SEASONAL ANALYSIS:

FROM CONCEPTS TO MODELS

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PRELIMINARY DRAFT

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1. Introduction

The aim of this paper is to provide an analytical framework for the analysis of seasonal time series. From this framework, a general class of seasonal stochastic processes is generated, which encompasses the majority of the seasonal processes commonly used today, and the model structure of such processes is analysed.

The seasonal analysis of stochastic processes was introduced in the early sixties to tackle two separate problems: the forecasting of a time series characterized by seasonality and the seasonal adjustment of economic time series. Those problems were studied independently of one another and distinct approaches were adopted. The modelling of a seasonal time series for forecasting purposes has been dominated by the introduction of the seasonal multiplicative model by Box, Jenkins and Bacon (1967). The seasonal adjustment analysis has mostly started from the unobservable components model in which the process is seen as the sum of three unobservable components, the trend-cycle, the seasonal, and the irregular or noise component. In this context two paths have been followed: Nerlove (1964) and others have proposed spectral criteria a seasonally adjusted series ought to follow, leading to Granger's (1976) definition of the seasonal, while Hannan (1964), Hannan, Terrell, and Tuckwell (1970), Couts, Grether, and Nerlove (1966), and Grether and Nerlove (1970) have proposed parametric stochastic models for the unobservable components.

In spite of extensive development, a great degree of confusion is still found even at the most elementary level of what constitutes a seasonal. Hence, if one starts from one of the simplest seasonal models, the model $(1-\Phi^s B^s) z_t = a_t$, the approaches proposed by Grether and Nerlove, Ashworth and Tunicliffe Wilson (1972), Brewer (1976), Hannan, Terrell and Tuckwell, and Box, Hillmer and Tiao (1976),... can be shown to all imply different decompositions[1]. Alternative approaches to the conceptualization and definition of seasonality in terms of spectral properties or in terms of a causal model are shown to be inadequate in providing a general definition of the unobservable seasonal component.

The basic problem with seasonality has been seen as "the close but not perfect approach to regularity",and this has led to the use of vague concepts such as "slowly" moving seasonality,... This view is somewhat misleading since slower changes in seasonality do not imply that the series is somehow more seasonal. Generally, it would only imply that the presence of seasonality in a given process can be detected more easily. In fact the basis for the periodicity of a seasonal series and the only periodicity which is exactly "regular" is the periodicity which is embodied in the concept of time and which reflects the relative position of the earth and the sun. The realization of this problem has led Guilbaud (1968) and Fontenay (1973) to present an alternative conceptualization of a seasonal time series. These ideas are developped and the model of a bi-dimensional time is proposed in which this periodic dimension is distinguished from the chronological dimension. Those dimensions are directly related to the unobservable components since the trend-cycle is a representation of the process which will depend solely upon the chronological time while the seasonal is a residual associated with the periodic dimension, the irregular being that unobservable component which is independent of time. Yet this time model does not imply a prior definition of the unobserved components. In bidimensional time, stationarity is immediately generalized in terms of seasonal stationarity, i.e. stationarity in the seasonal dimension, and chronological stationarity, i.e. stationarity

(1) Throughout this paper, in as much as possible, the notation introduced by Box and Jenkins (1970) will be used.

in the chronological dimension. The process being exactly periodic in its seasonal dimension, it can be represented by a Fourier expansion over the seasonal dimension. Such a representation enables one to disassociate the periodic aspect of the process which is associated with the frequencies from its chronological aspect which is associated with the amplitudes. Such amplitudes, which depend solely upon the chronological time, can be conceived as stochastic processes defined in the chronological time. If those are specified to be simple random walks, the model implicit to Hannan, Terrell, and Tuckwell is obtained. In general the aggregation of these Fourier sums, given seasonal stationarity, is shown to yield ARMA seasonal processes.

It is shown that the assumptions needed to derive Box, Jenkins and Bacon's multiplicative model, i.e. a model which is multiplicative both in its AR and in its MA polynomials are very restrictive - thus providing an explanation for Plosser (1977)'s observation to this effect. In other words, the approaches used for modelling time series and for seasonal adjustment are shown to be particular specifications all related to the same framework. The general seasonal model will yield a general form of Box, Jenkins, and Bacon's trigonometric model. In particular, in the eventuality that the full complement of seasonal frequencies is needed and that at least one root is common to the amplitude processes corresponding to the zero frequency and to each of the seasonal frequencies, a model multiplicative in its AR polynomial is obtained. This result does not depend upon the other roots of the AR polynomials, the variance of the various generating white noise processes, or even their respective MA polynomials.

It is shown that not all ARMA multiplicative seasonal models can thus be derived form the general seasonal model. However, even though processes which are clearly seasonal, and which are not thus derivable, can be obtained, such as $(1-\Phi^s B^s)z_t = (1+B^s)a_t$, $0 < \Phi \leq 1$, general criterion is found to establish whether a given model which cannot be obtained from the general class of models is properly a seasonal model.

In chapter 2, the literature is reviewed. In chapter 3, the bidimensional model of time is introduced and the general seasonal model is developped. In chapter 4, the general seasonal model is shown to yield in general a trigonometric seasonal model, and under certain assumptions a multiplicative model. It is shown, also, that not all multiplicative seasonal models can thus be obtained.

2. A CRITICAL REVIEW OF THE LITERATURE

2.1 Introduction

Two alternative approaches to the introduction of seasonality in a stochastic process are reviewed. The first, introduced by Bacon (1965), consists in an approach based on the premise that observations made in the same month years away should help explain today's observation. Even though it has the merit of being remarkably simple, in that a seasonal component needs not be defined - the process is modelled as a whole - , the class of models thus obtained, is based on a heuristic argument and has the added disadvantage that the itnroduction of even a small error will yield a model which does not belong to this class.

The second type of approach is based on the work which has been done toward developping a seasonal adjustment method. It implies both that the composition of a seasonal series as a sum of unobserved components and that the definition of the seasonal unobserved component are accepted by most researchers. It is shown that this is not the case with the second step: even the most inocuous seasonal AR model, $(1-\Phi^s B^s) z_t = a_t$, $0<\Phi\leq 1$, will lead to alternative decompositions, depending upon the analyst.

Alternative approaches to the definition of seasonality, by means of a spectral criterion and causal seasonal variables, are then reviewed and rejected.

From the existing literature one can only choose between the Box, Jenkins and Bacon class of models, which is shown to be inadequate, and classes of models based on the arbitrary acceptance of one of the various definitions of a seasonal component proposed in the literature.

2. The Box, Jenkins, and Bacon (BJB) Multiplicative Seasonal Model

The development of what is commonly known under the name of multiplicative seasonal ARIMA model can be traced back to Bacon (1965), Box, Jenkins, and Bacon (1967) and Box and Jenkins (1970). In the development of this model, two approaches were investigated.

The first approach considered to study the modelling of a seasonal process is based on a desired property of the eventual forecast function of any seasonal process; such an eventual forecast function should trace out the seasonal pattern of the series. In fact, Bacon suggests that

> "... its solution was a mixture of sines and cosines possibly mixed with polynomial terms to allow for changes in level and changes in seasonal pattern". (p.)

The first approach has been illustrated with various examples, both to show its eventual applicability and its supposed limitation. Its supposed limitation is that

> "... it is not true that periodic behavior is necessarily represented <u>economically</u> by mixtures of sines and cosines. Many sine-cosine components would, for example, be needed to represent sales data affected by Christmas..." (Box and Jenkins, 1970, p. 302).

Even though it has been revived recently (see, for instance, Abraham and Box, 1975), this approach has received little attention on the grounds that it is not parsimonious.

The second approach to be proposed starts from the Buys-Ballot table in which columns are used to denote months while rows denote years. It is hypothesized that the process at time t, i.e. in year r and month j, should not only be dependent upon past observations within the same row (same year), as for a non-seasonal process, but that it should simultaneously depend upon the observations within the same column, i.e. made in the same month, years away. A stepwise method was then proposed such that, considering any column, the following relationship was hypothesized:

$$\Phi(B^s)\, z_t = \Theta(B^s)\, e_t$$

Evidently e_t was not assumed to be white noise; rather it was assumed that $\Phi(B^s)\Theta^{-1}(B^s)$ was such as to filter all seasonal effects out of the original process. To that extent, it was further assumed that e_t could be modelled as any ordinary ARIMA process i.e.

$$\phi(B)\, e_t = \theta(B)\, a_t$$

where a_t is a white noise.

Combining the two models, the general multiplicative seasonal model is obtained:

$$\phi(B)\,\Phi(B^s)\, z_t = \theta(B)\,\Theta(B^s)\, a_t$$

In practice, however, most attention has been given to the particular multiplicative seasonal model (or Airlines model), also developped by Box, Jenkins, and Bacon, i.e. the (0, 1, 1) x (0, 1, 1) model. This model is clearly related to the first approach considered here since its eventual forecast function can alternatively be written as:

$$\hat{z}_t(\ell) = \beta_{0,m}^{(t)} + \beta_1^{(t)} k$$

or

$$\hat{z}_t(k) = b_0^{(t)} + \sum_{\ell=1}^{s/2} \left\{ b_{1,\ell}^{(t)} \cos \lambda_\ell k + b_{2,\ell}^{(t)} \sin \lambda_\ell k \right\} + b_1^{(t)} k$$

where $\lambda_\ell = 2\pi\ell/s$ (Box and Jenkins, 1970, p. 303 and p. 310).

The latter result follows from the decomposition of the polynomial $(1 - B^s)$ in terms of the roots of unity, i.e. from

$$(1-B^s) = (1-B)(1-\sqrt{3}B+B^2)(1-B+B^2)(1+B^2)(1+B+B^2)(1+\sqrt{3}B+B^2)(1+B)$$

It follows that the particular multiplicative seasonal model can be represented as a periodic function with adaptive coefficients provided the full complement of sines and cosines is used.

The argument that the process in time t should depend both upon observations month away (same row),and upon observations years away (same column) does not appear sufficient at,first sight, to justify the multiplicative model above a model such as

$$z_t = \phi z_{t-1} + \Phi z_{t-s} + a_t - \theta a_{t-1} - \Theta a_{t-s}$$

or the one proposed by Sims (1976). Both Sims' model and the above model however would have eventual forecasting functions which do not generally meet the desired criterion. Furthermore they will be shown not to be generally related to simple notions of seasonality.

2.3 Couts, Grether, and Nerlove's (C.G.N) Approach

A different approach to the development of seasonal time series models was pionneered by Couts, Grether, and Nerlove (1966) and expanded by Grether (1966), Nerlove (1967), Grether (1968), Grether and Nerlove (1970) and Pagan (1973). Starting from the classical decomposition of the seasonal time series ~~in~~ into unobserved (and unobservable) components, the trend-cycle (C_t), the seasonal, (S_t) and the irregular (I_t), such that, if z_t is the observed series,

$$z_t = C_t + S_t + I_t$$

stochastic models are proposed for each of the unobserved components. The specification of those models evolved, and, in Grether and Nerlove (1970), those models are given in their general forms as:

$$\phi(B)\, C_t = \theta_c(B)\, a_{c,t}$$

$$\Phi(B^s)\, S_t = \Theta_s(B)\, a_{s,t}$$

$$I_t = a_{I,t}$$

where $\phi(B)$, $\theta_c(B)$, and $\Theta(B)$ are assumed to be low order polynomials in B while $\Phi(B^S)$ is assumed to be a low order polynomial in B^S and where $a_{C,t}$, $a_{S,t}$, and $a_{I,t}$ are assumed to be independent white noises with variances σ^2_C, σ^2_S, and σ^2_I.

The proposed justification for S_t's model evolved over time, hence Couts, Grether, and Nerlove, having specified a non-stationary process for S_t:

$$(1-B^s)^2\, S_t = a_{s,t}$$

noted that: "the seasonal component S_t is trending as well; it follows a model much like the trend-cycle in the original Theil-Wage model if 's' is taken to be the number of observation periods in a year" (p. 11).

Grether (1968) shifted to a stationary seasonal process. In a first step, considering the simplest model:

$$(1-\Phi^s B^s)\, S_t = a_{s,t} \qquad 0 < \Phi < 1$$

he introduced the justification that S_t, to be a seasonal unobserved component, should peak at each of the seasonal frequencies. In other words, he shifted to a model consistent with Nerlove (1964)'s definition of a seasonal.

In fact, it was on the basis of this general justification that a non-seasonal moving average, i.e. a moving average characterized by a low order polynomial in B rather than in B^S, was proposed. Noting that, in the simplest model given above, all the peaks, in the spectrum, are of equal height, the following generalization was proposed:

$$(1-\Phi^{s}B^{s})\,s_t = (1-\Theta B)\,a_{s,t} \qquad \Theta < \Phi^{s}$$

Such a model does generate a seasonal the spectrum of which will have progressively smaller peaks, as one moves toward higher seasonal frequencies. Unfortunately, he overlooked the fact that, through this transformation, the spectrum will not peak anymore at the seasonal frequencies.

The intent of those authors was to propose a method to investigate seasonal adjustment and not to model stochastic seasonal processes. Even though both objectives are closely related, they are not identical. In fact, the implied model for z_t was not formally derived. Nevertheless all the results necessary to its derivation were provided; having specified

$$(1-\varphi_1 B-\varphi_2 B^2)\,c_t = a_{c,t}$$

the autocovariance generating function (AGF) of z_t, K_z (B), is obtained, on the assumption that the generating white noises are normally distributed, and independent, as the sum of the unobserved components'.AGF, so that

$$\begin{aligned}
K_z(B) = {} & (1-\varphi_1 B-\varphi_2 B^2)^{-1}(1-\varphi_1 F-\varphi_2 F^2)^{-1}\sigma_c^2 \\
& + (1-\Phi^{s}B^{s})^{-1}(1-\Theta B)(1-\Theta F)(1-\Phi^{s}F^{s})^{-1}\sigma_s^2 + \sigma_I^2 \\
= {} & \left\{(1-\varphi_1 B-\varphi_2 B^2)(1-\Phi^{s}B^{s})(1-\Phi^{s}F^{s})(1-\varphi_1 F-\varphi_2 F^2)\right\}^{-1} \\
& \times\Big\{(1-\Phi^{s}B^{s})(1-\Phi^{s}F^{s})\,\sigma_c^2 \\
& + (1-\varphi_1 B-\varphi_2 B^2)(1-\Theta B)(1-\Theta F)(1-\varphi_1 F-\varphi_2 F^2)\,\sigma_s^2 \\
& + (1-\varphi_1 B-\varphi_2 B^2)(1-\Phi^{s}B^{s})(1-\Phi^{s}F^{s})(1-\varphi_1 F-\varphi_2 F^2)\,\sigma_I^2\Big\}
\end{aligned}$$

Each of the three elements of the sum, within the last bracket on the right handside of the equation, can be viewed as the AGF of moving average processes, and, once they are specified on the unit circle, ($B = e^{-i\lambda}$) they are non-negative, and their sum will be non-negative at all frequencies.

It follows from Wold (1954) that the last bracket, on the left handside, can be rewritten in the form $\Theta_z(B)\Theta_z(F)\sigma_a^2$ where σ_a^2 is such that $\Theta_{z,0} = 1$ in $\Theta_z(B)$, and where $\Theta_z(B)$ will be (s-2)th degree polynomial. Then

$$K_z(B) = \{(1-\phi_1 B-\phi_2 B^2)(1-\Phi^s B^s)(1-\Phi^s F^s)(1-\phi_1 F-\phi_2 F^2)\}^{-1}$$
$$\times\ \Theta_z(B)\,\Theta_z(F)\,\sigma_a^2$$

The model implicit to Couts, Grether, and Nerlove's approach has thus been derived as

$$(1-\phi_1 B-\phi_2 B^2)(1-\Phi^s B^s)\,z_t = \Theta_z(B)\,a_t$$

The general model will be

$$\phi(B)\,\Phi(B^s)\,z_t = \Theta_z(B)\,a_t$$

It follows that a multiplicative seasonal class of models was implicit to Couts, Grether, and Nerlove's approach, and that this class of models, which will be called the C.G.N. class, differs from the B.J.B. class of models in its being multiplicative only with respect to the autoregressive portion of the model.

Finally, it can also be shown, not only that the C.G.N. model is more general than the B.J.B. model but also that the B.J.B. model is a most unlikely model. Assume that on the basis of an observed series, z_t, a multiplicative B.J.B. seasonal model is hypothesized, but that in fact, the real process z_t^* has been observed with an independent error, e_t, which will be assumed to be white noise, in such a way that

$$z_t = z_t^* + e_t$$

then, since z_t^* and e_t are independent,

$$K_z(B) = K_z^*(B) + K_e(B)$$

Had a C.G.N. multiplicative seasonal model originally been sepcified, then z_t^* would also be described by C.G.N. model, and possibly a

B.J.B. model, even though the latter possibility is highly unlikely, since

$$K_{\gamma}^{*}(B) = \{\phi(B)\Phi(B^{s})\Phi(F^{s})\phi(F)\}^{-1}\{\theta_{\gamma}^{*}(B)\,\theta_{\gamma}^{*}(F)\,\sigma_{a^*}^{2}\}$$

where

$$\theta_{\gamma}^{*}(B)\theta_{\gamma}^{*}(F)\sigma_{a^*}^{2} = \theta_{\gamma}(B)\theta_{\gamma}(F)\sigma_{a}^{2} - \phi(B)\Phi(B^{s})\Phi(F^{s})\phi(F)\sigma_{e}^{2}$$

It is clear that it is unlikely that $\theta_{\gamma}^{*}(B)$ can be written as the product of two low order, but non-null, polynomials, one in B and the other in B^{s}.

However, had a B.J.B. multiplicative seasonal model been specified for a_t, then

$$\theta_{\gamma}^{*}(B)\theta_{\gamma}^{*}(F)\sigma_{a^*}^{2} = \theta(B)\Theta(B^{s})\Theta(F^{s})\theta(F)\sigma_{a}^{2} - \phi(B)\Phi(B^{s})\Phi(F^{s})\phi(F)\sigma_{e}^{2}$$

and, again, it would clearly not generally be the case that $\theta_{\gamma}^{*}(B)$ could be written in the desired multiplicative form.

2.4 The Hannan, Terrell, and Tuckwell (H.T.T.) Approach

Hannan (1960, 1963, 1964, 1970), Hannan, Terrell, and Tuckwell (1970), and Pagan (1973) studied the same problem as Couts, Grether, and Nerlove; seasonal adjustment. For this same reason, they also opted for the unobservable components approach, and a seasonal model for the overall process, implicit to their approach, will be shown in this paper to be an ARIMA model multiplicative in its autoregressive part.

The H.T.T. approach is comparable to the CGN approach in another way; the corner stone of the approach is Nerlove (1964)'s seasonal definition in terms of the spectrum. The stable seasonal is associated with a periodic function, the Fourier represensation of which is favoured:

$$S_t = \sum_{\ell=1}^{s/2}\{b_{1,\ell}\cos\lambda_{\ell}t + b_{2,\ell}\sin\lambda_{\ell}t\}$$

where $b_{2,s/2}$ is identically zero.

The amplitudes could just as well be the realization of some random variables, and they find it natural to introduce a further generalization by letting these amplitudes be stochastic processes which vary with time. Denoting those stochastic processes by $S_{t,2\ell}$ and $S_{t,2\ell+1}$, Hannan assumed that they are AR (1) and independent processes, such that

$$(1+\Phi_\ell B)\, S_{t,\ell} = a_{t,\ell} \qquad \ell = 1,2,\ldots,s$$

Observing that even if Φ_ℓ if very close to 1, say .98, given a monthly series, the autocorrelation of the seasonal components, five years apart, would only be .3, which would imply a very unstable seasonal, Hannan further simplifies the model by letting those seasonal components be modelled as random walks, i.e. he sets $\Phi_\ell = 1$. The following model for the seasonal unobservable component is thus assumed:

$$S_t = \sum_{\ell=1}^{s/2} \left\{ S_{t,2\ell} \cos \lambda_\ell t + S_{t,2\ell+1} \sin \lambda_\ell t \right\}$$

where

$$(1-B)\, S_{t,\ell} = a_{t,\ell} \qquad \ell = 1,2,\ldots,s$$

$$\sigma^2_{2\ell} = \sigma^2_{2\ell+1} \qquad \ell = 1,2,\ldots,(s/2)-1$$

The approach adopted by those authors, in their seasonal adjustment procedure, consists in an iterative approach , in which the trend is first filtered out (partial prewhitening), and then the seasonal sub-components corresponding to each seasonal frequency are filtered, one frequency at a time. Three alternatives but related procedures are proposed to remove the trend; it will suffice at present to look only at the implications of the third method, in which the trend-cycle, C_t, is assumed to be of the form:

$$(1-B)^2 C_t = a_{c,t}$$

It will be one of the object of this paper to show that this model, together with the irregular implies that z_t is represented by an ARIMA (0, 1, 2 + s) x (0, 1, 0). It will follow from this result that the HTT approach generates a class of models generally similar to that generated by a CGN type of model.

2.5 The Seasonal Unobserved Component

Only the first of the various approaches reviewed attacks the problem of seasonal modelling without worrying about the seasonal composition of a time series. However it has been shown that the class of models thus entertained was not conceptually a fully appropriate class. To accept a seasonal class of model on analytical grounds, at this stage, either Couts, Grether, and Nerlove's approach or Hannan, Terrell and Tuckwell's one would have to be adopted. Both of these approaches depend crucially on a series of assumption, and in particular they both depend on accepting the process has been "composed" of a sum of three unobservable components C_t, S_t, and I_t, and on accepting the specification given to each of those processes.

The composition assumption, i.e. $z_t = C_t + S_t + I_t$, is quite generally accepted, hence, in as much as the class of models proposed depends upon this assumption, it will still be a very general class. Similarly the component $(C_t + I_t)$ does not raise, for most analysts, too many questions, since it can, without loss of generality, be assumed to be represented, for instance, by a standard ARIMA model. It is with S_t, the seasonal unobservable component, that a serious problem arises. To illustrate the disarray one can observe today in the literature, it is sufficient to study one of the simplest seasonal model:

$$(1 - \Phi^s B^s)\, y_t = a_t$$

and the hypothesis that

$$y_t = c_t + s_t + I_t$$

and to ask the question as what is the seasonal? It has already been shown that Grether and Nerlove's answer is simply that this is a seasonal unobserved component, hence that

$$c_t = I_t = 0$$

Ashworth and Tunnicliffe Wilson (1972) and Brewer (1976), however, have proposed also a seasonal adjustment procedure for ARIMA models. While Ashworth and Tunnicliffe Wilson's approach can only treat the situation in which $\Phi = 1$, Brewer's is not so constrained. In either case, the proposed procedure implies that the irregular alone would be zero, for this model, and that

$$(1 - \Phi B)\, c_t = k_c\, a_t$$

$$\Sigma_{\Phi}(B)\, s_t = k_s\, \Theta_s(B)\, a_t$$

where k_C and k_S are constant, $\Sigma_{\Phi}(B) = 1 + \Phi B + \Phi^2 B^2 + \ldots + \Phi^{s-1} B^{s-1}$, and $\Theta_s(B)$ is an $(s + 2)$th degree polynomial in B.

It follows from results which will be obtained below that, for Hannan, Terrell, and Tuckwell, none of the three unobserved components would be null, and that the following models can be derived:

$$(1 - \Phi B)\, c_t = (1 - \Theta_c B)\, a_{c,t}$$

$$\Sigma_{\Phi}(B)\, s_t = \Theta(B)\, a_{s,t} \qquad I_t = a_{I,t}$$

where $a_{c,t}$, $a_{s,t}$, and $a_{I,t}$ are independent white noises and where $\Theta(B)$ is a $(s-2)$th degree polynomial in B.

Recently Hillmer (1976) and Box, Hillmer, and Tiao (1976) have developped a method to deseasonalize the BJB particular multiplicative model. Their approach, translated in the present context, assuming $\Phi = 1$, would imply an answer generelly similar to Hannan, Terrell, and Tuckwell, but for the fact that both C_t and S_t's model would be non-invertible, hence that $a_{C,t}$ and $a_{S,t}$ would have smaller variances while $a_{I,t}$ would have a greater variance, and that Θ (B) would generally be a (s - 1)th degree polynomial in B.

The review of the state of the art can stop at this stage, even though this is far from an exhaustive list; it is sufficient to note that, unless some further clarification is offered, one would still face the fundamental problem of the conceptualization of the seasonal problem. When one starts with the Buys Ballot table, one is apparently misled in proposing the B.J.B. model, yet, to have some confidance in the alternative approaches, which amount to designing the model from its unobserved components, there appears a need to develop a general definition of seasonality.

2.6 The Spectral Definition

In the stochastic domain, just as in the deterministic one, seasonal modelling has evolved and "progressed" despite the obvious lack of agreement as to what constitutes the seasonal unobserved component. The problem of defining the seasonal is necessarily intertwined with the seasonal adjustment problem and most researchers have approached the question from the later point of view.

The general attitude toward the definition of the seasonal has been one of caution. This reluctance has led to purposedly vague proposed definitions. For instance, Baron (1973) defines seasonality as "the monthly fluctuations which recur every year with more or less the same timing and intensity" (p.21). In spite of this vagueness, two distinct but related themes appear time and again, mamely, first, that the seasonal is what could be called "quasi-periodic", i.e. that the same pattern "almost" repeats itself year after year, then that, as a component, it should average over a calendar year close to zero, hence that it is some sort of deviation about the trend. The first attempt to a systematic treatment of the seasonal composition of a series was proposed by Lovell (1963) in the form of a set of axioms. Unfortunately, the proposed axioms did not necessarily apply even to the simplest specifications (Lovell, 1966), and this failure can be traced back to two of Lovell's desired properties; orthogonality and symmetry, for which an inadequate rationalization had been developped (Fontenay, 1973). Most researchers have been content with proposing some a priori model for the seasonal and have concentrated their attention on the estimation problem of the specified model (Jorgenson, 1963; Grether and Nerlove, 1970 ; Box, Hillmer, and Tiao, 1976,...). Even though it would be foolish to imagine that *the* approach to seasonality exists, this shyness toward defining the problem renders much of the existing works of little use for the researcher who has to select one among the many methods proposed.

Nerlove (1964) was, with Lovell, one of the first to attempt to cope with this problem and to propose a definition for seasonality which be as general as possible. Since Nerlove's approach is still relevant today (Granger, 1976) it is useful to review it. The analysis begins from the concept of a stable seasonal with which is associated a periodic function, with the year as period, y_t. The Fourier representation of this function is

$$y_t = b_0 + \sum_{\ell=1}^{s/2} \{ b_{2\ell} \cos \lambda_\ell t + b_{2\ell+1} \sin \lambda_\ell t \}$$

He begins by letting b_ℓ, $\ell = 0, 1, \ldots, s$, since $b_{s+1} = 0$, be random variables. Then he lets b_0 be any stochastic process, as long as b_0 is independent of b_ℓ , $\ell = 1, 2, \ldots, s$. Denoting the component which corresponds to the ℓth frequency by $y_{\ell,t}$ where

$$y_{\ell,t} = d_\ell \cos(\lambda_\ell t + \gamma_\ell) \qquad \ell = 1, 2, 3, \ldots, (s/2) - 1$$

where γ_ℓ is a uniformely distributed random variable, he lets d_ℓ be itself a stochastic process $x_{\ell,t}$ with spectral density function $K_x(\lambda)$, i.e.

$$y_{\ell,t} = x_{\ell,t} \cos(\lambda_\ell t + \gamma_\ell)$$

It follows that $y_{\ell,t}$'s spectral density at the frequency will be proportional to the sum of $K_{x\ell}(\lambda + \lambda_\ell)$ and $K_{x,\ell}(\lambda - \lambda_\ell)$, and, provided, as it is generally assumed, that $K_x(\lambda)$ peaks at the zero frequency and has its power concentrated close to that frequency, $K_{y,\ell}(\lambda)$ will be approximately $K_{x,\ell}(\lambda)$ shifted to the frequency λ_ℓ at least in the neighbourhood of that frequency. Nerlove is thus led to propose the following definition:

> "In the more general case, then, we may define seasonality as the characteristic of a time series that gives rise to spectral peaks at seasonal frequencies". (p. 262).

This definition does not follow precisely from the theory of narrow band processes used to arrive at it. As long as $K_{x,\ell}(\lambda)$ does not reach a local maximum or minimum at the frequency $2\lambda_\ell$, $K_{x,\ell}(\lambda+\lambda_\ell)$ will have a non-null slope at that frequency, and, if $\lambda=\lambda$, $K_{y,\ell}(\lambda)$ will differ from $K_{x,\ell}(0)$ by $K_{x\ell}(2\lambda_\ell)$, hence $K_{y,\ell}(\lambda)$ will not peak at λ_ℓ. In practice, most of $K_{x,\ell}(\lambda)$'s power is expected to be concentrated close to $\lambda = 0$, and the above problem was cleared by Granger (1976) who allows $K_{y,\ell}(\lambda)$ to peak in an appropriately chosen neighbourhood of λ_ℓ.

Even though this definition is attractive, it has the shorthcoming to fail to explain how the seasonality of a process is generated. As such it only provides an ex post guideline on how to investigate seasonality. The added problem raised by non-stationary series has also been dealt with by Granger, however there remains other problems. For instance, Granger cites, as an example of a seasonal component, Grether and Nerlove's model:

$$(1-.9B^{12})S_t = (1+.6B)a_{s,t}$$

While, if neighbourhoods are properly chosen about the seasonal frequencies, S_t's spectrum will be concentrated in those neighbourhoods it will also be concentrated about the zero frequency. More important, S_t's spectrum does not peak at seasonal frequencies, and finally the selection of the neighbourhoods is very sensitive to the value of the AR and MA coefficients.

Similarly a model such as

$$\Sigma\varphi(B)\,S_t = a_{s,t}$$

would be rejected by Nerlove's spectral criterion as being the model of a seasonal unobserved component. It would only be accepted by Granger if the neighbourhood about the seasonal frequencies are not too small and if ϕ is close to 1. In other words, while a strict application of Nerlove's criterion would force us to reject processes which are intuitively seasonal, Granger's criterion leaves a lot of ambivalence.

However, regardless of the approach, this definition fails if, in fact, the changes in seasonality are subject to cyclical movements, since, then, the spectral density function rather than peaking at the seasonal frequencies, will peak about those frequencies. The possibility of a cyclical evolution in the seasonal is accounted for in the ARIMA model, since a possible model with cyclically moving seasonality would be

$$(1 - \varphi_1 B^s - \varphi_2 B^{2s}) z_t = a_t$$

where

$$\varphi_1 < 2\sqrt{\varphi_2}$$

i.e. such that $(1 - \varphi_1 B^s - \varphi_2 B^{2s})$ has complex roots. The possibility of such an occurence was explicitly investigated by Kuznets (1933) . Similarly, this definition does not help, when studying the problem of the seasonal "dip" which is observed in the spectrum of most seasonally adjusted series, since it could not be associated with seasonality. Even though the narrow use of the spectral criterion itself is rejected in this paper, Nerlove's original justification, in the form of the theory of narrow band processes, will be maintained to develop a general class of seasonal models.

2.7 Causal Analysis of Seasonality

For most researchers, the "ideal" approach is implicitly or explicitly what Granger (1976) has called, with respect to the seasonal adjustment problem, the "causal adjustment" approach (Grether and Nerlove, 1970). It is considered "ideal" for two distinct reasons; first of all, it is usually assumed that there does not exist a unique decomposition of a univariate series and that the components can only be specified uniquely if specific causes are associated with each component, then, even more fundamentally, the components are assumed to be the results of well defined "causes". Even though it has yet to be developped in the form of an operational method, in view both of its implication for the modelling of seasonal processes and of the underlying view on how seasonality is generated, it is important, at this stage, to review it.

Granger has probably provided the most explicit analysis of this approach and Granger's presentation will be the base of the present discussion. Granger states that

> "... ignoring consideration of causation can lead to imprecise or improper definitions of seasonality and consequently to misunderstanding of why series require seasonal adjustment, to improper criteria for a good method of adjustment and to have implications for the evaluation of the effects of adjustment..." (p. 1.).

Four classes of causes are listed: calendar, timing decision, weather, and expectation, and, given the decomposition

$$z_t = A_t + S_t$$

where A_t is the non-seasonal component

and S_t is the seasonal,

it is assumed that S_t and A_t are independent.

However, in as much as a univariate approach is chosen, and some model, say an ARIMA model, is specified for each of the unobserved components:

$$\varphi_A(B)A_t = \theta(B)a_{A,t}$$

$$\varphi_S(B)S_t = \theta_S(B)a_{S,t}$$

where $a_{A,t}$ and $a_{S,t}$ are two independent white noises,

φ_A (B) and θ_A (B) are such that A_t's spectrum has almost no power outside some neighbourhood about the seasonal frequencies.

and are selected such that S_t's spectrum has almost no power outside of some neighbourhoods about the seasonal frequencies.

Granger notes "that there is no unique decomposition" (p.15) unless very arbitrary and stringent conditions are imposed. In fact he notes rightly that the components are not only unobserved but also unobservable, and he suggests that, only through the use of a causal model, can they be rendered "observable". In the above example, to state that φ_A (B) and θ_A (B) must be such that A_t's spectrum does not peak at seasonal frequencies while φ_S (B) and θ_S (B) are such that S_t's spectrum's power is concentrated in the neighbourhood, of the seasonal frequency is insufficient to specify A_t and S_t's models. However, if in fact S_t is generated by causes, say the weather measured by the rainfall, and denoted by R_t, given that z_t is say the California tomato production, such that

$$S_t = v(B)\,R_t$$

then, in principle, S_t can be "observed" through R_t, since

$$\varphi_R(B)\,R_t = \theta_S(B)\,a_{S,t}$$

where

$$\varphi_R(B) = v(B)\,\varphi_S(B)$$

and

$$\varphi_A(B)\,\cdot_t = v(B)\,\varphi_A(B)\,R_t + \theta_A(B)\,a_{A,t}$$

i.e. since S_t can be retrieved from z_t through a simple transfer function model.

This approach raises serious questions, in as much as it is seen as a method to model seasonality, or to deseasonalize. There is however no doubt that whenever proper causal variables such as R_t can be found, z_t's estimation from its transfer function model should often be superior to an estimation based on a univatiate model, i.e. one does not have to accept "$S_t = \nu(B)\ R_t$" to model z_t's transfer function. Questions relating to seasonal adjustment proper being beyond the scope of this paper, it is sufficient to consider the transfer function model of z_t.

First of all this model implies that z_t's seasonality enters solely through R_t, but whether the seasonality enters solely through some exogeneous variables is exactly the question which was studied by Plosser (1976). Hence, this approach to seasonality would restrict one to a subclass of transfer functions, since there is no <u>a priori</u> reasons to exclude the possibility that either $\nu(B)\ \varphi_A\ (B)$ or $\theta_A\ (B)$ be themselves "seasonal" filters. That $\nu(B)\ \varphi_A\ (B)$ might be a seasonal filter would imply, in Granger's example, that the production of tomato is not affected in every season the same way by the rainfall, i.e. that a fluctuation in rainfall will affect the output of tomatoes differently depending upon the stage of growth of the tomatoes. That $\theta_A\ (B)$ might also be seasonal would only imply that not all of z_t's seasonality can be accounted for by R_t. Whereas whether $\theta_A(B)$ might also be seasonal appears academic, it seems very plausible that in general $\nu(B)\ \varphi_A(B)$ will be a seasonal filter.

If the modeling problem is considered, it must be noted that, even if exogeneous "seasonal causative" variables such as R_t are given, seasonality in the transfer filter might have to be modeled and that no guidelines on how to model "seasonal Causative" variables have been generated. In fact, the very principle of a seasonal causative variable is questionable. If R_t, the rainfall, is taken as an example, even though there is little doubt that, usually, seasonal movements will dominate (this is not necessarily the case, as can be seen by observing rainfall in desertic or equatorial regions) there are also unseasonal rainfalls, and both long "cyclical" movements and long run trends in rainfall. The only "cause" which is unchanging, as far as seasonality, is the calendar, i.e. time, and this is the object of the next section.

3. A GENERAL CLASS OF SEASONAL MODELS

3.1 Introduction

On the observation that the periodicity associated with seasonality is above all a property of time, the periodic aspect of time is isolated from its chronological nature and a bidimensional model is developped.

The concept of seasonality, in this broader concept of time, is reconsidered, and two distinct stationarities, seasonal and chronological stationarities, are defined. Henceforth the analysis is restricted to seasonally stationary processes.

The ARMA model is generalized to the multidimensional model of time, and an example is given to illustrate how a simple seasonal multiplicative model can be generated. Nevertheless this approach does not enable the researcher to discriminate, in general, between non-seasonal and seasonal models.

A Fourier transformation of the white noise process from which any process is generated is shown to be itself a white noise process. The Fourier representation being an equivalent representation of the generating white noise process, a seasonal process can be viewed as linearly generated from this transformed white noise. Successive transformations of the white noise process into equivalent forms in the complex domain enables one to represent the seasonal process as a sum of indepedent real and complex processes, each being symmetrical about either the zero frequency or a seasonal frequency. Those processes are the elementary components of a seasonal time series.

The Fourier representation and the complex representation are alternative forms of the general seasonal model.

3.2 A Bi-dimensional Time Model

The concept of seasonality is fundamentally tied to the concept of periodicity, however a seasonal series is not in general a periodic series nor is there any reason to expect it to be periodic. The problem in conceptualizing seasonality, therefore, is that, even though one knows that there is periodicity, a strict periodicity is not observed:

> "It is the close but not perfect approach to regularity which renders the precise definition of seasonality so difficult." (Nerlove, 1964, p. 259).

The problem, however, is, at least conceptually, not all that intractable: a time series is an observation made through time, and the periodicity in question is function of the periodicity of time. When a seasonal time series is considered, the underlying time on which the series is indexed is conceived as having two completely distinct properties; on the one hand we think of the past, present and future, i.e. we conceive of time as a chronology, a flow which passes but never comes back, and, on the other hand, we think of time as the seasons which pass away only to come back faithfully year after year. With each conceptualization of time we associate a conceptualization of the series; with chronological time we associate the concept of a trend-cycle and with periodic time, we associate that of a seasonal. Both are abstract concepts which describes the series as if the other were held constant. A trend-cycle, therefore, is some path the series would follow were it not for the periodic change in seasons; it is an imaginary construct since, in real life, the process cannot be observed independently of the periodic change of the seasons. A seasonal, similarly, is the periodic path the series would follow were it not for the chronological time. It follows immediately that, while a trend-cycle should be indexed in terms of the season, a seasonal should be indexed in terms of the date. Such a decomposition would be of limited utility since at the limit we would have as many trend-cycles as we have seasons and as many seasonals as there are dates, and given one series, no way to estimate those components.

Now, for an observed process, chronological time and periodic time pass simultaneously, hence the evolutionary changes in the seasonal, over chronological time, should be expected to be related to the evolution of the series itself, hence to the trend-cycle. This evolutionary change in the seasonal is what gives rise to the concept of "moving" seasonality, of "slowly evolving" periodicity,... The periodicity in time, which is directly attributable to the relative positions of the sun and the earth, is taken as exact and fixed, even though the process itself will appear only more and less periodic.

The dichotomization of time to conceptualize within the observed process a trend-cycle and a seasonal implies that the time one observes, this time which simultaneously both passes never to come back and comes back periodically year after year, is to be seen as a subset of the time within which trend-cycle and seasonal are conceptualized. This subset of time may be called real time, its complement being called imaginary time. Whereas a point in real time is well known, say (1 June 1978, 1 June), an example of a point in imaginary time would be, say, (1 June 1978, 14 January). There 1 June 1978 indicates the date, independently of the season and 14 January indicates solely the season.[1]

1. The roots of this approach can be traced back to the work of Buys Ballot (1847). It is Guilbaud (1968) who was the first to point out that the Buys Ballot array, as used traditionally, was inadequate, because the first month of the year, say January 1979, should be as close to the last month of the preceding year, December 1978, as to the second month of the year, February 1979. This led to the indexation of the series both in terms of the chronological date and in terms of the season (Calot, 1969). The concept of a multidimensional time was proposed by Fontenay (1973).

Given such a view of the time series, the observed process is an observation of the realization of the process only over real time. The process itself would be defined over the overall time, i.e. over imaginary time as well as over real time. However the very nature of periodic time simplifies the specification of the process. Real time can be rewritten as

$$t = \hat{t} \pmod{s} + \tilde{t}$$

such that t denotes the chronological date while $\hat{t}$ denotes the season. This leads to denote the process in terms of both t and $\hat{t}$ subscripts, i.e. to write $z_{t,\hat{t}}$. In generalized time, that is, when the process is not constrained to real time, the above relation between chronological time t and periodic time $\hat{t}$ needs not hold anymore. For instance, we could have t to be the chronological date 1 July 1978 while the season, $\hat{t}$, is 14 January (see Figure 1).

If now the process $z_{t,\hat{t}}$ is observed while holding the chronological date constant, say at t_0, then $z_{t_o,\hat{t}}$ depends only upon t and it is a periodic process

$$\gamma_{t,\hat{t}} = \gamma_{t,\hat{t}+ks} \qquad k = 0, \pm 1, \pm 2, \ldots$$

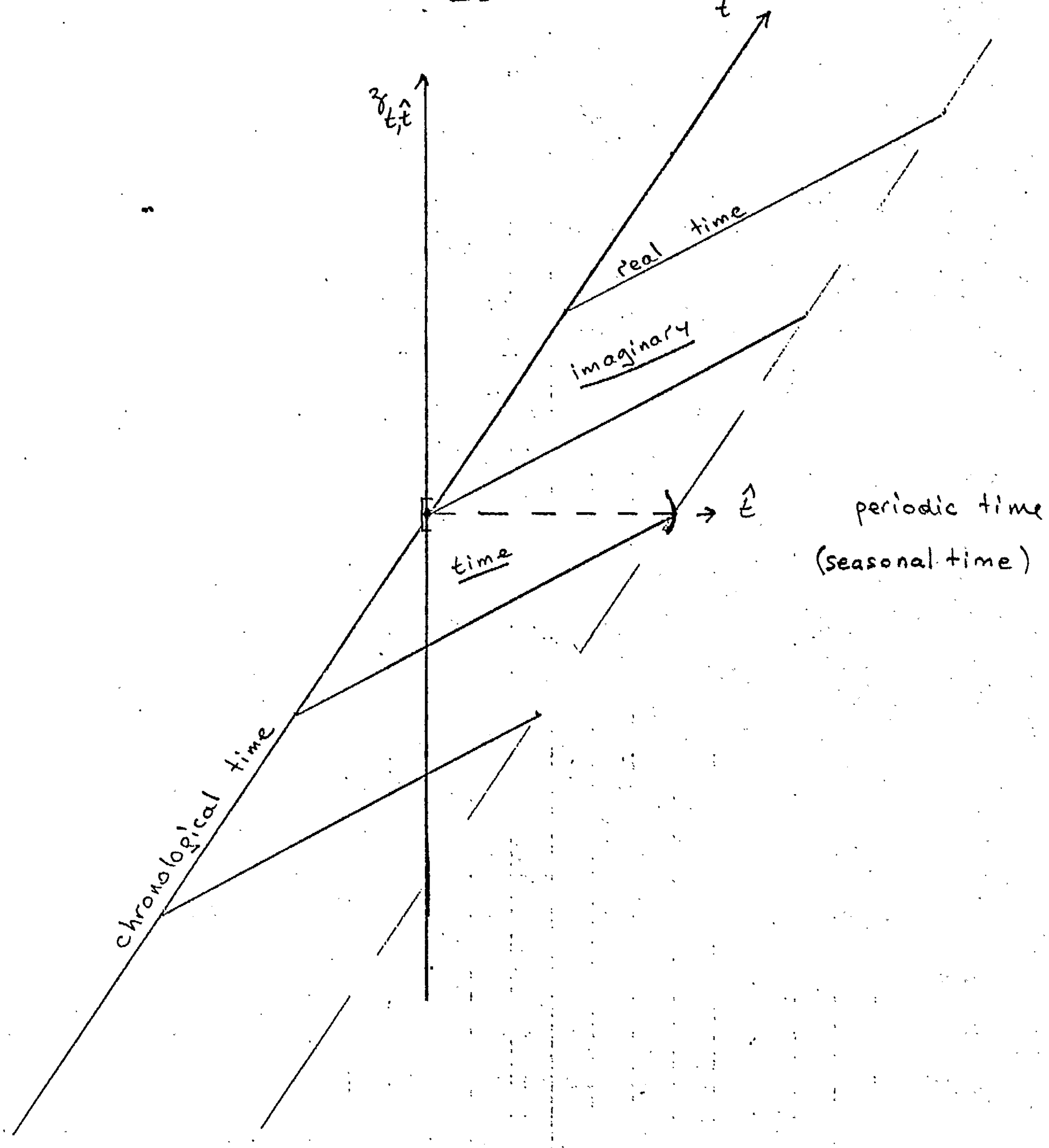

Figure 1 : Bidimensional time model

3.2 Bi-dimensional Time and Stationarity

As the process if defined in terms of two time dimensions, t and $\hat{t}$, it is necessary to consider stationarity in each dimensions. The present paper considers only weak stationarity and weak stationarity and stationarity will be used interchangeably. Stationarity in the seasonal dimension will be called seasonal stationarity while stationarity in the chronological dimension will be called chronological. A series which is both seasonally stationary and chronologically stationary is stationary.

Seasonal stationarity implies that both the mean and the variance-covariance of the process are independent of the seasons and that they depend only on the relative position of the seasons, in other words that the impact of a given shock in January on an observation in June is similar to that of the same shock in June on an observation made in November. Seasonally stationary process do not need to be indexed by the season.

B.J.B., C.G.N., and H.T.T. models are all seasonally stationary, however in recent years Cleveland (1972), Froeschle (1975), Cleveland and Tiao (1978) Havenner and Swamy (1979) have investigated processes which were not seasonally stationary. Cleveland and Tiao have called seasonally stationary process homogeneous. Since, before passing to the more general form of the seasonally non-stationary processes, the properties of the seasonally stationary processes should be established, and since this is the objective of this paper, seasonally non-stationary series will not be considered.

An additional motivation to restrict the analysis to seasonally stationary processes was provided by Froeschle:

> "The theoretical motivation for this [using non-seasonally stationary model] is good but in many cases the number of observations in a time series is not sufficient to permit good analysis" (p. 272).

Chronological stationarity parallels much more closely standard stationarity and it has been considered by Pagano (1976)'s under the name of periodic correlation. In fact Grether (1966)'s had already established that stationarity implied that a process could not be indexed by the season, i.e., in the present terminology, that it is seasonally stationary.

3.4 Time Representation of a Seasonal Process

A white noise process can be defined over the bidimensional grid $(t, \hat{t})$. If it is denoted by $a_{t,\hat{t}}$, we may define linear filters on this white noise to generate stochastic processes. The general form of the linear processes thus obtained would be

$$z_{t,\hat{t}} = \sum_{j=0}^{(s-1)} \sum_{\tau=0}^{\infty} \psi_{\tau,j}^{\hat{t}} \; a_{t-\tau,\hat{t}-j}$$

and seasonal stationarity would imply $\psi_{\tau,j}^{\hat{t}} = \psi_{\tau,j}$.

It is convenient to define subprocesses in the chronological dimension as $x_{t,d}$, where $d = \hat{t}-j$ and where $d + s = d$. Then

$$z_{t,\hat{t}} = \sum_{d=0}^{(s-1)} x_{t,d}$$

where $x_{t,d} = \psi_d(B)\, a_{t,\hat{t}-j}$ and $B\, a_{t,\hat{t}-j} = a_{t-1,\hat{t}-j}$

The s processes $x_{t,d}$ are simple unidimensional stochastic processes and if $\psi_d(B)$ is rational so that

$$\psi_d(B) = \frac{\theta_d(B)}{\phi_d(B)} \cdot \psi_{0,d}$$

$x_{t,d}$ is a standard ARIMA process.

Intuitively since, B.J.B. multiplicative models were generated from a Buys Ballot table, it should also be possible to generate them this way. An example can be given, at this stage, to illustrate a possible procedure. Let

$$\psi_d(B) = (1-\varphi^s B^s)^{-1}(1-\Phi^s B^s)^{-1} \varphi^d B^d$$

Given that a B.J.B. model is defined in real time, let $a_{t,\hat{t}} = a_t$, then $z_{t,\hat{t}} = z_t$ and

$$z_t = \sum_{d=0}^{(s-1)} (1-\varphi^s B^s)^{-1} (1-\Phi^s B^s)^{-1} \varphi^d B^d a_t$$

$$(1-\varphi^s B^s)(1-\Phi^s B^s) z_t = \Sigma_\varphi (B) a_t$$

However, since $(1-\varphi^s B^s) = (1-\varphi B) \; \Sigma_\varphi(B)$, a B.J.B multiplicative seasonal AR model is obtained:

$$(1-\varphi B)(1-\Phi^s B^s) z_t = a_t$$

A second example will be used to show that this framework is more general than models in which the seasonality is generated solely by introducing mutiples of lag s, and that B.J.B. trigonometric and multiplicative models have common foundations.

Let $\Psi_d(B)$ be such that

$$\Psi_d(B) = (1-\varphi^s B^s)^{-1} \delta_d B^d$$

where
$$\delta_0 = 1, \; \delta_1 = \sqrt{3}\,\varphi, \; \delta_2 = 2\varphi^2, \; \delta_3 = \sqrt{3}\,\varphi^3,$$
$$\delta_4 = \varphi^4, \; \delta_5 = 0, \; \delta_6 = -\varphi^6, \; \delta_7 = -\sqrt{3}\,\varphi^7$$
$$\delta_8 = -\varphi^8, \; \delta_9 = -\sqrt{3}\,\varphi^9, \; \delta_{10} = -\varphi^{10}, \; \delta_{11} = 0$$

As in the preceeding example, let $a_{t,\hat{t}} = a_t$, hence $z_{t,\hat{t}} = z_t$

and

$$z_t = (1-\varphi^s B^s)^{-1} \delta_\varphi(B) a_t$$

where $$\delta_{\varphi}(B) = \sum_{k=0}^{11} \delta_k B^k$$

But then, since $\delta_{\varphi}(B)\cdot(1-\sqrt{3}\,\varphi B+\varphi^2B^2) = (1-\varphi^4 B^4)$

$$(1-\sqrt{3}\,\varphi B+\varphi^2 B^2)\, z_t = a_t$$

which is, when $\varphi = 1$, the B.J.B. first example of a seasonal process. In fact, then, the coefficients of $\delta_{\varphi}(B)$ are the values taken by a sinusoidal with a period of one year and an amplitude of 2.

It follows however from those two examples that, short of adopting a definition for the seasonal component, there is no direct way to establish whether, in general, a model will be seasonal, thus in the first example, one could have had $\Phi = 0$. The only models which appear unambiguously seasonal are those in which $z_{t,\hat{t}}$ depends only on observations made in month $\hat{t}$. In real time, those models will be:

$$\Phi(B^4)\, z_t = \Theta(B^4)\, a_t$$

3.5 The Fourier Representation of a Seasonal Process

Even though the link between the time representation of a stochastic process and a simple seasonal ARIMA model was illustrated in the previous section, the relationship is not particularly intuitive and the kind of interpretation one can hope to make appears rather limited. An alternative is to follow the path of Hannan and Nerlove and to work with the Fourier representation. The Fourier representation, given the bi-dimensional model of time, is particularly logical since the process, in its seasonal dimension is periodic, hence exactly representable by a Fourier series. Thus, given the chronological date t, the white noise process $a_{t,\hat{t}}$ is periodic with respect to $\hat{t}$. Its Fourier representation is:

$$a_{t,\hat{t}} = \sqrt{2/s}\left\{\frac{a_{t,0}}{\sqrt{2}} + \sum_{\ell=1}^{(s/2)-1}\left(a_{t,2\ell}\cos\lambda_\ell\hat{t} + a_{t,2\ell+1}\sin\lambda_\ell\hat{t}\right) + \frac{(-1)^{\hat{t}}a_{t,s}}{\sqrt{2}}\right\}$$

where

$$\lambda_\ell = \frac{2\pi\ell}{s}$$

Then

$$a_{t,0} = \sqrt{\frac{1}{s}}\sum_{\hat{t}=1}^{s} a_{t,\hat{t}}$$

$$a_{t,2\ell} = \sqrt{\frac{2}{s}}\sum_{\hat{t}=1}^{s} a_{t,\hat{t}}\cos\lambda_\ell\hat{t} \qquad \ell = 1,2,\ldots,(s/2)-1$$

$$a_{t,s/2} = \sqrt{\frac{1}{s}}\sum_{\hat{t}=1}^{s} a_{t,\hat{t}}(-1)^{\hat{t}}$$

$$a_{t,2\ell+1} = \sqrt{\frac{2}{s}}\sum_{\hat{t}=1}^{s} a_{t,\hat{t}}\sin\lambda_\ell\hat{t} \qquad \ell = 1,2,\ldots,(s/2)-1$$

$$E\left(a_{t,\hat{t}}\right) = 0 \quad \text{implies} \quad E\left(a_{t,\ell}\right) = 0 \qquad \ell = 0, 2, 3, \ldots, s$$

$$E\left(a_{t,\hat{t}_1} \cdot a_{t,\hat{t}_2}\right) = \begin{cases} \sigma_a^2 & \hat{t}_1 = \hat{t}_2 \\ 0 & \hat{t}_1 \neq \hat{t}_2 \end{cases} \qquad \hat{t}_1, \hat{t}_2 = 1, 2, \ldots, s$$

implies

$$E\left(a_{t,\ell_1} \cdot a_{t,\ell_2}\right) = \begin{cases} \sigma_a^2 & \ell_1 = \ell_2 \\ 0 & \ell_1 \neq \ell_2 \end{cases} \qquad \ell_1, \ell_2 = 0, 2, 3, \ldots, s$$

In other words, $\{a_{t,\ell};\ t=0,\pm 1,\pm 2,\ldots,\ \ell=0,1,\ldots,s\}$ is a white noise process and, furthermore, $a_{t,\ell}$ and $a_{t,\hat{t}}$ are two equivalent representations of the same white noise process. Since they are equivalent, a process $z_{t,\hat{t}}$ can just as well be represented as being generated linearly from $a_{t,\ell}$ such that

$$z_{t,\hat{t}} = \sqrt{\frac{2}{s}}\left\{\frac{1}{\sqrt{2}}x_{t,0} + \sum_{\ell=1}^{(s/2)-1}\left(x_{t,2\ell}\cos\lambda_\ell \hat{t} + x_{t,2\ell+1}\sin\lambda_\ell \hat{t}\right) + \frac{(-1)^{\hat{t}}x_{t,s}}{\sqrt{2}}\right\}$$

where

$$\varphi_\ell(B)\,x_{t,\ell} = \gamma_\ell\,\theta_\ell(B)\,a_{t,\ell} \qquad \ell = 0,2,3,\ldots,s$$

and where γ_ℓ is some constant depending only upon ℓ.

The desirability of this representation compared to the previous representation becomes obvious when it is noted that the seasonal dimension has been isolated, in the trigonometric terms, from the chronological dimension, in which are defined the amplitude processes $x_{t,\ell}$.

Seasonal stationarity implies immediately that $\varphi_{2\ell}(B) = \varphi_{2\ell+1}(B)$, $\theta_{2\ell}(B) = \theta_{2\ell+1}(B)$, and $\gamma_{2\ell} = \gamma_{2\ell+1}$, $\ell = 1,2,\ldots,(s/2)-1$

Such seasonal models can be seen to be the sum of sine and cosine waves at each of the seasonal frequencies and of one year width, in periodic time. The amplitudes of those curves are described, in chronological time, by ARIMA processes generated by linear combinations of the white noises $a_{t,\hat{t}}$. The underlying idea of analysing seasonality in terms of amplitude can be traced back to Kuznets (1933). Wald, implicitly, introduced the stochastic formulation. If the subset of this process defined on real time is considered, since $\cos\lambda_\ell\hat{t} = \cos\lambda_\ell t$ and $\sin\lambda_\ell\hat{t} = \sin\lambda_\ell t$, and if it is assumed that the amplitude processes are random walks, Hannan, Terrell and Tuckwell's model is obtained.

An alternative but more useful representation of the process can be developped as follows:

$$a'_{t,0} = a_{t,0}$$
$$a'_{t,l} = \frac{1}{2}(a_{t,2l} + i\ a_{t,2l+1}) \qquad l = 1, 2, \ldots, (s/2)-1$$
$$a'_{t,s} = a_{t,s}$$

where $\bar{a}$ is the complex conjugate of a

Then
$$a_{t,\hat{t}} = \left(\frac{1}{\sqrt{s}}\right)\left\{a_{t,0} + \sum_{l=1}^{(s/2)-1}\left(a'_{t,l}\, e^{-i\lambda_l \hat{t}} + \bar{a}'_{t,l}\, e^{i\lambda_l \hat{t}}\right) + a'_{t,s}\,(-1)^{\hat{t}}\right\}$$
$$E(a'_{t,l}) = 0 \qquad l = 0, 1, \ldots, (s/2)$$
$$E(a'_{t,l_1} \cdot \bar{a}'_{t,l_2}) = \begin{cases} \sigma_a^2 & l_1 = l_2 \\ 0 & l_1 \neq l_2 \end{cases} \qquad l_1, l_2 = 1, 2, \ldots, (s/2)-1$$

Once again a new and equivalent representation of the white noise process, $\{a'_{t,l}\ ;\ t = 0, \pm 1, \pm 2, \ldots, l = 0, 1, \ldots, s/2\}$ is obtained, which can be used to generate the process $z_{t,\hat{t}}$, which can be represented as

$$z_{t,\hat{t}} = \frac{1}{\sqrt{s}}\left\{x'_{t,0} + \sum_{l=1}^{(s/2)-1}\left(x'_{t,l}\, e^{-i\lambda_l \hat{t}} + \bar{x}'_{t,l}\, e^{i\lambda_l \hat{t}}\right) + x'_{t,s}\,(-1)^{\hat{t}}\right\}$$

where
$$\varphi'_l(B)\, x'_{t,l} = \gamma'_l\, \theta'_l(B)\, a'_{t,l} \qquad l = 0, 1, \ldots, s/2$$

and where φ'_l (B) and θ_l (B) are polynomials in B with real coefficients and γ_l is a real scalar.

In fact, since $z_{t,\hat{t}}$ is seasonally stationary, i.e. since $\varphi_{2\ell}(B) = \varphi_{2\ell+1}(B)$, $\theta_{2\ell}(B) = \theta_{2\ell+1}(B)$, and $\gamma_{2\ell} = \gamma_{2\ell+1}$, $\ell = 1,2,\ldots,(s/2)-1$, then $\varphi'_\ell(B) = \varphi_{2\ell}(B)$, $\theta'_\ell(B) = \theta_{2\ell}(B)$, and $\gamma'_\ell = \gamma_{2\ell}$, i.e.

$$\varphi_{2\ell}(B)\, x'_{t,\ell} = \gamma_{2\ell} \cdot \theta_{2\ell}(B)\, a'_{t,\ell} \qquad \ell = 0,1,\ldots,s/2$$

This form will henceforth be written as

$$\varphi_\ell(B)\, x'_{t,\ell} = \gamma_\ell \cdot \theta_\ell(B)\, a'_{t,\ell} \qquad \ell = 0,1,\ldots,s/2$$

In fact this is not a new representation of the process $z_{t,\hat{t}}$ since one can pass directly from the representation of $z_{t,\hat{t}}$ in terms of $x_{t,\ell}$ to that in terms of $x'_{t,\ell}$. The present formulation would also be equivalent to Godfrey's own formulation had he assumed seasonal stationarity.

It is useful however to further transform the representation of $z_{t,\hat{t}}$ so as to look at $\left(x'_{t,\ell}\, e^{-i\lambda_\ell \hat{t}}\right)$ as one elementary process, provided the analysis is restricted to real time, i.e. $\hat{t} = t$. Let

$$a^*_{t,\ell} = a'_{t,\ell}\, e^{-i\lambda_\ell t}$$

Once again the same properties are maintained and $\{a^*_{t,\ell};\ \ell = 0,2,\ldots,s,\ t = 0,\pm 1,\pm 2,\ldots\}$ is an equivalent representation of the same white noise process, since

$$E(a^*_{t,\ell}) = 0$$

$$E(a^*_{t,\ell_1} \cdot a^*_{t,\ell_2}) = \begin{cases} \sigma_a^2 & \ell_1 = \ell_2 \\ 0 & \ell_1 \neq \ell_2 \end{cases} \qquad \ell_1, \ell_2 = 0,2,3,\ldots,s$$

$a^*_{t,\ell}$ has nevertheless some distinct characteristics. Hence a lag operator B^* defined on $a^*_{t,\ell}$ such that

$$B^* a^*_{t,\ell} = a^*_{t-1,\ell}$$

on the unit circle, will be specified as $e^{-i(\lambda-\lambda_\ell)}$ if B, the standard lag operator is specified as $e^{-i\lambda}$. This follows either from the Cramer representation or simply by observing that

$$\begin{aligned} a^*_{t-1,\ell} &= a'_{t-1,\ell}\, e^{-i\lambda_\ell (t-1)} \\ &= e^{i\lambda_\ell} B \cdot a'_{t,\ell}\, e^{-i\lambda_\ell t} \\ &= B^* \cdot a^*_{t,\ell} \end{aligned}$$

The general model $z_{t,\hat{t}}$, restricted in the real time, is now represented as the sum of elementary processes $x^*_{t,\ell}$ and their conjugates where

$$z_t = \frac{1}{\sqrt{s}} \left\{ x^*_{t,0} + \sum_{\ell=1}^{(s/2)-1} \left(x^*_{t,\ell} + \bar{x}^*_{t,\ell} \right) + x^*_{t,s/2} \quad (-1)^t \right\}$$

$$x^*_{t,0} = x_{t,0}$$

$$\bar{\varphi}^*_\ell (B^*)\, x^*_{t,\ell} = \gamma_\ell\, \bar{\theta}^*_\ell (B^*)\, a^*_{t,\ell} \qquad \ell = 1, 2, \ldots, (s/2)-1$$

$$x^*_{t,s/2} = x_{t,s}$$

and where $\bar{\varphi}^*_\ell (B^*)$ and $\bar{\theta}^*_\ell (B^*)$ are polynomials in B^* with complex coefficients $\varphi_{\ell,j}\, e^{-i\lambda_\ell j}$ and $\left(\theta_{\ell,j}\, e^{-i\lambda_\ell j}\right)$ respectively.

In fact, once again, this is not a new representation of z_t but only a simple transformation of the previous representation since, considering $\left(x'_{t,\ell}\, e^{-i\lambda_\ell t}\right)$,

$$x'_{t,\ell}\, e^{-i\lambda_\ell t} = e^{-i\lambda_\ell t} \sum_{j=0}^{\infty} \psi_{\ell,j} \cdot a_{t-j,\ell}$$

$$x'_{t,\ell}\, e^{-i\lambda_\ell t} = \sum_{j=0}^{\infty} \left(\psi_{\ell,j} \cdot e^{-i\lambda_\ell j}\right) \cdot \left(a_{t-j,\ell} \cdot e^{-i\lambda_\ell (t-j)}\right)$$
$$= \Psi^*_\ell (B^*)\, a^*_{t,\ell}$$
$$= x^*_{t,\ell}$$

Similarly

$$\bar{x}'_{t,\ell}\, e^{i\lambda_\ell t} = e^{i\lambda_\ell t} \sum_{j=0}^{\infty} \psi_{\ell,j}\, \bar{a}'_{t-j,\ell} \qquad \ell = 1,2,\ldots,(s/2)-1$$
$$= \sum_{j=0}^{\infty} \left(\psi_{\ell,j}\, e^{i\lambda_\ell j}\right) \cdot \left(\bar{a}'_{t-j,\ell} \cdot e^{i\lambda_\ell (t-j)}\right)$$
$$= \bar{x}^*_{t,\ell}$$

where $\varphi^*_\ell (B^*)\, \bar{x}^*_{t,\ell} = \gamma_\ell\, \theta^*_\ell (B^*)\, \bar{a}^*_{t,\ell}$

and where $\varphi^*_\ell (B^*)$ and $\theta^*_\ell (B^*)$ are polynomials in B* with respectively $\left(\varphi_{\ell,j}\, e^{i\lambda_\ell j}\right)$ and $\left(\theta_{\ell,j}\, e^{i\lambda_\ell j}\right)$ as jth coefficient.

Given that $x_{t,\ell}$ is an ARMA process, its autocovariance generating function, AGF, $K_\ell(B)$, will be

$$K_\ell(B) = \gamma_\ell^2\, \frac{\theta_\ell(B)\, \theta_\ell(F)}{\varphi_\ell(B)\, \varphi_\ell(F)}\, \sigma_a^2 \qquad \ell = 0,2,3,\ldots,s$$

Given seasonal stationarity, this would also be $x'_{t,\ell}$'s AGF. On the other hand, $x^*_{t,\ell}$'s AGF can be written as

$$K^*_\ell(B) = \gamma_\ell^2\, \frac{\bar{\theta}^*_\ell(B)\, \theta^*_\ell(F)}{\bar{\varphi}^*_\ell(B)\, \varphi^*_\ell(F)}\, \sigma_a^2$$

while $\bar{x}^*_{t,\ell}$'s AGF will be

$$\bar{K}^*_\ell(B) = \gamma_\ell^2\, \frac{\theta^*_\ell(B)\, \bar{\theta}^*_\ell(F)}{\varphi^*_\ell(B)\, \bar{\varphi}^*_\ell(F)}\, \sigma_a^2 .$$

This result follows from observing that the AGF is a z-transform.

Noting that, given any polynomial P, with jth coefficient p_j,

$$P^*(B) = P(B^*)$$

where P*'s jth coefficient is $\left(p_j\, e^{i\lambda_\ell j}\right)$,

the following relation is obtained between $x'_{t,\ell}$ and $\bar{x}^*_{t,\ell}$'s AGFs:

$$K^*_\ell(B) = K'_\ell(B^*)$$

By passing from $x^\ell_{t,\ell}$ to $\bar{x}^*_{t,\ell}$, the shape of the spectral density function has been left unchanged but a lateral translation of the spectrum by λ_ℓ has been performed, the center of symmetry passing from 0 to λ_ℓ. Similarly, if $\bar{B}^*$ is defined on the unit circle as $e^{-i(\lambda+\lambda_\ell)}$,

$$\bar{P}^*(B) = P(\bar{B}^*)$$

and the following relation is obtained between $x'_{t,\ell}$ and $x^*_{t,\ell}$'s AGFs:

$$\bar{K}^*_\ell(B) = K'_\ell(\bar{B}^*)$$

Specified on the unit circle,

$$K^*_\ell(\lambda+\lambda_\ell) = K'_\ell(\lambda)$$

$$\bar{K}^*_\ell(\lambda-\lambda_\ell) = K'_\ell(\lambda)$$

Even though this approach to the construction of a seasonal process is closely related to the theory of narrow band processes, it differs from it by the fact that, since complex elementary processes are used, the spectrum of the transform process is solely the translation of the original process. If one were to return to Nerlove (1964), instead of worrying with $\{K_\ell(\lambda+\lambda_\ell)+K_\ell^{\#}(\lambda-\lambda_\ell)\}$ collectively, one studies them individually. Just as $x^{1}_{t,\ell}$'s spectrum is symmetrical about the zero frequency, since $x_{t,\ell}$ is real, $x^*_{t,\ell}$'s spectrum will be symmetrical about the ℓth seasonal frequency, and the intuitive idea of Nerlove and Granger has been recovered in a meaningful way. A seasonal process is a sum of elementary complex processes which are symmetrical about the origin and about seasonal frequencies respectively.

An alternative way to look at the seasonal model, if one uses the signal extraction terminology, is to conceive of a seasonal process as a signal, possibly with noise, made up of a series of messages, each sent at the zero frequency or at a seasonal frequency.

Those components are said to be elementary since for them and for them alone the symmetry property holds. A component such as $(x_{t,\ell}\cos\lambda_\ell t)$ is notstationary while a component such as $\{(x^*_{t,\ell}+\bar{x}^*_{t,\ell}) = (x_{t,2\ell}\cos\lambda_\ell t + x_{t,2\ell+1}\sin\lambda_\ell t)\}$ while being real and hence symmetric about the zero frequency, neither is symmetric about nor peaks at the seaosnal frequency λ_ℓ.

The distinction is not frivolous if its' implications are considered, in the case, say, of Hannan, Terrell, and Tuckwell's seasonal adjustment method. If the variance of the white noise generating the random walks corresponding to each seasonal frequency is the same regardless of that frequency, by smoothing each frequency at a time there is the possibility to extract more noise from the seasonal frequencies closest to $\pi/2$ than from those farthest - a spurious result since all are generated the same way a similar white noise process.

4. The General Seasonal Model and the Multiplicative Seasonal Models

4.1 Introduction

Starting from elementary notions of what constitutes seasonality, a general class of seasonal models has been developed. H.T.T.'s seasonal model has been shown to be a particularisation of this model obtained by allowing the amplitude of each of the seasonal sine and cosine waves to be random walk processes. In this chapter the relationship between the general seasonal model and the multiplicative seasonal models is investigated. Provided that attention is restricted to seasonally stationary processes, the general seasonal model will be shown to include not only both the BJB and CGN multiplicative models but also the trigonometric models as developped by Bacon(1965) and Box and Jenkins (1970). On the other hand, it will be shown that the proposed class is in another way less general since both multiplicative and trigonometric models can be designed which cannot be derived from the class of general seasonal models. The class of models derived in this paper will nevertheless still be called a general class of models since models thus designed are related to fundamental notions of seasonality while multiplicative models which cannot thus be derived can generally only be justified as seasonal models on intuitive grounds.

4.2 Derivation of the Multiplicative Seasonal Model

The general seasonal model is a sum of stochastic processes, and, to deduce the implied overall model for z_t, these processes must be aggregated. This will be done in three stages, the first stage consisting in deriving the component which corresponds to the ℓth seasonal frequency, the next one being use to aggregate either over all seasonal frequencies, or, if the AR polynimial $\phi_\ell(B)$, $\ell = 1,2,\ldots,s/2$, is simultaneously independent of ℓ and a factor of ϕ_0 (B), over all seasonal frequencies together with the element of $x_{t,o}$ which has ϕ_ℓ (B) as AR polynomial. The last step consists in aggregating all the components thus obtained to generate z_t's model.

Let

$$y_{t,\ell} = x_{t,2\ell} \cos \lambda_\ell t + x_{t,2\ell+1} \sin \lambda_\ell t$$

$$= x^*_{t,\ell} + \bar{x}^*_{t,\ell}$$

then, denoting $y_{t,\ell}$'s ACGF by $K_{y,\ell}(B)$,

$$K_{y,\ell}(B) = K^*_\ell(B) + \bar{K}^*_\ell(B)$$

$$= \left\{ \frac{\theta^*_\ell(B)\,\bar{\theta}^*_\ell(F)}{\phi^*_\ell(B)\,\bar{\phi}^*_\ell(F)} + \frac{\bar{\theta}^*_\ell(B)\,\theta^*_\ell(F)}{\bar{\phi}^*_\ell(B)\,\phi^*_\ell(F)} \right\} \gamma^2_\ell\, \sigma^2_a$$

$$= \left\{ \frac{\theta_{y,\ell}(B)\ \theta_{y,\ell}(F)}{\phi_{y,\ell}(B)\ \phi_{y,\ell}(F)} \right\} \sigma^2_{y,\ell}$$

where $\phi_{y,\ell}(B) = \phi^*_\ell(B)\ \bar{\phi}^*_\ell(B)$

$$k^2_{y,\ell}\ \theta_{y,\ell}(B)\ \theta_{y,\ell}(F)\ \sigma^2_a = \left\{ \theta^*_\ell(B)\,\bar{\theta}^*_\ell(F)\,\bar{\phi}^*_\ell(B)\,\phi^*_\ell(F) + \bar{\theta}^*_\ell(B)\,\theta^*_\ell(F)\,\phi^*_\ell(B)\,\bar{\phi}^*_\ell(F) \right\} \gamma^2_\ell\, \sigma^2_a$$

where $k_{y,\ell}$ is some constant.

For instance, if $s = 12$, $\ell = 1$,

$$\phi^*_\ell(B) = (1 - \phi\, e^{i\pi/6} B)$$

$$\phi_{y,\ell}(B) = (1 - \sqrt{3}\,\phi B + \phi^2 B^2)$$

If $\phi = 1$, this is Box and Jenkins's trigonometric model.

$\theta_{y,\ell}(B)$ will be a real polynomial in B of degree $(p_\ell + q_\ell)$ where p_ℓ and q_ℓ are respectively $\phi_\ell(B)$ and $\theta_\ell(B)$'s degrees. It should be noted that as the sum corresponding to $\{\theta_{y,\ell}(B)\ \theta_{y,\ell}(F)\}\sigma^2_{y,\ell}$ is a complex polynomial plus its complex conjugate, $\theta_{y,\ell}(B)$ will not be of a degree smaller than $(p + q)$. The aggregation used to obtain $\theta_{y,\ell}(B)$ is possible since, as $\theta^*_\ell(B)\,\bar{\theta}^*_\ell(F)\,\bar{\phi}^*_\ell(B)\,\phi^*_\ell(F)$ and its conjugate are AGFs of complex moving average processes, their spectrum will be non-negative for all λ_ℓ, the sum of their spectrum will be non-negative, and, being real, symmetrical about the zero frequency, hence the sum of their AGFs will be the AGF of some real MA process and it can be written as the product of a real polynomial in B multiplied by the same polynomial in F. It is this polynomial which is denoted by $\theta_{y,\ell}(B)$.

$y_{t,\ell}$'s model can immediately be obtained as:

$$\phi_{y,\ell}(B)\, y_{t,\ell} = k_{y,\ell} \cdot \theta_{y,\ell}(B)\ a_{y,\ell,t}$$

where $a_{y,\ell,t}$ is a white noise process with variance σ^2_a.

As $x^*_{t,o}$ and $x^*_{t,s/2}$ are real processes, $y_{t,o} = x^*_{t,o}$ and $y_{t,s/2} = x^*_{t,s/2}$, hence the procedure presented below needs be applied only for $\ell = 1, 2, \ldots, (s/2) - 1$

Finally it should be noted that $\phi_{y,\ell}(B)$ is the product of quadratic simplifying operators with complex roots at the frequency λ_ℓ:

$$\phi_{y,\ell}(B) = \prod_{k=1}^{p_{\ell,1}} (1 - 2\rho_k \cos\lambda_\ell B + \rho_k^2 B^2)^{r_k} \prod_{k=1}^{p_{\ell,2}} (1 - 4\rho_k \cos\lambda_\ell \cos\gamma_k B - 2\rho_k(2\cos\lambda_\ell + \rho_k) B^2 - 4\rho_k^3 \cos\lambda_\ell \cos\gamma_k B^3 + \rho^4 B^4)^{r_k}$$

where $p_{\ell,1}$ is the number of distinct complex roots at the frequency λ_ℓ, r_k being the multiplicity of the kth such root and $p_{\ell,2}$ is the number of distinct complex roots in terms of both the frequencies λ_ℓ and γ, r_k being the corresponding multiplicity.

The aggregation of the $[(s/2)+1]$ $y_{t,\ell}$ into z_t is a straightforward operation, however two particular cases deserve attention.

First, let's consider the model, given s = 12,

$$\varphi_0(B)\, y_{t,0} = \theta_0(B)\, a_{t,0}$$

$$(1-\sqrt{3}\,\varphi_1 B + \varphi_1^2 B^2)\, y_{t,1} = \theta_{y,1}(B)\quad a_{t,1}$$

$$y_{t,\ell} = 0 \qquad \ell = 2,3,\ldots,6$$

$$z_t = \sum_{\ell=0}^{6} y_{t,\ell}$$

Then

$$K_z(B) = K_0(B) + K_{y,1}(B)$$

$$= \frac{\theta_0(B)\,\theta_0(F)}{\varphi_0(B)\,\varphi_0(F)}\,\gamma_0^2\,\sigma_a^2 + \frac{\theta_1(B)\,\theta_1(F)}{\varphi_1(B)\,\varphi_1(F)}\,\gamma_1^2\,\sigma_a^2$$

$$= \left\{ \frac{\theta_0(B)\theta_0(F)\varphi_1(B)\varphi_1(F)\gamma_0^2 + \theta_1(B)\theta_1(F)\varphi_0(B)\varphi_0(F)\gamma_1^2}{\varphi_0(B)\;\varphi_0(F)\;\varphi_1(B)\,\varphi_1(F)} \right\} \sigma_a^2$$

Hence z_t will be an $\left[p_0+2,\ \max\left(2+q_0,\, p_0+q_1\right)\right]$ ARMA process with

$$\varphi_0(B)\left(1-\sqrt{3}\,\varphi B + \varphi^2 B^2\right) z_t = \theta_z(B)\, a_{z,t}$$

i.e. a trigonemetric model similar to those developped by Box Jenkins, and Bacon, except that it is stationary.

Alternatively, a model generally similar to the CGN multiplicative model may be derived. Assuming that

$$\varphi_{y,\ell}(B) = \varphi_{1,\ell}(B)\,\varphi_{2,\ell}(B) \qquad \ell = 0,1,\ldots,s/2$$

where $\varphi_{1,\ell_1}(B)$ and $\varphi_{1,\ell_2}(B)$, $\ell_1, \ell_2 = 0,1,\ldots, s/2$, differ solely by the frequency of the roots, the arguments being the same for all seasonal frequencies, i.e. for instance $\ell_1 = 1$ and

and if $\ell_2 = 0$, then

$$\varphi_{1,0}(B) = (1 - \varphi B)$$

as long as $\varphi_{1,\ell}$ (B) has at least one root, assuming that $\gamma_\ell \neq 0$, $\ell = 0, 1, \ldots, s/2$ and regardless of γ_ℓ, $\varphi_{2,\ell}(B)$, $\ell = 0, 1, \ldots, s/2$, and $\Theta_{y,\ell}(B)$, $\ell = 0, 1, \ldots, s/2$ since

$$\gamma_t = \sum_{\ell=0}^{s/2} y_{t,\ell}$$

$$= \sum_{\ell=0}^{s/2} k_\ell \left(\frac{\Theta_{y,\ell}(B)}{\varphi_{1,\ell}(B)\,\varphi_{2,\ell}(B)} \right)$$

then γ_t's AGF will be

$$K_\gamma(B) = \sum_{\ell=0}^{s/2} k_\ell^2 \left(\frac{\Theta_{y,\ell}(B)\ \Theta_{y,\ell}(F)}{\varphi_{1,\ell}(B)\,\varphi_{2,\ell}(B)\,\varphi_{1,\ell}(F)\,\varphi_{2,\ell}(F)} \right) \sigma_a^2$$

The denominator of K_z (B) will contain an element

$$\prod_{\ell=0}^{s/2} \varphi_{1,\ell}(B)\,\varphi_{1,\ell}(F)$$

and, since this element contains the full compleemnt of the roots of unity for the arguments, it may be rewritten as a polynomial in B^s, with $\varphi_{1,\ell}$ (B)'s arguments at the sth power, i.e.

$$\prod_{\ell=0}^{s/2} \varphi_{1,\ell}(B)\,\varphi_{1,\ell}(F) = \Phi(B^s)\,\Phi(F^s)$$

where, if $(1 - \Phi_1 B)$ is an element of $\varphi_{1,0}(B)$, then $(1 - \Phi_1^s B^s)$ is an element of Φ (B^s).

Denoting $\prod_{\ell} \phi_{2,\ell}(B)$ by $\phi(B)$, the denominator of $K_z(B)$ may be rewritten as $\phi(B)\,\Phi(B^s)\,\Phi(F^s)\,\phi(F)$, i.e., since the numerator, as the sum of AGF's of MA processes, is the AGF of an MA process, denoting this numerator by $\theta(B)\theta(F)$,

$$K_z(B) = \left\{ \frac{\theta(B)\ \theta(F)}{\phi(B)\,\Phi(B^s)\,\Phi(F^s)\,\phi(F)} \right\} \nu^2 \sigma_a^2$$

$$\phi(B)\,\Phi(B^s)\, z_t = \theta(B)\,\nu\, a_{z,t}$$

z_t will be a stochastic process described by a CGN multiplicative seasonal model, and $\theta(B)$ will be a $\max_{\ell} \left\{ q_\ell + \sum_{\substack{k=0 \\ k\neq \ell}}^{s/2} p_k \right\}$ degree polynomial in B.

To illustrate the process, a very simple example may be given. Let

$$(1-\Phi B)\, y_{t,0} = a_{t,0}$$

$$(1+\Phi^2 B^2)\, y_{t,1} = 2(1+\Phi^2)\, a_{t,1}$$

$$(1+\Phi B)\, y_{t,2} = a_{t,2}$$

where $E(a_{t,0})^2 = E(a_{t,1})^2 = E(a_{t,2})^2 = \sigma_a^2$

then, given

$$z_t = \sum_{\ell=0}^{2} y_{t,\ell}$$

$$K_z(B) = \left\{ \frac{1}{(1-\Phi^4 B^4)(1-\Phi^4 F^4)} \right\} 4(1+\Phi^2)(1+\Phi^4)\,\sigma_a^2$$

i.e.

$$(1-\Phi^4 B^4)\, z_t = \nu_z \cdot a_{z,t}$$

where $E(a_{z,t})^2 = \sigma_a^2$

$$\nu_z^2 = 4(1+\Phi^2)(1+\Phi^4)$$

If now, z_t also includes a component y_t such that

$$(1-\varphi B)\, y_t = a_{y,t}$$

and $$E\,(a_{y,t})^2 = \nu_y^2 \cdot \sigma_a^2$$

then

$$K_y(B) = \nu_y^2 \left\{ \frac{-\Phi^4 B^4 - \varphi B + [2+\varphi^2+\Phi^8] - \varphi F - \Phi^4 F^4}{(1-\varphi B)(1-\Phi^4 B^4)(1-\Phi^4 F^4)(1-\varphi B)} \right\} \sigma_a^2$$

Denoting the numerator by $\theta(B)\,\theta(F)\,k_y^2$, y_t's model is obtained as a (1, 4) x (1, 0) multiplicative seasonal model:

$$(1-\varphi B)(1-\Phi^4 B^4)\, y_t = k_y\, \theta(B)\, a_{y,t}$$

To illustrate the role of a MA, it is sufficient to return to the original example, and assume, now,

$$E\,(a_{t,0})^2 = E\,(a_{t,2})^2 = Q^2\, E\,(a_{t,1})^2 = Q^2\, \sigma_a^2$$

then

$$K_y(B) = \left\{ \frac{(1-Q^2)\Phi^2 B^2 + (1+Q^2)(1+\Phi^4) + (1-Q^2)\Phi^2 F^2}{(1-\Phi^4 B^4)(1-\Phi^4 F^4)} \right\} 2(1+\Phi^2)\,\sigma_a^2$$

i.e. z_t will now be an ARMA (0, 2)×(4,0) multiplicative seasonal model rather than a AR (0)×(4), ... ∴ may be denoted by

$$(1-\Phi^4 B^4)\, y_t = (1-\theta^2 B^2)\, \nu_y \cdot a_{y,t}$$

4.3 Non-decomposable multiplicative seasonal models

It has been shown in the previous section that both the BJB trigonemetric seasonal model and the multiplicative seasonal models can be generated from the general seasonal model developped in the previous chapter. Even though, provided the assumption of seasonal stationarity is made, all general seasonal models can be expressed in either the trigonometric or the multiplicative form, the converse does not necessarily hold. This last contention is best established by using a simple example.

Let $s = 2$ and

$$(1-\Phi B)\, y_{t,0} = (1+B)\, a_{t,0}$$

$$(1+\Phi B)\, y_{t,1} = (1-B)\, a_{t,1}$$

$$E\left(a_{t,0}\right)^2 = E\left(a_{t,1}\right)^2 = \sigma_a^2$$

The two processes are non-invertible, hence they are smoothest (Hillmer, 1976; Hillmer and Tiao, 1978). Then z_t, where $z_t = y_{t,0} + y_{t,1}$,

can be derived to be

$$(1-\Phi^2 B^2)\, z_t = (1-\Theta B^2)\, \nu_z\, a_{z,t}$$

where $E\left(a_{z,t}\right) = \sigma_a^2$

and both Θ and ν_z are obtained by solving the quadratic polynomial in B^2,

$$2\left\{\phi B^2 + \left[1+\phi^2 + (1+\phi)^2\right] + \phi F^2\right\}$$

z_t is an invertible process since $y_{t,o}$ and $y_{t,1}$ are independent processes and since, as z_t's spectrum is the sum of their spectrum, and as $y_{t,o}$'s zero on the unit circle is at the frequency π, and $y_{t,1}$'s is at the frequency 0, the sum is positive at all frequencies. It follows that there exists a process v_t where

$$v_t = \gamma_t - \varepsilon_t$$

and where ε_t is some independent white noise process the variance of which is small enough to ensure that

$$K_\gamma(\lambda) - \sigma_\varepsilon^2 \geq 0 \qquad 0 \leq \lambda \leq \pi$$

Then there exists a polynomial $\theta_v(B)$ and a constant ν_v such that

$$K_v(B) = K_\gamma(B) - \sigma_\varepsilon^2$$

$$(1 - \Phi^2 B^2) v_t = \nu_v \cdot \theta_v(B)\, a_{v,t}$$

where

$$E(a_{v,t}) = \sigma_a^2$$

However v_t cannot be partitioned into processes equivalent to $y_{t,o}$ and $y_{t,1}$ since in fact $y_{t,o}$ and $y_{t,1}$ were already the smoothest processes one could obtain, hence z_t was the smoothest seasonal model which could be generated from the general class of seasonal model.

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