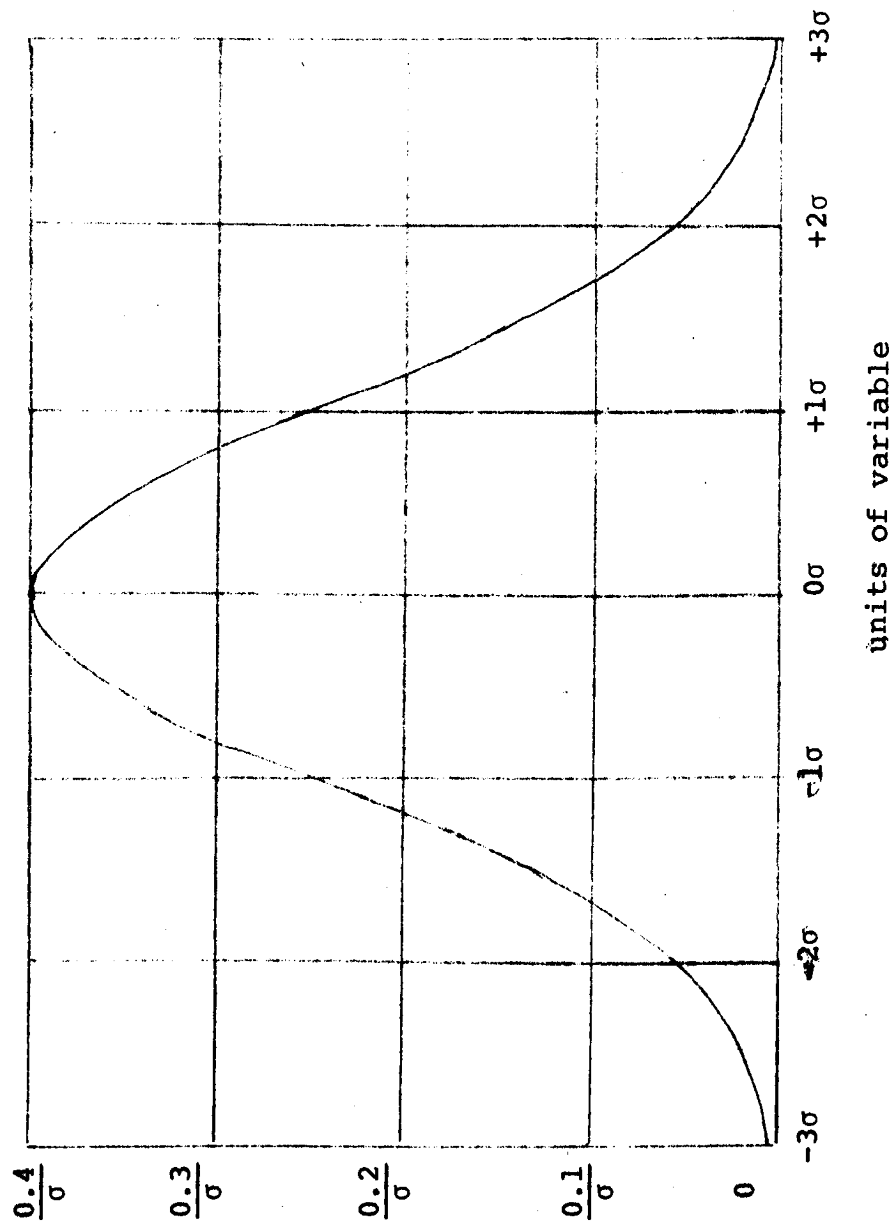
$$P(v_o) = \int_{-\infty}^{v_o} p(v)dv$$

The graphs of P(v) and p(v) for the gaussian distribution are shown in Fig. 1. For a more complete discussion of the concept of probability functions, see Bennett [1956].

The normal or gaussian distribution shown in Fig. 1 is written

$$p(v) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp(-v^2/2\sigma^2)$$

This is the probability density function that describes random noise.



probability density, p(v)

FIGURE 1(a)

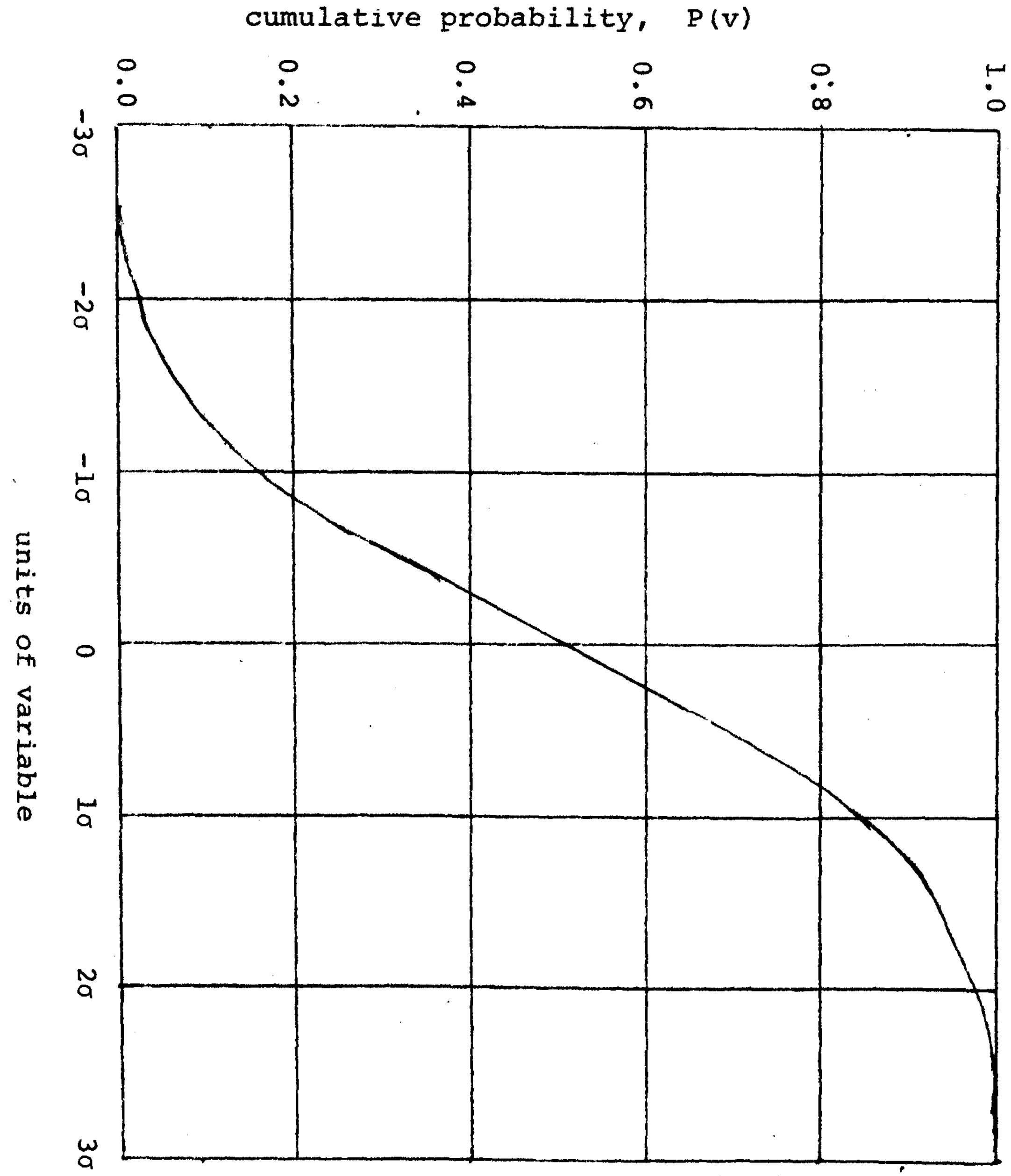

cumulative probability, P(v)
0.0
0.2
0.4
0.6
0.8
1.0
-3σ
-2σ
-1σ
0
1σ
2σ
3σ
units of variable

FIGURE 1(b)

Random noise is generated in conductors by thermal action. From thermodynamics and statistical physics, the equipartition theorem gives the energy per degree of freedom of a system as (1/2)kT, where k is Boltzmann's constant and T is the absolute temperature of the system in kelvins. In a conductor there are two degrees of freedom, electric and magnetic, giving a thermal energy of kT. The average power the conductor transfers to an ideal infinite transmission line is kTdf per frequency interval. The voltage is given by

$$d\langle v^2 \rangle = 4kTRdf$$

where R is the resistance of the conductor. This results in (1) the power is independent of the resistance provided matching is obtained, and, (2) the (ideal) system is frequency independent. In practice real systems have some frequency dependence, hence the total noise power can be found by integration. The formula

$$d\langle v^2 \rangle = 4kTRdf$$

is called Nyquist's formula [H. Nyquist, 1928].

The link between these two ideas - the noise mean square voltage as a temperature function and the gaussian distribution - was provided by Einstein [1905] who discovered the theory of Brownian motion. From the general case of Brownian motion which includes electric charge fluctuations comes the concept of a random varying voltage, v, having a probability density function

$$p(v) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-v^2/_{2\sigma^2})$$

where σ^2 is the mean square voltage having the property

$$d\sigma^2 = 4kTRdf$$

1.1.2. Band Limited White Noise.

Since noises do not behave ideally in the sense of having an infinite spectum, the simple case outlined above cannot be used indiscriminately. Although many noises are white well into the microwave region and beyond, a limit is imposed by quantum effects. Before the quantum region is reached circuit impedences become significant. Because of this, the band-limited cases are most often encountered.

If a noise is confined to a passband then the voltage fluctuations lose some of their randomness. In the unlimited case the voltage at any time was considered to be completely independent of the voltage at any other time. If it is physically possible for the voltage to fluctuate very rapidly, then this ideal independence is approached. To achieve this, a very wide bandwidth will be required. On the other hand, if the bandwidth is narrow the voltage must fluctuate at a slower rate. This means the voltage at any time depends to some extent upon the voltage at an earlier thme. Only for much earlier times will the voltage again be independent. This dependence of the voltage on the time interval is called autocorrelation, and it is expressed mathematically in the autocorrelation function.

The autocorrelation function measures the independence of a random variable upon itself at an earlier time. It is a function of time interval. Several examples of autocorrelation functions are given in Fig. 2. The importance of the

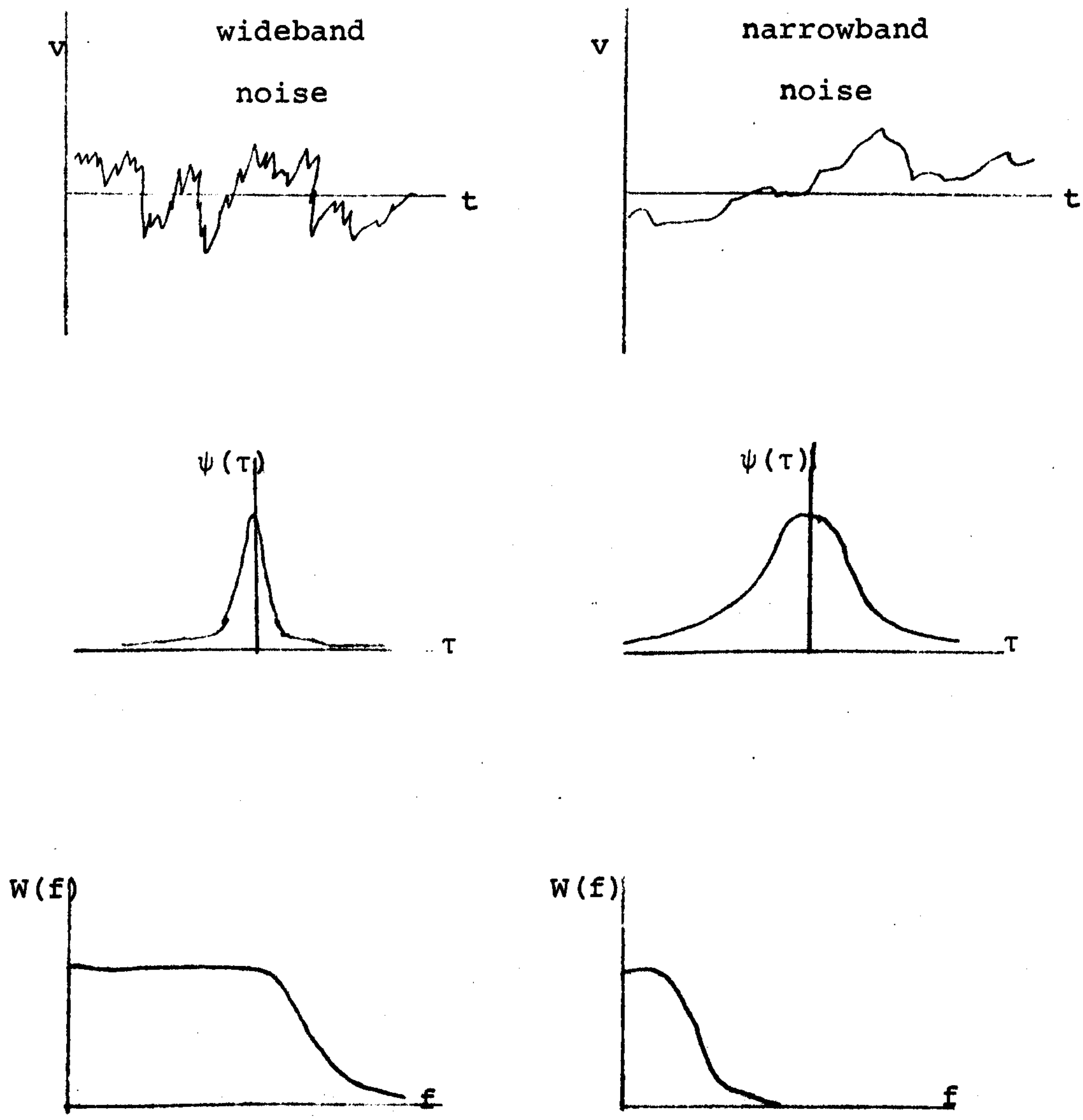


FIGURE 2.

autocorrelation function in noise work lies in the Weiner--Kinchine theorem which relates the autocorrelation function to the power spectrum as a Fourier cosine transform pair. Essentially, this means that the power spectrum may be found from the autocorrelation function, and vice versa, just as the voltage spectrum of a regular waveform may be found from the waveform profile. The spectra for several autocorrelation functions are tabled in Fig. 3. Note particularly the inverse property of the autocorrelation function - power spectrum pair: the longer the autocorrelation time, the lower the high frequency cutoff. For a zero autocorrelation time the spectrum is infinite; this is just the ideal white noise case first described.

1.1.3. Narrowband Noise and its Fluctuating Envelope.

It is practically important to consider the particular narrowband noise case. Narrowbands occur often, particularly in radio receivers and RIFI meters, making the results obtained useful in later chapters.

The mathematical solution to the narrowband noise case is given by S. O. Rice [1945] who considered white noise being tuned by a narrowband filter. The output of such a filter may be thought of as having a sinusoidal component with the frequency of the filter midband, and modulated by an irregularly fluctuating amplitude. The rapidity if the fluctuations is determined by the filter bandwidth.

Several important statistics may be calculates from the noise at the output of a narrowband filter. One important result is the probability density function of the

Description	$\psi(t)$	$W(f)$
random noise square spectrum	$W_o f_o \cos(2\pi f_c t)\,\mathrm{sinc}(f_o t)$	$W_o,\ -\frac{f_o}{2} < f-f_c < +\frac{f_o}{2}$ 0, otherwise
random noise gaussian spectrum	$W_o f_o \cos(2\pi f_c t)\exp(-\pi f_o^2 t^2)$	$W_o \exp[-\ \frac{\pi(f-f_c)^2}{f_o^2}$
random pulses of equal duration, τ	$\frac{Ie}{\tau}(1 - \frac{t}{\tau}),\ 0 < t < \tau$ 0, otherwise	$2Ie\ \mathrm{sinc}^2(f\tau)$
strongly clipped random noise	$a^2\exp(-\mu\|t\|)$	$\frac{2a^2\mu}{\pi^2 f^2+\mu^2}$

FIGURE 3.

envelope current:

$$p(R) = \frac{R}{\psi_o} \exp\left(\frac{-R^2}{2\psi_o}\right)$$

The envelope is not gaussian, but follows a Rayleigh distribution. The quantity ψ_o is the mean square current, and for the rectangular passband,

$$\psi_o = W_o(BW)$$

where W_o is the spectral density within the band and BW is the bandwidth.

Rice has also given the distribution for the number of maxima of the envelope as well as the energy fluctuation for various time constants of smoothing filters in his classic paper [Rice, 1945]. A couple of interesting figures are expected number of maxima for the ideal bandpass filter

$$N = 0.641(BW)$$

and for the gaussian-shaped bandpass filter

$$N = 1.006(BW)$$

The effect of the shape of the bandpass filter on noise will be discussed more fully in later chapters.

1.2. Impulsive Noise

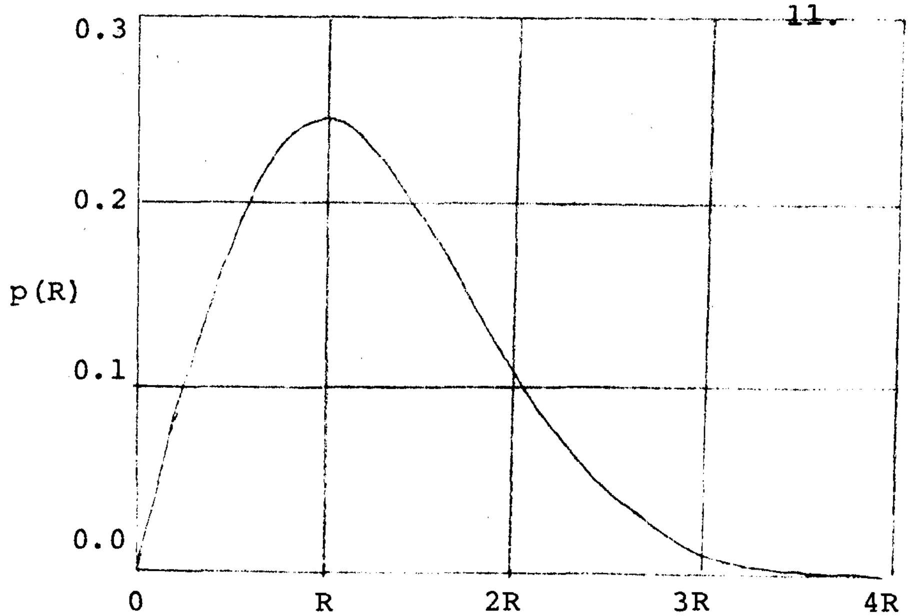
1.2.1. Definitions.

Random noise is one extreme type of interference while periodic pulses and impulses are another. The pulses described here are regular periodic with a constant amplitude.

FIGURE 4.

The Rayleigh

Distribution.



Since both pulses and impulses are described, it is important to distinguish between them.

A pulse is a burst of current, with or without a dicernable fine structure, having a finite duration. An impulse is a pulse whose duration is too short to be measured.

Several important consequences of this definition are:

1) The pulse is described by at least two independent parameters - current and time. In addition, fine structure parameters such as envelope shape and carrier frequency may be given.

2) The impulse is a subjective concept. It is used to imply a near-zero duration with respect to the relaxation time of the observer (instrument).

3) Since the impulse does not have a known lifetime it must be described by one less parameter than the pulse.

The minimum number of parameters needed to describe a pulse is two, therefore only one parameter is used to describe the impulse.

4) Since the impulse duration is too brief to measure the current will not be present long enough to measure. The only physical quantity accessable to measurement is the charge, i.e. the current multiplied by the time.

In order to measure the charge of an impulse, a narrowband meter - a RIFI meter - is often used. This leads to the use of a voltage-type unit in the following way.

If an impulse is passed through a resistor, then the voltage across this resistor is given by Ohm's law

$$Q = \frac{V\tau}{R}$$

where τ is the lifetime of the pulse. Since R is constant, the impulse may be given by

$$S = V\tau = QR$$

which may be defined as the impulse parameter, S. The impulse may be found from the impulse parameter by

$$Q = \frac{S}{R}$$

once S and R are known. Note that the unit of the impulse is the ampere-second (or coulomb) and the unit of the impulse parameter is the volt-second.

The impulse parameter S should not be confused with the spectral intensity of the impulse which will be described later. Although a simple relationship exists between the impulse parameter and its spectral intensity, the relation does not hold for pulses in general - each pulse waveform has a different spectrum.

With these definitions in mind, the balance of this chapter will treat the general pulse and then the impulse as a special case.

1.2.2. Pulse Trains.

A pulse train consists of an infinite number of pulses, each following the other by the same time, T. In practice pulses are not produced indefinitely but are started and stopped at particular times. However, if they occur continuously during the measurement time, then they may be considered as infinite in number.

Before a pulse train is described, the ideal case of a single pulse will be described. The single pulse, although real, is not a physical quantity. This is because any measurement made of a single pulse cannot be confirmed by another independent measurement. To do so would require a second pulse exactly the same as the first, but at a later time. This is a pulse group however, having a period between them. However for most purposes, a pulse behaves as a single pulse if the period T is much greater than the measuring instrument relaxation time; this ensures that successive pulses do not interfere with each other (Fig. 5).

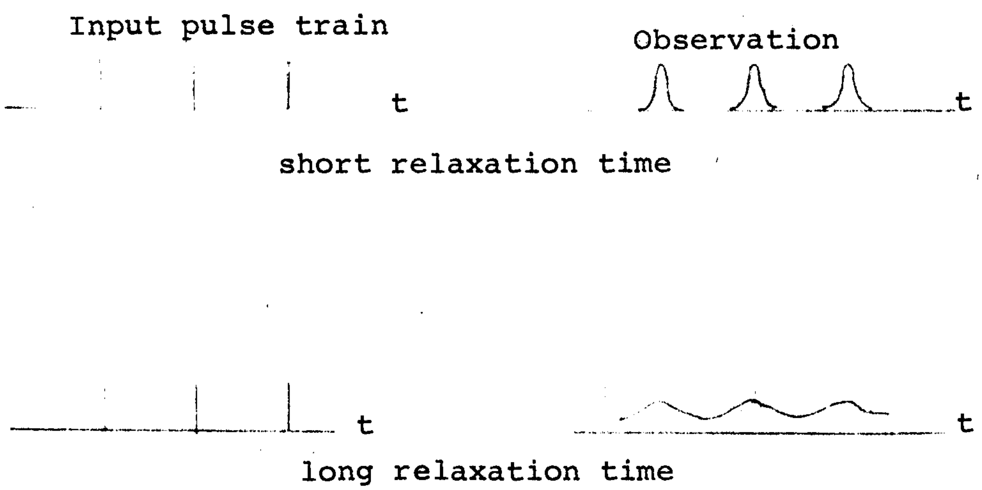


FIGURE 5. Effect of relaxation time on pulse train observations.

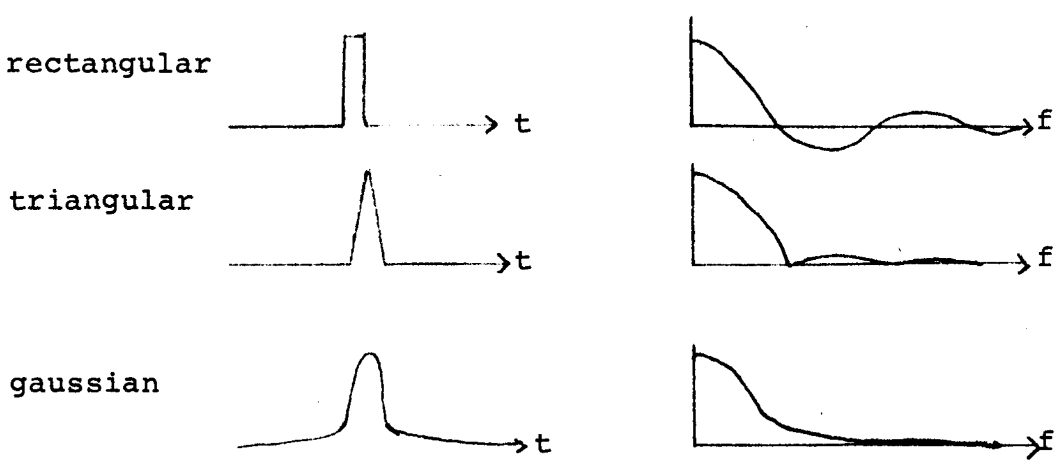


FIGURE 6. Spectra of various single pulses.

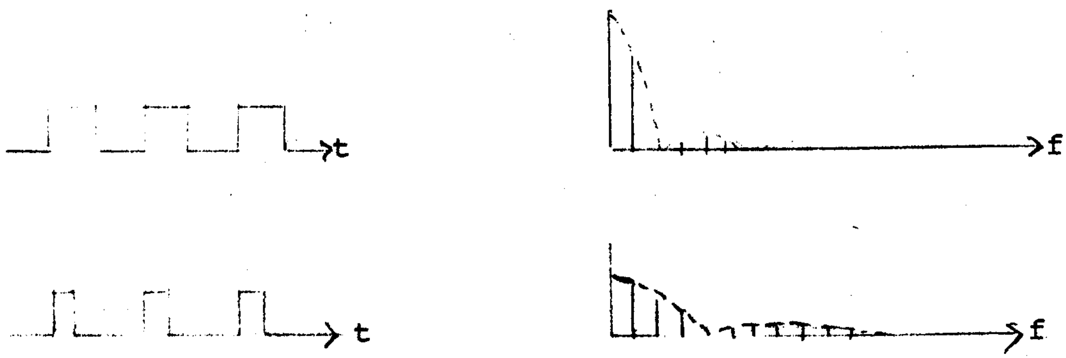
It is the voltage profile of a pulse that determines the unique spectrum of the pulse. The spectral intensity is the variable, and it has units of volts per hertz. The spectra for several waveforms are shown in Fig. 6. These spectra are the Fourier cosine transforms of the profiles. Note that they are continuous rather than discrete like the corresponding Fourier series of the infinite pulse trains.

The infinite pulse train has a pulse repetition frequency, PRF = $f_p = 1/T$. It is composed of harmonics; each one at some multiple of f_p. The envelope of the spectrum is, however, independent of f_p and is determined by the size and shape of the individual pulse. Several examples illustrating this dependence are shown in Fig. 7.

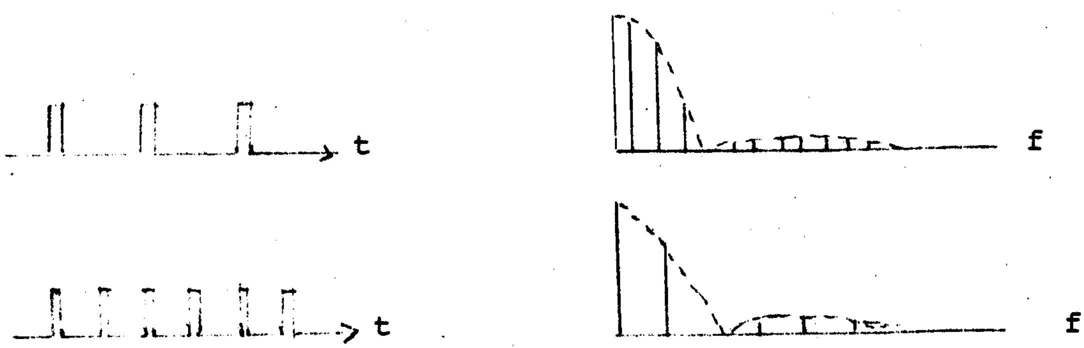
The relationship between pulse width and spectrum is important to an understanding of the impulsive case. For an impulse the pulse width is very small, hence the spectrum is very broad. Considering impulses as having almost zero width, they may be seen to have a spectrum practically flat. For the single impulse, this is a constant value throughout the entire spectrum observable by the instrument concerned. For an impulse train, this would be a spectral line each f_p throughout the band; each line having equal intensity. The height of each line is $2f_pS$. The quantity

$$\sqrt{2}S$$

has historically been taken as the spectral intensity of the impulse, and is called the RMS spectral intensity.



Decreasing the pulse width extends the spectrum of the pulse train. The position of the spectral lines remains unchanged.



Decreasing the interval between pulses increases the interval between spectral lines. The envelope of the spectrum remains unchanged.

FIGURE 7. The dependence of a square wave spectrum on the pulse interval T and the pulse width τ.

1.2.3. Random Occuring Pulses.

Although the general case of random occuring pulses has been solved, [Lawson and Uhlenbeck, 1950], it is much simpler to consider only the case where the pulses do not overlap. This is a valid assumption where the pulse width is much less than the average interval between pulses. If the pulse width is veryshort, the impulsive case may be found.

Consider a current pulse

$$i' = \frac{e}{\tau}, \text{ for } 0 < t < \tau$$
$$= 0, \text{ otherwise.}$$

By applying Campbell's theorem, the average current is

$$\langle i \rangle = N\int_0^\infty i'(t)dt$$
$$= N(\frac{e}{\tau})\tau$$
$$= Ne = I$$

whose mean square fluctuation is

$$\langle \delta i^2 \rangle = N\int_0^\infty i'^2(t)dt$$
$$= N(\frac{e}{\tau})^2\tau$$
$$= \frac{Ne^2}{\tau} = \frac{Ie}{\tau}$$

where N is the average rate or pulse repitition frequency. The autocorrelation function is

$$\psi(t) = N\int_0^\infty i'(T)\,i'(T+t)\,dT$$

$$= N\int_0^{T-t} \left(\frac{e}{\tau}\right)^2 dT$$

$$= \frac{Ie}{\tau}\left(1 - \frac{t}{\tau}\right), \text{ for } 0 < t < \tau$$

$$\psi(t) = 0, \quad \text{for } t > \tau$$

The spectral density is found by applying the Weiner-Kinchine theorem:

$$W(f) = 4\int_0^\infty (t)\cos(2\pi ft)\,dt$$

$$= 4\frac{Ie}{\tau}\int_0^\tau \left(1 - \frac{t}{\tau}\right)\cos(2\pi ft)\,dt$$

$$= 2Ie\left(\frac{\sin\pi f\tau}{f\tau}\right)^2$$

$$= 2Ie\cdot \text{sinc}^2 f\tau$$

For the impulsive case, $\text{sinc}^2 f\tau = 1$, and the result is the Schottky formula

$$W(f) = 2Ie$$

which describes the shot effect.

It should be noted that the Weiner-Kinchine theorum gives the spectral density for random processes, while the Fourier transform of the square of the function profile gives the spectral density for regular functions. The spectral density should not be confused with the spectral intensity which is found by the Fourier transform of the profile itself.

Random occuring pulses arise in other ways than by the shot effect. For example, if white noise is strongly clipped, then the result is a rectangular waveform whise length varies according to an exponential law. The correlation function and spectral density are [Rice, 1945]

$$\psi(t) = a^2 \exp(-\mu|t|)$$

$$W(f) = \frac{2a^2}{\pi^2 f^2 + \mu^2}$$

The strongly clipped narrowband case is more complex, and is treated by Lawson and Uhlenbeck [1950].

CHAPTER TWO

NOISE RECEPTION

2.1. Noise in AM Receivers.

2.1.1. Linear and Square Law Detection.

The output of random noise from a narrowband filter was given in chapter 1 as the Rayleigh distribution

$$p(R)\,dR = \frac{R}{\psi_o}\exp\left(-\frac{R^2}{2\psi_o}\right)dR$$

where R is the amplitude of the envelope. In the case of an AM receiver, the tuner may be considered as a narrowband filter followed by a detector. The output of the narrowband filter is first considered as noise only, and later as signal with noise.

The probability density function of the envelope is given above. This gives the density function of the detected noise. In addition, Rice [1945] gave the probability density function of the maxima as

$$\frac{1.13(y^2-1)}{\sqrt{\psi_o}}\exp\left(-\frac{y^2}{2}\right)$$

where $y = R/\sqrt{\psi_o}$. From the Rayleigh distribution may be found the average

$$\overline{R} = \sqrt{\psi_o \pi/2}$$

and the mean square

$$\overline{R^2} = 2\psi_o$$

of the envelope.

If a signal (CW) is being received, then the distribution changes. The envelope probability density then depends on the signal-to-noise ratio

$$x = \frac{P^2}{2\psi_o} = \frac{\text{ave. sine wave power}}{\text{Ave. noise power}}$$

In this case, the envelope probability density is

$$\frac{R}{\psi_o}\exp[-\frac{R^2+P^2}{2\psi_o}]I_o[\frac{RP}{\psi_o}]$$

Here, R is the envelope composed of signal and noise, P is the peak sine wave voltage, and I_o is a bessel function of the second kind. A graph of the envelope probability density is shown in Fig. 8. For x = 0 the Rayleigh distribution is obtained, as expected, since this is the noise--only case. For large x, the distribution becomes gaussian with mean $P/\sqrt{\psi_o}$ and variance ψ_o. The distribution for large x is approximately

$$\frac{1}{\sqrt{2\pi\psi_o}}\exp[-\frac{(R-P)^2}{2\psi_o}]$$

It should be noted that the mean square envelope now becomes

$$\overline{R^2} = P^2 + 2\psi_o$$

With these distributions the action of various kinds of detectors may be evaluated.

The first detector considered here is the square law or quadratic detector. Let the detector be described by

$$I = aV^2$$

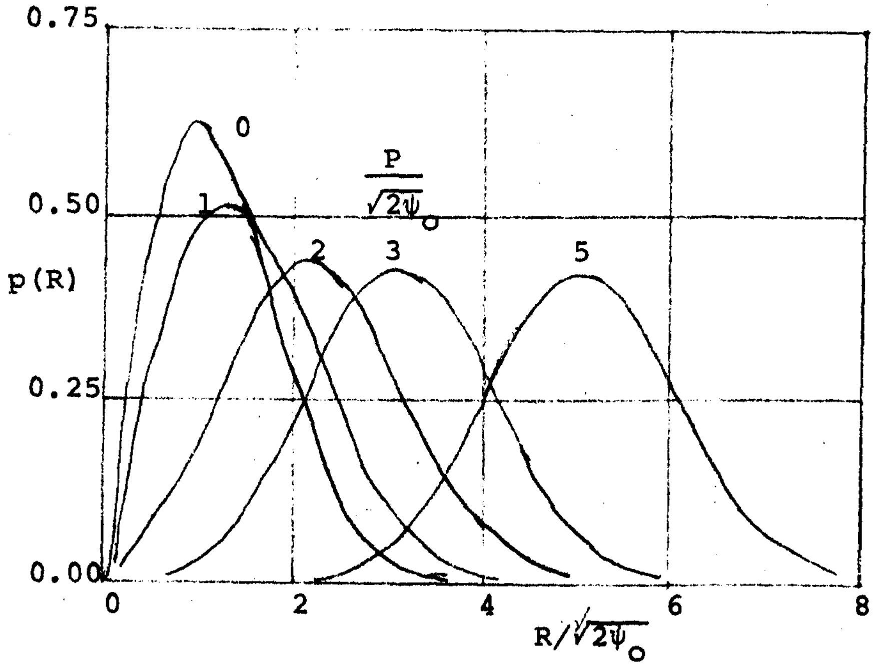


FIGURE 8.

and the input to the detector by

$$V = V_N + P\cos(pt)$$

where V_N is the noise and $P\cos(pt)$ is the signal. The envelope will be given by

$$R^2 = V_N^2 + P^2$$

If V is substituted into the rectifier equation, and if the expression for R together with its distribution are used, then the current may be found [Rice, 1945]. The current is composed of two parts: a DC component and an (LF) AC component. These are

$$I_{DC} = a(\psi_o + \frac{P^2}{2})$$

$$I_{LF} = a^2\psi_o(\psi_o + P^2)$$

The total mean square current is

$$\overline{I^2} = \frac{a^2}{4}\overline{R^2} = a^2(2\psi_o + 2P^2\psi_o + \frac{P^4}{4})$$

The linear detector may be solved in much the same manner. The equation of the ideal linear rectifier is

$$I = 0 \text{ , whenever } V < 0$$
$$= aV, \text{ whenever } V \geq o$$

from which the DC and LF components may be found:

$$I_{DC} = \frac{a}{\pi}\overline{R} = a\frac{\sqrt{\psi_o}}{\sqrt{2\pi}} e^{-x/2}[(1+x)I_o(\frac{x}{2}) + xI_1(\frac{x}{2})]$$

$$I_{LF} = \frac{a^2}{\pi^2}\psi_o(2 - \frac{\pi}{2})$$

The total mean square current is

$$\overline{I^2} = (\frac{a}{\pi})^2\overline{R^2} = (\frac{a}{\pi})^2(P^2 + 2\psi_o)$$

2.1.2. Minimum Detectable AM.

Before proceeding to the detectability of AM it might be worthwhile to first consider the spectrum of the detector output. This may be found from the probability distribution of the envelope by employing the proper rectifier equation - linear or quadratic - to find the distribution of the detector output. From the distribution, the autocorrelation function may be found, and hence the spectrum by the Weiner--Kinchine theorem. A solution has been obtained for the quadratic detector [Lawson and Uhlenbeck, 1950] which contains four terms.

The first term is the direct current term. This is composed of the square of the mean signal power, the square of the mean noise power, and a cross term of the average value of the beats between signal and noise. The second term is the signal at the modulating frequency and its second harmonic. The third term is a continuous spectrum due to cross-modulation of signal and noise. The fourth term is caused by beats among the noise components. Lawson and Uhlenbeck [1950[gives the equation on p. 158, and it is illustrated and explained on p. 159 for the rectangular bandpass case.

The minimum detectable signal is a statistical quantity. Any steady signal may be detected, no matter what the noise level, if there is sufficient time available. This is because the noise term tends to be self-cancelling when a long average is taken. Unfortunately, this averaging or integrating of the detector current will mask the variation of signal during the averaging time. Hence the signal can be detected only if it is constant during the integrating time necessary to seperate it from the noise. This integrator is usually a resistance-capacitance network placed across the detector output, and having some specified time constant.

To define the minimum detectable signal, several approaches may be taken. One approach by Lawson establishes a detectability criterion based on either a betting curve which expresses the ability of an observer to guess the presence or absence of a signal pulse, or as a signal-to--noise ratio employing signal peak power to bandwidth integrated noise power ratio of the order of unity, but requiring empirical determination. This second criterion is called the power criterion, and it is described in Ch. 7 of Lawson and Uhlenbeck [1950]. The results of this approach includes the specification of the minimum detectable modulation, ε_{min}, as a function of x, the ratio of unmodulated signal power to noise power. Significant in the results is equations [4] on p.369 [Lawson and Uhlenbeck, 1950] which show

$$(\varepsilon^2_{min})_{AM} \sim 1/x^2$$

for small signal-to-noise, x, and

$$(\varepsilon^2_{min})_{AM} \sim 1/x$$

for large x.

While the above approach may be useful in some cases, it has its shortcomings. For instance, the constant is empirical, hence it must be measured for each system. Also, the criteria are not directly applicable to interference study in their present form, except in radar perhaps, where they were first established. Another approach which would give the signal-to-noise acceptable as a "clean" detector output is required. This would involve somewhat larger signal-to-noise ratios; a minimum modulation level and band-width for a minimum signal strength. This problem will be discussed after the methods of the reception and measurement have been developed.

2.1.3. Inpulses in AM Receivers.

If impulses are received and tuned by an AM receiver, the output to the detector will usually consist of a pulse train. If the impulses occur randomly, then the output will be a noise. On the other hand, if the impulses are regular then the output will have regular features.

The response of a receiver to an impulse train depends a great deal on the pulse repitition frequency, f_p. If f_p is low, then there is a long time between pulses and each pulse behaves like a single pulse in the detector. Further, the output will be independent of the receiver tuning. However, if f_p is high, then the pulses will interfere with each other in the tuner; sometimes constructively to produce an enhanced output, sometimes destructively to produce a reduced output, depending on the receiver tuning and the bandwidth.

The effect of a high f_p on impulse reception can perhaps best be seen as a Fourier series. An impulse has a spectrum extending beyond the tuning range of the receiver, each component having a constant amplitude. The spectrum is composed of a series of discrete lines, each f_p apart. Hence, if the bandwidth of the receiver is less than the f_p then the receiver will display a series of peaks and nulls as it is tuned across the band. When tuned to a peak, the output to the detector will be a constant sine wave of midband frequency.

On the other hand, if the f_p is quite low then it is easier to view the situation in the time domain. The pulses are formed by the tuned circuits of the receiver independently of each other and they may therefore be treated as single impulses.

The impulse response of a cascade of series-tuned circuits is given by Sabaroff [1944] who gives the peak value of the output as

$$E \approx SG(\omega_2 - \omega_1)/2$$

where S is the impulse parameter and G is the midband gain. This is Sabaroff's equation which may be written as

$$E = \sqrt{\frac{2\pi}{\ln 2}}(3DB)SG$$

where $(3DB) = (\omega_2 - \omega_1)/2\pi$ is the 3 db bandwidth, and

$$\sqrt{\frac{2\pi}{\ln 2}} \simeq 3.0$$

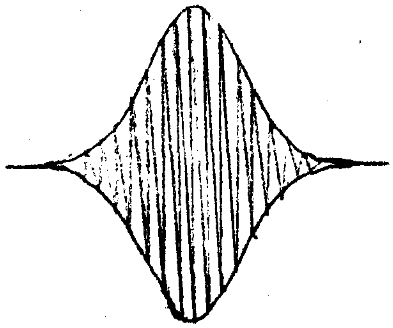


FIGURE 9. Response of a cascade of series-tuned circuits to an impulse.

Sabaroff derived his equation by cascading several tuned circuits and obtained a general expression for the output containing both amplitude and phase terms for n tuned circuits. The equation was found by first finding the maximum value of the amplitude and then making the appropriate symplifying assumptions. If it is assumed there are enough tuned circuits to ensure a gaussian frequency response, then

$$E = \sqrt{\frac{2\pi}{\ln 2}}(3DB)SG$$

is exact.

2.2. Noise in FM Receivers.

2.2.1. The Random Noise Spectrum.

When signal-to-noise ratios are high enough, an expression may be found fairly easily. The main parameters involved are

f_s, the maximum carrier deviation at 100% modulation,

f_a, the accepted output band,

W_c, the mean carrier power,

W_n, the equivalent mean noise power, $2f_a W(f)$,

The solution for S^2/N^2, where $S^2 = f_s^2/2$ is given by almost any textbook of FM as

$$\frac{S^2}{N^2} = \frac{1}{2}\left(\frac{f_s}{f_a}\right)^2\left(\frac{W_c}{W_n}\right)$$

Bennett [1956] shows the derivation and also gives the case for smaller S/N ratios. This solution is

$$\frac{S^2}{N^2} = \frac{1}{2}\left(\frac{f_s}{RN}\right)^2 (\int Y dX)^{-1}$$

where $\int YdX$ is the integral over the power spectrum of the sine wave plus noise. Graphs of the power spectrum are given in Fig. 10. Note that the noise increases with frequency for the audio frequencies. For this reason, the broadcasting industry employs a combination of FM and PM; the PM taking the form of a 6db per octave pre-emphasis of the treble frequencies on transmission. For a more specific analysis of FM, the reader is recommended to texts.

The other feature to note from the curves of Fig. 10 is the dependence of S/N on carrier intensity. As the carrier (W_c) is increased, the noise decreases. The combination of these factors make any realistic analysis of FM noise somewhat cumbersome. Undoubtedly, the spectrum gives the easiest picture to grasp. Some further analysis may be found in Ch. 13 of Lawson and Uhlenbeck [1950].

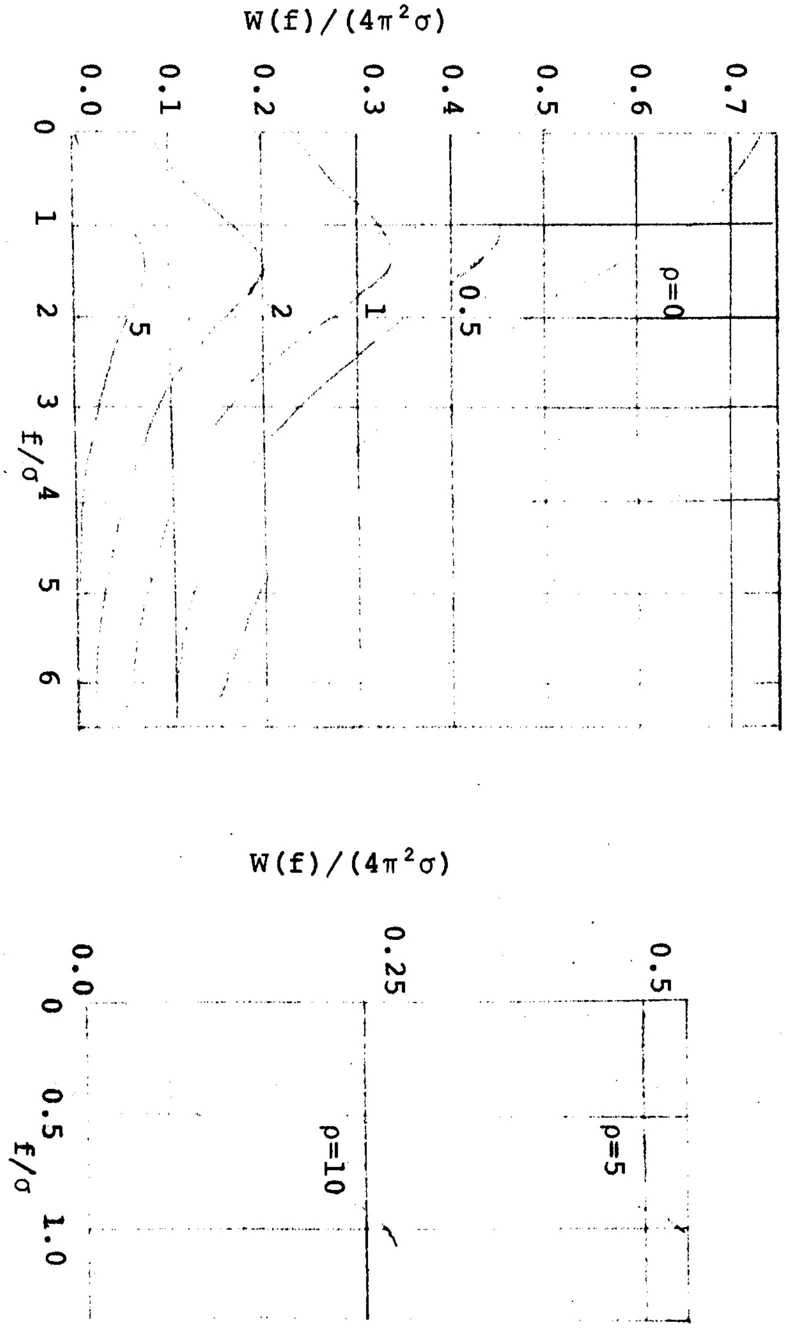


FIGURE 10. Power spectra of sine wave plus noise as given by S.O. Rice. [after Bennett, 1956]

Impulses in FM Receivers.

The case of impulses in FM receivers is essentially the same as random noise. When a carrier is present quieting occurs, but in the absence of a carrier, the audio output is determined by the limiter action only. It is the combination of limiting and detection - each being a non-linear process - that makes analytic solutions difficult in the FM case. Some work has been done in this direction as cited earlier [Lawson and Uhlenbeck, 1950].

2.3 Noise in TV and Radar.

2.3.1. Radar.

The AM receiver has already been discussed; radar and TV are further applications of the general case. The particular case considered here is the establishment of a minimum detectable signal.

Consider a radar signal being received. The signal may be a gaussian-, square-, or other shape pulse of RF. Let f_p be the pulse repitition frequency and n be the number of pulses observable in a time, t. Hence

$$n = f_p t$$

Now, a detector may be considered as an energy-measuring device having an uncertianty of the order kT where k is Boltzmann's constant and T is the (noise) temperature.

This gives a noise power measurement of

$$N^2 = kT(RN)$$

where (RN) is the bandwidth. If each pulse has energy E, then the signal power is

$$S^2 = \frac{E}{\tau} f_p \tau$$
$$= Ef_p$$

providing the bandwidth admits the entire pulse spectrum. The signal-to-noise ratio is then given by

$$\frac{S^2}{N^2} = \frac{Ef_p}{kT(RN)}$$

after the IF amplifier. If z is the signal-to-noise ratio before reception then

$$z = \frac{E}{kT} = 1.$$

gives the detectability criteria for a single pulse. This gives

$$\frac{S^2}{N^2} = \frac{f_p}{(RN)}$$

where the signal may be detected with near-certianty. Solving for the minimum power:

$$\overline{P_m} = Ef_p = kTf_p$$

for a single pulse, or

$$\overline{P_m} = \frac{kT}{\sqrt{n}} f_p$$

for n pulses. Recalling $n = f_p t$, the minimum detectable pulse train has an average power

$$\overline{P_m} = kT \frac{f_p}{t}$$

This is Eqn. 7-37 of Lawson and Uhlenbeck [1950]. It is important to note that the formula assumes a sufficient bandwidth to accept the entire pulse. The dependence on bandwidth is easy to see; if the band is too narrow then signal will be lost, if it is too wide then unnecessary noise will be included. It can be shown that the lowest value of the minimum detectable signal occurs when

$$(RN)\tau \approx 1$$

τ being the pulse width. A thorough discussion of noise and radar signals may be found in Lawsom and Uhlenbeck[1950].

2.3.2. Television.

In television as in radar, there is an AM receiver with a cathode-ray tube display. Television is a different application however, so the noise requirements are also different. For instance, in radar it is necessary to determine whether or not a signal is present, while in television the depth of modulation in each small time interval must also be discerned.

To illustrate the effect of noise on television, two problems will be discussed. The first problem is the specification of ~~mi~~maximum allowable internal receiver noise, while the second problem is the measurement of noise accompanying a television signal.

The ~~min~~*max*imum allowable internal receiver noise may be specified in at least two different ways. One is simply to state the noise figure [see Appendix] of the receiver; the other is to specify the maximum signal input required to produce an acceptable picture. The latter method requires careful measurement of the effect of noise upon the picture. By varying the input consisting of some common television test signals to several domestic receivers, some interesting results may be obtained. These tests were performed at the Telecommunications Engineering Lab. of the Dept. of Communications in Ottawa during the summer of 1970.

One standard signal - the staircase - gives interesting results. If the signal is slowly decreased until the noise first appears, the noise is seen first in the grey steps. For very low signal levels, the noise is also seen in shadow and highlight steps while the grey steps deteriorate. From the staircase test can be seen the loss of contrast and the degradation of the middle tones by graininess.

Another standard signal is the multiburst. This consists of several bursts of tone at different frequencies, appearing on the raster as vertical bars arranged in groups of different line densities. As the signal level is decreased, noise begins destroying the resolution of the signal.

Now, resolution is defined either by the number of discernable lines the frame can contain, or by the video bandwidth - there being 75 lines per MHz. Using a multiburst of 4.2 or 3.6 MHz, the noise of the receiver could easily be specified by stating the maximum signal to give a specified resolution - 315 or 270 lines.

The second problem - measuring the noise level of a television signal on a line - is more difficult. Here, the standard signal cannot be used; the noise must be distinguished from an arbitrary television program. One line of attack is to examine only the blanking and syncronization portions of the signal. By analyzing an oscilloscope trace the noise distribution may be found. Another line of attack would be through autocorrelation measurement. Noise would be more random than the video signal, hence the autocorrelation time would be lower. This approach at first appears to be hampered by lack of low cost instrumentation.

CHAPTER THREE

NOISE MEASUREMENT

3.1. Voltmeters

3.1.1. Average, RMS, and Peak.

Most commercial voltmeters respond to the average voltage. They are usually calibrated with sinusoidal voltages to read RMS (sine). Before discussing average and RMS noise voltages, the sine wave will first be described.

For a sine wave, the voltage at any time, t, is

$$v(t) = v_o \sin(\omega t)$$

where v_o is the peak, or maximum, voltage. To find the average over a time T, the following integral is found:

$$\langle v \rangle = \frac{\int_0^T v_o \sin(\omega t)\,dt}{\int_0^T dt}$$

Choosing T to be the first quarter-cycle,

$$\langle v \rangle = \frac{2}{\pi} v_o$$

Similarly, the mean square voltage is found by integrating over the square:

$$\langle v^2 \rangle = \frac{1}{T} \int_0^T v_o^2 \sin^2(\omega t)\,dt$$

Again, choosing T to be the first quarter-cycle,

$$\langle v^2 \rangle = v_o^2 / 2$$

hence, the root mean square voltage is

$$\text{rms(sine)} = \sqrt{\langle v^2 \rangle} = V_o/\sqrt{2}$$

The ratio of these two is sometimes called the form factor,

$$\frac{\text{rms(sine)}}{\text{ave(sine)}} = \frac{\pi}{2\sqrt{2}}$$

These quantities may also be found for noise voltages. The average (rectified) noise voltage is

$$\langle v \rangle = \frac{1}{2} \int_0^\infty x p(x) dx$$

where p(x) is the probability density function. When this integral is evaluated for gaussian noise, the average noise voltage is seen to be

$$\langle v \rangle = \frac{\sqrt{2}\sigma}{\sqrt{\pi}}$$

where σ is the standard deviation. Similarly, the mean square voltage is found,

$$\langle v^2 \rangle = \frac{1}{2} \int_0^\infty x^2 p(x) dx$$

which when solved for gaussian noise is seen to be

$$\langle v^2 \rangle = \sigma^2$$

This gives the form factor as

$$\frac{\text{rms(noise)}}{\text{ave(noise)}} = \sqrt{\frac{\pi}{2}}$$

The statistical quantities average and root mean square are easily found. Other quantities such as the peak may depend on a conventional definition. For regular functions such as the sine wave, it is clearly defined. For statistical functions such as noise, the only so-called peak voltage depends on its particular definition. Usually it is chosen as some voltage that is exceeded only some small fraction of the time. Specifying this quantity then gives the multiple of σ that is defined as the peak voltage. Whenever "peak voltage" of a noise is asked for, the peculiar definition must be given.

Sometimes however, "peak voltage" refers simply to the scale reading of a peak-responding (quasi-peak) or peak-to-peak responding voltmeter. In this case, a calibration curve must be found, usually empirically. Readers interested in peak responding meters are referred to Broderick [1965]. Examples of noise measurements with various meter types may be found in Noise Measurements, REL Report 251(1968) by the present author.

3.1.2. Bandwidths.

Instruments used in noise measurements may be roughly divided into two classes: narrowband and broadband. Narrowband instruments include radio noise meters, and they have bandwidths often much less than the noise. Broadband instruments include VTVM's and they have bandwidths that may or may not exceed the actual noise bandwidth. In either case, a knowledge if the instrument bandwidth is necessary to interpet the scale reading.

The random noise bandwidth is defined as the area under the normalized power response curve, i.e.

$$RN = \frac{1}{V_o^2} \int_0^\infty V^2(f)df$$

where V_o is the maximum voltage response. This gives the bandwidth of a rectangular passband having the same power gain. It is important to realize that the power gain is proportional to the bandwidth but the voltage gain is proportional to the root bandwidth.

A narrowband instrument such as an RIFI meter reads voltage per root bandwidth. A wideband instrument such as a VTVM reads voltage. If the bandwidth of a wideband instrument is less than the noise bandwidth, then a correction factor of

$$\sqrt{\frac{\text{noise bandwidth}}{\text{instrument bandwidth}}}$$

must be multiplied by the instrument voltage reading.

3.2 RIFI Meters.

3.2.1. Response to Impulses.

The input to any instrument may be considered as the sum of many individual impulses just as it may be considered as the sum of sines and cosines. This is just another kind of Fourier series, but one that uses impulses instead of sines. For impulse-type noises, this representation has the advantage of simplicity once the response of the instrument is known.

Impulses have already been described [see Sect. 1.2] and an idea of their impact upon an ideal AM receiver has been discussed. The effect of an impulse on an instrument may be analyzed more generally using Fourier transform techniques.

From the Appendix comes the spectrum and time responses to an impulse train by a gaussian bandpass:

$$F(f) = \frac{2S}{T}\exp[-\frac{\pi(f-f_c)^2}{f_o^2}]\cdot III(\frac{1}{T})$$

$$V(t) = 2Sf_o\sum_{-\infty}^{+\infty}\exp[-\pi f_o^2(t-nT)^2]\cos[2\pi f_c(t-nT)]$$

where f_o is the area under the normalized frequency response curve and $T = 1/f_p$ is the interval between pulses. From these two expressions may be found the readings of different detectors - average, rms, and peak - and the effect of the pulse repitition frequency, f_p, and bandwidth,

f_o, on the readings.

The first case to be considered is that of a high pulse repetition frequency, f_p. Each sine Fourier component may be seperated by tuning the instrument since the lines are far apart. Because each line is sinusoidal with peak value $2Sf_p$, it follows that a meter calibrated on CW to read rms sine will read $\sqrt{2}Sf_p$.

The second case is of a low f_p. Here the time view is taken since each pulse is seperate. The peak envelope of the pulse is

$$2Sf_o \exp[-\pi f_o^2 t^2]$$

and each pulse is $1/f_p$ apart. If the gaussian filter is followed by a peak detector, then the reading will be

$$\sqrt{2}Sf_o$$

If the detector is average responding, then the results from Sect. 3.1.1. are used to give a reading of

$$\sqrt{2}Sf_p$$

The rms value will now be derived. By definition, the mean square is

$$\begin{aligned} ms &= \frac{1}{2T} \int_{-\infty}^{+\infty} [2Sf_o \exp(-\pi f_o^2 t^2)]^2 dt \\ &= 4f_p f_o{}^2 S^2 \int_0^{\infty} \exp(-2\pi f_o^2 t^2) dt \\ &= \frac{4f_p f_o^2 S^2}{\sqrt{2\pi}\, f_o} \int_0^{\infty} \exp(-u^2) du \end{aligned}$$

$$= \frac{4f_p f_o S^2}{\sqrt{2\pi}}\left(\frac{\sqrt{\pi}}{2}\right)$$

Hence, the rms value is

$$\text{rms} = S\sqrt{\sqrt{2} f_o f_p}$$

3.2.2. Response to Random Noise.

The results of Section 2.1.1. may be used to determine the instrument readings of random noise. Again, the instrument is assumed to have a gaussian response and an average-responding meter calibrated in rmssine. Both the quadratic and the linear detection cases have been solved.

From the IF amplifier having a gaussian response, the mean square voltage is $2\psi_o$ where ψ_o is the value of the autocorrelation function at zero time. This value may be shown to be

$$2\psi_o = W_o(RN)$$

where W_o is the spectral noise density and RN is the random noise bandwidth. The noise may be applied to either a quadratic ($I=aV^2$) or a linear ($I=aV$) detector.

If the detector is quadratic, then the mean square envelope gives the output:

$$\langle R^2 \rangle = \int_0^\infty R^2 p(R)\,dR = 2\psi_o$$

while if the detector is linear, the mean envelope gives the output:

$$\langle R \rangle = \int_0^\infty Rp(R)dR = \sqrt{\frac{\pi\psi_o}{2}}$$

where p(R) is the Rayleigh probability density function. In using these results it should be borne in mind that a meter responding to average, but calibrated in rms sine, will read

$$\frac{\pi}{2\sqrt{2}}$$

of the noise voltage. Hence, for such a meter the linear detector gives a reading of

$$\frac{\pi}{2\sqrt{2}}\sqrt{\frac{\pi\psi_o}{2}} = \frac{\pi}{2\sqrt{2}}\frac{\sqrt{\pi}}{2}\sqrt{W_o(RN)}$$

$$= \frac{\pi^{\frac{3}{2}}}{4\sqrt{2}}\sqrt{W_o(RN)}$$

$$= 0.984\sqrt{W_o(RN)}$$

It has been common practice to simplify the noise voltage readings by ignoring the 1.6% correction. The actual value was calculated here to giva a complete view of what is actually being measured - the noise average voltage on an RMS sine scale.

The results of Sect. 3.2.1. and the above are summarized in the table of Fig. 11.

input	peak	linear	quadratic
CW $P\sin(2\pi f_c t)$	$\frac{1}{\sqrt{2}}P$	$\frac{P}{\sqrt{2}}$	$\frac{P}{\sqrt{2}}$
impulse train S, f_p <<IMP	$\sqrt{2}S(IMP)$	$\sqrt{2}S(PRF)$	$\sqrt{2}S^2(IMP)(PRF)$
random noise $F(f)$	undefined	$\frac{\pi^{3/2}}{4\sqrt{2}} F(f)\sqrt{RN}$	$F^2(f)\cdot(RN)$

FIGURE 11. RIFI meter readings for various detectors and inputs.

	3DB	RN	6DB	IMP
3DB	1	$\frac{1}{2}\sqrt{\frac{\pi}{\ln 2}} = 1.07$	$\sqrt{2} = 1.41$	$\sqrt{\frac{\pi}{2\ln 2}} = 1.51$
RN	$2\sqrt{\frac{\ln 2}{\pi}} = 0.93$	1	$2\sqrt{\frac{2\ln 2}{\pi}} = 1.32$	$\sqrt{2} = 1.414$
6DB	$\frac{1}{\sqrt{2}} = 0.707$	$\frac{1}{2}\sqrt{\frac{\pi}{2\ln 2}} = 0.72$	1	$\frac{1}{2}\sqrt{\frac{\pi}{\ln 2}} = 1.07$
IMP	$\sqrt{\frac{2\ln 2}{\pi}} = 0.66$	$\frac{1}{\sqrt{2}} = 0.707$	$2\sqrt{\frac{\ln 2}{\pi}} = 0.93$	1

FIGURE 12. Ratios of bandwidths for a gaussian response bandpass curve.

3.3. Bandwidth and Uncertianty.

3.3.1. Bandwidth.

Various kinds of bandwidths have been defined and assumed. These definitions will now be formally made.

The first bandwidth to be considered is f_o, the well-normalized voltage bandwidth. The gaussian band-pass was written

$$\exp[-\frac{\pi(f-f_c)^2}{f_o^2}]$$

where f_c is the centre frequency and f_o is the well--normalized bandwidth. This bandwidth is also referred to as the impulsive bandwidth, IMP. Hence, the author defines the impulsive bandwidth as

$$IMP = \frac{1}{G(f_c)} \int_0^\infty G(f)df$$

which is the well-normalized bandwidth. For example, in the gaussian case

$$IMP = \int_0^\infty \exp[-\frac{\pi(f-f_c)^2}{f_o^2}] = f_o$$

The second bandwidth of interest is the random noise bandwidth, RN. This is defined as the well-normalized power bandwidth:

$$RN = \frac{1}{G^2(f_c)} \int_0^\infty G^2(f)df$$

which, in the case of the gaussian bandpass, is

$$RN = \int_0^\infty \exp[-2\pi\frac{(f-f_c)^2}{f_o^2}]$$

Two other bandwidths are now treated, because of their ease of measurement: the 3 dB and the 6 dB bandwidths. By definition, the 3 dB bandwidth is the frequency difference between two points where the voltage gain is $1/\sqrt{2}$ of the centre frequency. For the gaussian bandpass, this is

$$\frac{1}{\sqrt{2}} = e^{-\pi x^2}$$

$$x = \sqrt{\frac{\ln 2}{2\pi}} = \frac{(3DB)}{(IMP)}$$

Similarly, the 6 dB bandwidth is defined where the voltage gain is 1/2 of the centre. Again, solving for the gaussian case,

$$\frac{1}{2} = e^{-\pi x^2}$$

$$x = \sqrt{\frac{\ln 2}{\pi}} = \frac{(6DB)}{(IMP)}$$

where, x has been taken as the bandwidth with respect to lMP. The notation (3DB) and (6DB) are used for the 3 dB and 6 dB bandwidths, respectively.

The results of this section summarizing the ratios of various bandwidths are given in FIG. 12.

3.3.2. Uncertainty Principle.

Up to now, concepts of time and frequency domains have been used almost interchangeably to find expressions of tuner outputs. However, it should be noted that there is a basic uncertainty principle underlying any discussion of filtering. Here, the principle states that the impulses being observed cannot be simultaneously distinguished and tuned by the same instrument.

The uncertainty principle means that an impulse train cannot be resolved into individual pulses and individual frequency line components at the same time. If the pulses are well-seperated, the spectrum appears continuous; while if the bandwidth is narrowed so as to resolve the frequency components, the pulses are no longer distinguishable.

To see how the principle operates, consider a "classical" model of an impulse train. This can be represented in a three-dimensional graph of amplitude vs. time and frequency (see Fig. 13). On such a graph, a CW sine wave appears as Fig 13 (a), while a single impulse appears as Fig 13 (b). The impulse train can look like a set of sine waves from a frequency view, or as a set of single impulses from a time view (Fig. 13 (c)). However, in reality, both views cannot be taken together.

As an illustration, consider the gaussian response to an impulse. Call the impulse bandwidth, f_o = IMP = BW, the bandwidth. The width of the pulse in the time domain at the same level would be

$$\frac{1}{f_o} = PW$$

which is the pulsewidth. Then the product

$$(BW)(PW) = 1$$

With this restriction, consider the display of pulses, each being seperated. To do this, the pulsewidth must be less than the period:

$$PW < T, \text{ or } PW < \frac{1}{PRF}$$

But, if (BW)(PW) = 1, then

$$PW = {}^{1}/BW$$

so that PW < T becomes

$$PW < \frac{1}{PRF}$$
$$\frac{1}{BW} < \frac{1}{PRF}$$

which may be re-written as

$$PRF < BW$$

But, if the PRF is less than the BW, the frequency components cannot be seperated.

Similarly, starting with the requirement for frequency resolution,

$$PRF > BW$$

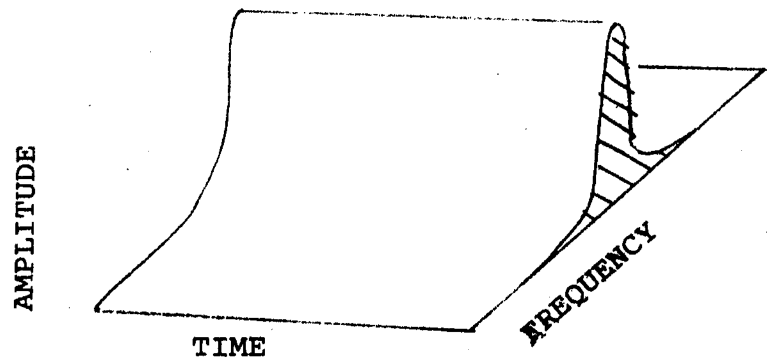
the result is obtained that

$$PW > T$$

which means the pulses cannot be seperated.

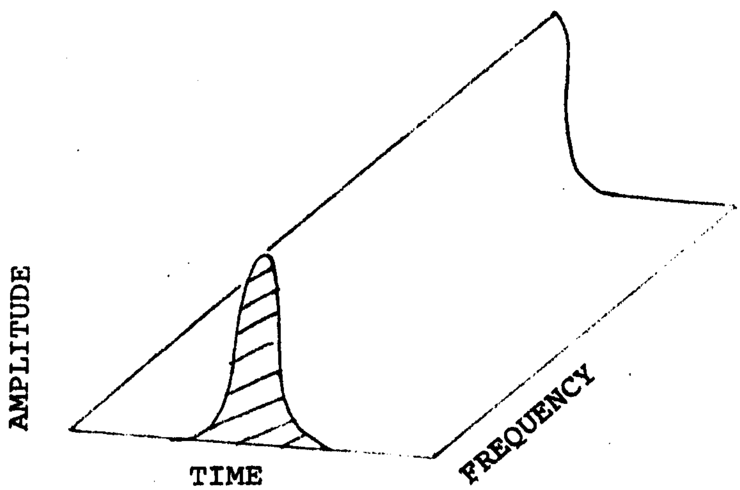
The illustration just given does not prove the uncertainty principle generally. For a general proof, the mathematical reader is referred to Bendat [1958], pp 53-5, where the case of the Fourier-transform pair is treated.

The radio noise worker should be alert to uncertainty, for example sampling for brief times will require a corresponding wide bandwidth. Such uncertainty, it should be noted, is a characteristic of the process of observation, but is independent of any particular observer (ie. a FIM). Another more fundamental aspect of uncertainty is the effect of relaxation time as receiver noise. If very short times are used, the bandwidth is wide, hence the receiver noise increases. Reducing receiver noise by using long relaxation times is often done (radio telescopes, quasi-peak RIFI meters, etc.) but at the expense of being able to follow a varying signal. Here, it can be shown that the signal strength and its (time) rate of change is constant. [Furth, 1950].



sine wave

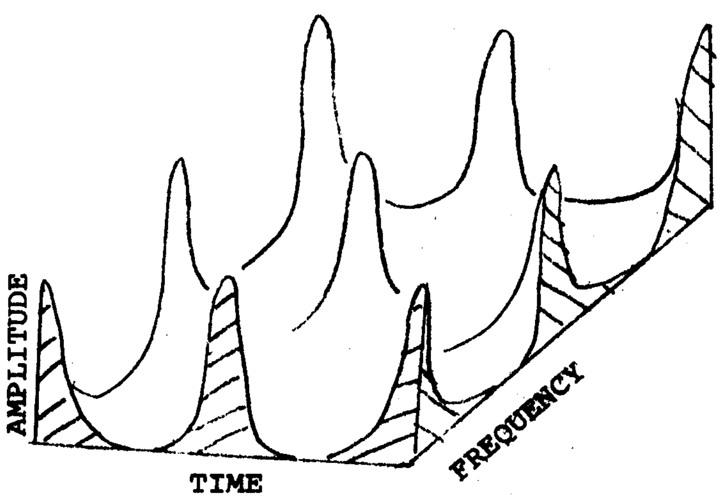
FIG.13(a)



single

impulse

FIG.13(b)



impulse

train

FIG.13(c)

FIGURE 13. Time and frequency views of a tuner. The 'classical' view.

APPENDIX A

NOISE FIGURE

The noise figure, F, of a network has been defined as the ratio of the available signal-to-noise ratio at the signal generator terminals to the available signal-to-noise ratio at the output terminals. F is also called the excess noise ratio. The modern definition gives the temperature-equivalent formula which is derived below:

$$F = \frac{Si/Ni}{So/No}$$

$$= \frac{Si/kTB}{So/No}$$

$$= \frac{SiNo}{SokTB}$$

Recall that power gain G = So/Si, then

$$F = \frac{No}{GkTB}$$

Total noise output is

$$No = FGkTB$$

The noise output due to the network only is

$$N = (F-1)GkTB$$

The temperature definition of noise figure depends upon the conventional temperature of the generator impedence [Friis,1944]. If T = 290K, then kT = 4×10^{-21}watts/hertz, and

$$F = \frac{No}{GB(4\times10^{-21}\,w/Hz)}$$

Note that definitions depend on a linear gain, G. Since gains of receivers arenot linear due to AGC action, care must be taken in the use and specification of noise figure.

APPENDIX B.

INSTRUMENT RESPONSE.

Let $G_1(\tau)$ = amplitude of input at time τ

$G_2(t-\tau)$ = amplitude of output at time $t-\tau$ after being excited by a unit impulse

then the instrument response to $G_1(t)$ is given by

$$G(t) = \int_{-\infty}^{+\infty} G_1(\tau)G_2(t-\tau)d\tau$$

If F_1 and F_2 are the Fourier transforms of G_1 and G_2 respectively, then

$$F_1(f) = \int_{-\infty}^{+\infty} G_1(t)\exp(-2\pi ift)dt$$

$$F_2(t) = \int_{-\infty}^{+\infty} G_2(t)\exp(-2\pi ift)dt$$

and the Fourier transform of $G(t)$ is $F_1(f)F_2(f)$:

$$F_1(f)F_2(f) = \int_{-\infty}^{+\infty} G(t)\exp(-2\pi ift)dt$$

by the convolution theorem. Hence, the inverse transform is

$$G(t) = \int_{-\infty}^{+\infty} F_1(f)F_2(f)\exp(-2\pi ift)df$$

where $F_1(f)$ is the spectrum of the input,

$F_2(f)$ is the spectrum of the instrument

$G(t)$ is the output function of the system.

1. Square spectrum-Random noise.

$$W(f) = \begin{cases} W_o, & f_1 < f < f_2 \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi(t) = \int_0^{\infty} W(f)\cos(2\pi ft)\,df$$

$$= \int_{f_2}^{f_1} W_o \cos(2\pi ft)\,df$$

$$= \frac{W_o}{2\pi t}(\sin 2\pi f_2 t - \sin 2\pi f_1 t)$$

$$= \frac{W_o}{\pi t}(\cos\pi[f_2+f_1]t \cdot \sin\pi[f_2-f_1]t)$$

$$= W_o(f_2-f_1)\cos\pi(f_2+f_1)t \cdot \mathrm{sinc}(f_2-f_1)t$$

$$= W_o f_o \cos(2\pi f_c t)\,\mathrm{sinc}(f_o t)$$

2. Gaussian spectrum-Random noise.

$$W(f) = W_o \exp[- \frac{\pi(f-f_c)^2}{f_o^2}$$

$$\psi(t) = W_o\cos(2\pi f_c t) \int_{-\infty}^{+\infty} \exp[- \frac{\pi f^2}{f_o^2}]\cos(2\pi ft)\,df$$

$$= W_o f_o \cos(2\pi f_c t]\exp[-\pi f_o^2 t^2]$$

3. Square spectrum-Impulse input.

$$F_1(f) = 2S$$

$$F_2(f) = F_o, \ |f-f_c| < f_o/2$$
$$= 0, \text{ otherwise.}$$

$$G(t) = \int_{-\infty}^{+\infty} F_1 F_2 \exp(2\pi i f t)\,df$$

$$= \int_{f_c+f_o/2}^{f_c+f_o/2} 2SF_o \exp(2\pi i f t)\,df$$

$$= 2SF_o \int_{-f_o/2}^{f_o/2} \exp(2\pi i f t)\exp(-2\pi i f_c t)\,df$$

$$= 2SF_o \frac{[\exp(\pi i f_o t) - \exp(-\pi i f_o t)]}{2\pi i t} \exp(-2\pi i f_c t)$$

$$= 2SF_o f_o \frac{\sin(\pi f_o t)}{\pi f_o t} \exp(-2\pi i f_c t)$$

$$= 2SF_o f_o \operatorname{sinc}(f_o t)\cos(2\pi f_c t)$$

For the impulse train, $F_1(f) = \frac{S}{T}III(\frac{1}{T})$ follows from $G_1(t) = SIII(T)$ where $III(x)$ means the unit function is repeated each x throughout the domain. Using this replicating property of $III(T)$, the output function for the impulse train is the convolution

$$G(t)*III(T) = \sum_{-\infty}^{+\infty} G(t-nT)$$

$$= 2SF_o f_o \sum_{-\infty}^{\infty} \operatorname{sinc}[f_o(t-nT)]\cos[2\pi f_c(t-nT)]$$

The spectrum is

$$F_1F_2 = \frac{SF_o}{T}III(\frac{1}{T}),\ |f-f_c| < f_o/2$$

$$= 0,\ \text{otherwise}$$

$$= \frac{SF_o}{T} \sum_{-f_o/2}^{+f_o/2} \delta(f-nf_p)$$

where $f_p = 1/T$ is the pulse repitition frequency, and δ is the Dirac delta function.

4. Gaussian spectrum-Impulse input.

$$F_1(f) = 2S$$

$$F_2(f) = F_o\exp[-\frac{\pi(f-f_c)^2}{f_o^2}]$$

$$G(t) = \int_{-\infty}^{+\infty} F_1F_2\exp(2\pi ift)dt$$

$$= \int_{-\infty}^{+\infty} 2SF_o\exp[-\frac{\pi(f-f_c)^2}{f_o^2}]\exp(2\pi ift)df$$

$$= 2SF_o\exp(-2\pi if_ct)\int_{-\infty}^{+\infty}\exp(\frac{-\pi f^2}{f_o^2})\exp(2\pi ift)dt$$

$$= 2SF_of_o\exp(2\pi if_ct)\exp(-\pi f_o^2t^2)$$

$$= 2SF_of_o\exp(-\pi f_o^2t^2)\cos(2\pi f_ct)$$

For the impulse train, the output function is found as for the square response case, by the convolution

$$G(t)*III(T)$$

This may be evaluated directly without integration by using the replicating property of III(T):

$$G(t)*III(T) = \Sigma_{-\infty}^{+\infty} G(t-nT)$$

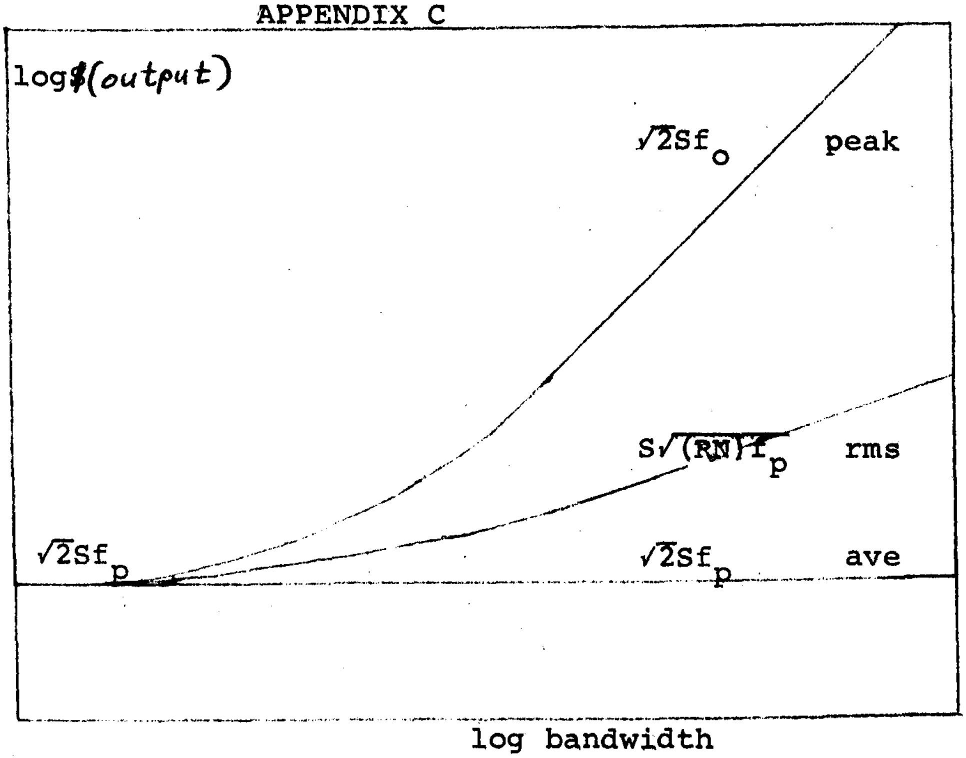
$$= 2SF_o f_o \Sigma_{-\infty}^{+\infty} \exp(\pi f_o^2 [t-nT]^2) \cos(2\pi f_c [t-nT])$$

It may be noted that the spectrum is given by

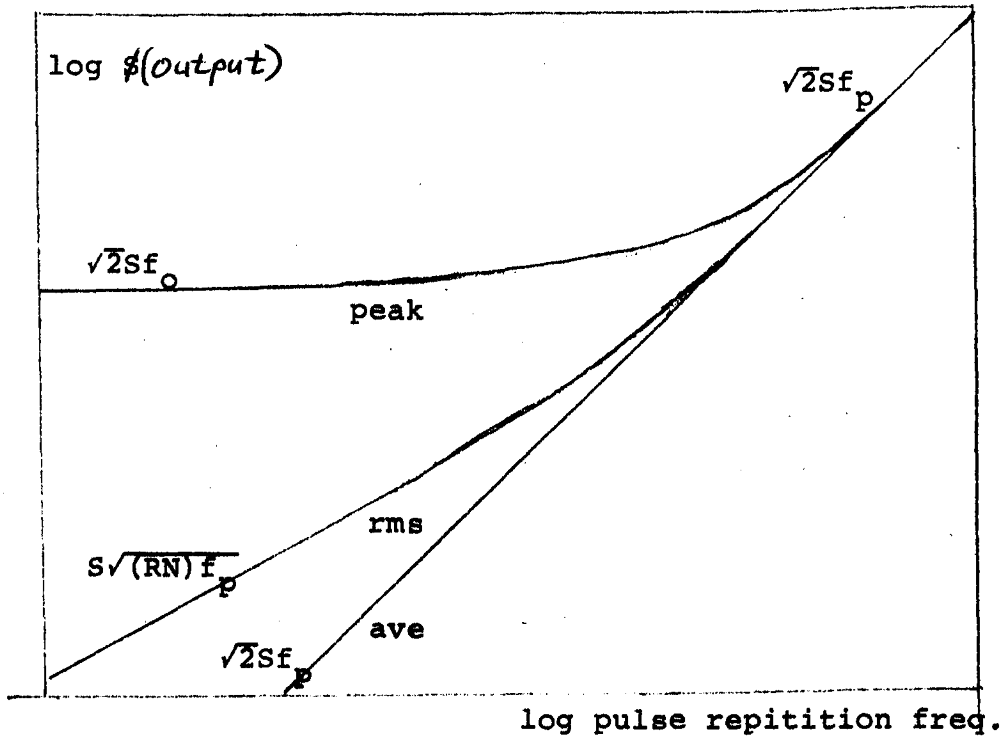
$$F_1 F_2 = \frac{2SF_o}{T} \exp[- \frac{\pi(f-f_c)^2}{f_o^2}] III(\frac{1}{T})$$

$$= 2SF_o f_p \Sigma_{-\infty}^{+\infty} \exp[- \frac{\pi(f-f_c)^2}{f_o^2}] \delta(f-nf_p)$$

APPENDIX C



ACTION OF DETECTORS AS A FUNCTION OF THE BANDWIDTH.



ACTION OF DETECTORS AS A FUNCTION OF THE PRF.

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