Fundamentals of

METROLOGICAL STATISTICS

Course given in 1986-7

by

M. Romanowski

Consulting metrologist

Department of Consumer and Corporate Affairs Canada
Product Safety Branch / Legal Metrology Branch
Holland Avenue, Ottawa, Canada
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FOREWORD

In 1971, I was given the responsibility of establishing a consumer product testing laboratory in the then Standards Branch of Consumer and Corporate Affairs. During this time I had frequent occasions to be in contact with Mr. Romanowski who, after his retirement from the Division of Physics of the National Research Council, was acting as consulting metrologist in the Standards Branch. Being a mathematician by education, Mr. Romanowski has always been highly interested in the role mathematical statistics plays in the analysis of the results of precise measurements, particularly from the standpoint of their accuracy and reliability. Since Mr. Romanowski possesses a very keen and inquisitive mind he became interested in the work performed in the new laboratory and we became involved in long discussions on the applications of statistical analysis to the measurement of the degree of inherent hazard in consumer products. These in-depth discussions produced a number of collaborative publications in various international journals.

During the last few years I became aware that the professionals in our laboratories have not acquired sufficient knowledge in the fields of calculus of probability and statistics from their respective universities and are therefore ill prepared to solve certain problems they encounter. In order to bridge the gap in their knowledge a proposal was made to Mr. J.W. Black, Director, Product Safety Branch (since retired) and Mr. R.G. Knapp, Director, Legal Metrology Branch, that Mr. Romanowski be requested to hold a workshop on statistics, in our laboratories. With the blessings from the two Directors I approached Mr. Romanowski and he, with some reluctance, agreed to hold a few workshops. These "few workshops" continued every week from April 1986 to June 1987.

The reader must bear in mind that the participants of the workshops have had very little preliminary knowledge of statistics and a poor understanding of the application of differential and integral calculus which, of course, plays a fundamental role in statistics. In spite of all these difficulties the author not only completed the course but compiled his lectures in a monograph form. This monograph is an attempt to adapt mathematical statistics to the kind of problems the present day metrologist encounters in his work and require a good knowledge of the advanced procedures of mathematical statistics.

K. Anwer Mehkeri
Special Advisor

PREFACE

In the near future the world will celebrate the second centennial of one of the most important and influential events in the world's history: the French Revolution. It is difficult to predict what aspects of this event will be the most described and discussed as it was so rich in great and glorious achievements and also in dismal failures and tragedies. It is however quite safe to foresee that one, particularly successful achievement is likely to be always quoted, namely the creation of the "Metric System".

If the year 1790 is considered as the first year of the "Metre-era", is it also legitimate to consider it as the beginning of a new branch of physical sciences, namely that of "metrology"? From a certain point of view, the answer is "yes". Although the problem of how to treat observations had already been raised half a century earlier (see Chapter II), the Metric System was so grandiose and ambitious that it deeply shook all scientific spheres. This system was supposed to be based on a so-called "natural unit", namely the length of an Earth Meridian, and its execution was supposed to be so accurate that it could be recommended for adoption "by all nations and for all times". This emphasized the role of geodesy which thus became closely connected with metrology and ... still is today.

It is very likely that the shock produced by the creation of the Metric System has been responsible for the generation of the wide spread interest in "accuracy" and "errors", in particular in "random errors". Now, the notion of "randomness" automatically leads to that of "probability". Up to the middle of the 19th century the term "probability" was used almost exclusively in the discussions concerning games of chance, particularly those which use dice and cards. The geodesists and the metrologists had to fabricate their own theory of probability in order to be able to apply it in the treatment of the observed results. While some of them were engaged in the herculean task of measuring the length of meridians, those who were more mathematically oriented (Legendre, Gauss, Hagen) were constructing the probabilistic theory of observational errors. It was not until the end of the first quarter of the nineteenth century that this theory took the form that remained practically unchanged until the beginning of the twentieth century when it became an integral part of a vast new scientific domain which, perhaps not very adequately, is termed now "Statistics".

The text that follows treats that part of this Statistics which is the most useful to those who take care of the objects that are called "Standards" and are of such importance to the activities of any nation.

The reader, when perusing the monograph, must take into consideration that it is neither a treatise, nor a textbook; in any case it is not a meticulously written, well polished product. It is presented to the reader exactly as it was presented to the participants of the workshop. It is a sort of living matter with all its peculiarities and imperfections. The author apologizes for all the inconveniences this may cause to the reader and solicits his magnanimity. He is, naturally, anxious to know the opinions of all those who may enter in contact with this, rather unorthodox, selection of topics in Statistics. Being the quintessence of a personal experience, this selection may be of interest to other metrologists-statisticians. The author has here particularly in mind those scientists of the younger generations who try to find their way in the modern metrology which is, no doubt, a rather peculiar brand of applied physics.

M. Romanowski

ACKNOWLEDGEMENTS

The authors of the Foreword and of the Preface offer their sincere thanks to Mr. Andrew Babcock (who was a participant of the workshop) for undertaking the arduous task of typing the notes distributed during the lectures, and then, for assisting the author to compile them into a coherent book form.

Special thanks are expressed to Dr. Carl Swail, from the N.R.C. National Aeronautical Establishment, who authorized Mr. Babcock to use the N.A.E. computer for printing the manuscripts of these lectures and the final manuscript of the book.

K.A.M. and M.R.

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PARS PRIOR

AUCTORE

CAROLO FRIDERICO GAUSS

SOCIETATI REGIAE SCIENTIARUM EXHIBITA 1821. FEBR. 15.

Commentationes societatis regiae scientiarum Gottingensis recentiores. Vol. V.
Gottingae MDCCCXXIII.

The famous work on the role calculus of probability plays in the adjustment of observations

Chapter I
General Theory

Section 1. Axioms and Basic Laws

The object of the calculus of probability (as a branch of mathematics) is a simple thing called an "event" or "trial" which possesses the specific property of having a result which can be either a "success" (positive) or a "failure" (negative). The result must be absolutely clear and, although the definition of the term "success" is purely arbitrary, it must be unambiguous.

The term "probability" designates a primary notion i.e. it belongs to the same category of terms as "point", "time", "space" etc. These terms designate those entities which cannot be defined in terms of simpler or more basic ones. They are directly created and conceived by the human mind.

The events are considered either individually or as forming groups designated by terms such as "series" or "sets" of trials. The study of the results of events starts with a certain number of axioms. An *axiom* is a statement that does not have to be demonstrated by means of some more basic entities. It is considered as self-evident and is substantiated by all experimental facts observed by mankind since time immemorial.

First Axiom

This axiom, often considered as the most important and fundamental, states that in certain events all possible results are *equiprobable*. A typical case is the throwing of a perfectly cubical die on a flat table top: all faces of the die are equiprobable.

Second Axiom

This axiom states that in certain well defined cases and in spite of the fact that the first axiom cannot be applied, each result of a trial possesses a fixed probability. A typical case is here the throwing of a rigid irregular die.

Third Axiom

In those events which conform either to the first or to the second axioms, the probability of a successful result is expressed numerically by the limit to which tends the ratio of the number of successes to the total number of events, when the latter becomes larger and larger. For instance, if in the throwings of an irregular die, a large number n of throwings are performed, of which m are considered as favorable, (the coming of a selected face) then the approximate value of the chance of obtaining a success is equal to $\frac{m}{n}$. The probability p is then defined as being equal to:

$$p = \lim_{n \to \infty} \frac{m}{n}$$

and it is considered as a number of the same nature as the irrational numbers e, π etc.

Many different problems may be examined in which the notion of probability plays an important role. Some of them can be treated by the axiom of equiprobality but some may be analysed only by means of the second and the third axioms.

There are two basic theorems on probability: the theorem of *total probability* and the theorem of *compound probability.* They are directly and easily deducible from the axioms.

Theorem of total probability

If in an event the success may be obtained in several different (mutually exclusive) ways, the probability of a success is equal to the sum of probabilities of a success in all possible ways. This theorem is an immediate consequence of the third axiom and is self-evident in all cases involving equiprobability.

Theorem of compound probability

This theorem requires some introductory remarks. An "event" does not always consist of one single trial. It may contain several distinct trials each of which possesses its own specific definition of the success. Compound events can be divided into two classes:

a) events in which the probability of a success in each component trial is totally independent of the results obtained in all other trials.

b) events in which there are relations between the probabilities attached to the component trials.

Let us examine the following example. A box contains 10 objects: 7 black and 3 white. The probability of drawing a black or a white object are thus 7/10 and 3/10, respectively. If, after the first draw, the object is put back into the box, the probabilities in a second draw remain the same as in the first draw. But, if the object is not put back after the first draw, the probabilities in a second draw are modified. If, for instance, the first draw gave a black object, the probability of obtaining a second black object is 6/9; but if the first draw gave a white object, the probability of a black object in the second draw is 7/9. The trials are not independent. In the sequel we will have to deal mainly with independent trials so let us examine this case first.

Now, an event will consist of two consecutive trials (drawing with replacement) the success being, by definition, the obtention of two consecutive black objects. If a very large number n of events are performed, then the number m' of events in which the first trial gave a black object is, with a high approximation:

$$m' = \frac{7}{10} \times n.$$

Among all m' events, the number of events in which the second draw has also brought a black object is designated by m'' and is equal to

$$m'' = \frac{7}{10} \times m'.$$

Hence,

$$m'' = \frac{7}{10} \times \frac{7}{10} \times n = \frac{49}{100} n = 0.49n .$$

This relation is readily adapted to the case where the object is not put back into the box. Clearly here:

$$m'' = \frac{6}{9} \cdot m'$$

$$m'' = \frac{7}{10} \times \frac{6}{9} \cdot n = \frac{42}{90} \cdot n = 0.467n .$$

All this is summarized as follows in the second theorem:

The probability of a success in an event that consists of two trials, is the product of the probability of a success in the first trial and the probability of a success in the second trial, the second probability being established on the assumption that the first trial has been successful.

The second part of this theorem may be simply omitted if the trials are independent.

Section 2. Introduction to the Theory of Permutations

Suppose that we have a deck of 12 cards numbered 1, 2, ... 12. Well shuffled and placed in a row they form a "permutation". Any interchange of any two cards leads to a new permutation. It can be readily shown that the number of different permutations that it is possible to form is equal to:

$$N = 1 \times 2 \times 3 \times \cdots \times 11 \times 12 = 12\,!$$

The general formula for k cards numbered 1, 2, $\cdots$ $k-1$, k is

$$N = k!\,.$$

If we remove one of the cards, say, 12 and replace it by any of the other numerals, eg. 5, the number of permutations will become $\frac{12!}{2}$ as the interchange of two identical 5's does not create a new permutation. If we remove the card 11 and replace it by another 5 the number of permutations will further decrease to $\frac{12!}{2\times 3} = \frac{12!}{3!}$.

This process of increasing the number of times the numeral 5 is repeated can be continued and it will lead to the following convenient and simple rule: if in a set of $N=12$ symbols, a certain symbol is repeated t times the number of permutations becomes equal to $\frac{12!}{t!}$.

Suppose now that the number of different symbols is reduced to two: 3 and 8. One of the permutations may be, for instance,

3 8 3 3 8 3 8 3 8 8 8 8

so that "3" figures five times and "8" figures seven times. The number of permutations will be equal to

$$\frac{12!}{5!7!} = 792.$$

Note that, if all 12 cards bear the same symbol, the number of permutations is reduced to

$$\frac{12!}{12!} = 1\,.$$

If all symbols are different from each other, then, of course,

$$\frac{12!}{1!\times 1!\times \cdots \times 1!} = \frac{12!}{1} = 12!$$

Section 3. Bernoulli Trials*

The term "Bernoulli trials" is used to designate a set of a fixed number of consecutive trials (the whole set is then called an "event"). All trials of a set have the same constant probability p to produce a success. The most common way of illustrating Bernoulli trials is to make a deck of, say $k = 12$ identical cards each of which has a "plus" (+) on one side and a minus (−) on the opposite side. To produce complete randomness it is recommended to throw the deck in a large box and shake it vigorously.

Suppose that, in a row, the succession of symbols contains 4 plusses and 8 minusses:

$$+ - + - - - + - - + - -$$

As here the probability of a "plus" is $p = \frac{1}{2}$ and that of a "minus" is also $q = 1-p = \frac{1}{2}$, the theorem of compound probability indicates that the probability

* James Bernouilli (1654 - 1705) Swiss mathematician (Basel) of Dutch origin.

of the row is equal to

$$P = \frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2} = \left(\frac{1}{2}\right)^{12} = \frac{1}{4096} = 0.000244 .$$

It must be strongly underlined that all rows of a constant number of cards (here $k = 12$) are equiprobable.

According to the formula of permutations the number of combinations is here $\frac{12!}{4!\,8!}$. The general expression for this number denoted by the symbol N_X is

$$N_X = \frac{k!}{X!(k-X)!} , \qquad ...(1)$$

k being the number of trials (number of cards) and X being the number of plusses (number of successes).

The combination of the theorem of total with that of compound probability leads to the following expression of the probability of obtaining X plusses in a set of k Bernoulli trials:

$$P_X = \frac{k!}{X!(k-X)!} p^X (1-p)^{k-X} . \qquad ...(2)$$

All trials possessing the same X form a "class" that is denoted by the symbol such as (4 +, 8 −) and represented by its "standard form":

$$+ + + + - - - - - - - -$$

Numerical Example: $k = 12, x = 0, 1, 2, ...5, 6$.

$N_0 = 1 \qquad P_0 = 0.0002$

$N_1 = 12 \qquad P_1 = 0.003$

$N_2 = 66 \qquad P_2 = 0.016$

$N_3 = 220 \qquad P_3 = 0.054$

$N_4 = 495 \qquad P_4 = 0.121$

$N_5 = 796 \qquad P_5 = 0.193$

$N_6 = 924 \qquad P_6 = 0.226$.

At this point it is necessary to introduce an important term *viz.* that of *variate*. This term designates that *variable the probability of which is a function of its value*. We will be concerned only with functions having simple algebraic forms.

The method for treating the problem of the largest group (practically adequate for simple cases of Bernoulli trials) is given in Appendix I. It is legitimate only for obviously unimodal distributions of probabilities. Here, it will be sufficient to say that the largest group is the one in which X_m satisfies, as closely as possible, the equation

$$\frac{X_m}{k-X_m} = \frac{p}{1-p}.$$

Hence,

$$X_m = kp .$$

A rigorous relation (as indicated in Appendix I) is

$$kp+p > X_m > kp-q .$$

It is now appropriate to make a very important remark on X_m. Bernoulli trials constitute a fundamental background to the theory of random errors, particularly through the properties they acquire when k grows to infinity. It does not matter what kind of number is k and how it tends towards infinity. It can be legitimately assumed, without creating any loss of generality, that (for a specific value of p) the value of kp remains an integer. For instance, if $p = \frac{1}{2}$ we can assume that k is an even number and remains always even; if $p = \frac{1}{6}$, we assume that k stays always divisible by 6, etc. Of course, if kp is an integer then kq is also an integer.

The quantity V, defined by the relation

$$V = X-X_m = X-kp , \qquad \text{...(3)}$$

is termed *deviation* (i.e. deviation from its most probable value). The formulae (1) and (2) can now be given the following forms:

$$N_V = \frac{k!}{(kp+V)!(kq-V)!} \quad \text{...(4)}$$

$$P_V = \frac{k!}{(kp+V)!(kq-V)!} p^{(kp+V)} q^{(kq-V)} . \quad \text{...(5)}$$

It is under this form that the expression P_V will be used in the theory of errors as this theory has been established by Hagen. With $p = q = \frac{1}{2}$, (5) becomes

$$P_V = \frac{k!}{\left(\frac{k}{2}+V\right)!\left(\frac{k}{2}-V\right)!}\left(\frac{1}{2}\right)^k . \quad \text{...(6)}$$

It is strongly recommended to avoid ambiguous expressions and symbols. Thus, for instance a symbol such as P_0 may mean either $P(X = 0) or P(V = 0)$. Also, concerning k, it must be noted that, except in some cases, (which should be clearly identified) k is always a large number. Thus the question whether kp is (or is not) an integer is irrelevant. It will be always assumed that kp is numerically rounded to the nearest integer value.

Section 4. Binomial Expansion

A remarkable feature of Bernoulli trials is that they can be so directly related to the binomial expansion. Let us, as an introduction, consider the formula:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 .$$

It can be written as follows:

$$(a+b)^3 = \frac{3!}{0!(3-0)!} a^3b^0 + \frac{3!}{1!(3-1)!} a^2b^1 + \frac{3!}{2!(3-2)!} a^1b^2 + \frac{3!}{3!(3-3)!} a^0b^3$$

and is identical to the form above, since according to algebra, $0! = 1$.

According to the general Binomial Theorem (established by Newton) the above development can be generalized for any value of the exponent. Comparing the forms of the second terms of $(a+b)^3$ with the terms of the formula (5) one readily notices the reasons why it is of a particular interest to consider the expansion of the expression $(q+p)^k$.

The expansion can be presented in the following classical form:

$$(q+p)^k = \frac{k!}{0!(k-0)!}\, p^0 q^k + \frac{k!}{1!(k-1)!}\, p^1 q^{k-1} + \frac{k!}{2!(k-2)!}\, p^2 q^{k-2} + \cdots$$

$$+ \frac{k!}{(k-2)\ 2!}\, p^{k-2} q^2 + \frac{k!}{(k-1)!\ 1!}\, p^{k-1} q^1 + \frac{k!}{(k-0)!\ 0!}\, p^k q^0 \qquad \text{...(7a)}$$

Each term can be simplified and the expansion takes a form that is easy to remember:

$$(q+p)^k = q^k + \frac{k}{1}\, pq^{k-1} + \frac{k(k-1)}{1\times 2}\, p^2 q^{k-2} + \frac{k(k-1)(k-2)}{1\times 2\times 3}\, p^3 q^{k-3} + \cdots$$

$$+ \frac{k(k-1)(k-2)}{1\times 2\times 3}\, p^{k-3} q^3 + \frac{k(k-1)}{1\times 2}\, p^{k-2} q^2 + \frac{k}{1}\, p^{k-1} q + p^k \qquad \text{...(7b)}$$

By comparing each term of (7a) with the relation (2) it is easy to notice that if, in k Bernoulli trials p is the probability of a "plus", the series (7a) or (7b) are equivalent to the fundamental relation:

$$(q+p)^k = P_0 + P_1 + P_2 + \cdots + P_k = \sum_{X=0}^{k} P_X = 1 \,. \qquad \text{...(8)}$$

In accord with the definitions presented in *Appendix II* the expression for the moment $\omega_1 = \bar{x}$ can be written in the form:

$$\omega_1 = 0P_0 + 1P_1 + 2P_2 + \cdots + kP_k = \sum_{X=0}^{k} XP_X = 1.$$

Therefore (as X takes all the values from $X = 0$ to $X = k$):

$$\omega_1 = 1\frac{k}{1}\, pq^{k-1} + 2\frac{k(k-1)}{1\times 2}\, p^2 q^{k-2} + 3\frac{k(k-1)(k-2)}{1\times 2\times 3}\, p^3 q^{k-3} + \cdots$$

$$+ (k-3)\frac{k(k-1)(k-2)}{1\times 2\times 3}\, p^{k-3} q^3 + (k-2)\frac{k(k-1)}{1\times 2}\, p^{k-2} q^2 + (k-1)\frac{k}{1}\, p^{k-1} q + kp^k \,.$$

The factor kp is common to all terms in the development and thus can be put out of the brackets:

$$\omega_1 = kp\left[q^{k-1} + \frac{k-1}{1}pq^{k-2} + \frac{(k-1)(k-2)}{1\times 2}p^2q^{k-3} + \dots\right.$$

$$+ \frac{(k-1)(k-2)(k-3)}{1\times 2\times 3}p^3q^{k-4} + \frac{(k-1)(k-2)(k-3)}{1\times 2\times 3}p^{k-4}q^3 + \dots$$

$$\left.\dots \frac{(k-1)(k-2)}{1\times 2}p^{k-3}q^2 + \frac{(k-1)}{1}p^{k-2}q + \frac{k}{1}p^{k-1}\right].$$

Close examination of the expression in the square brackets shows that it represents the development of the binomial $(q+p)^{k-1}$. It is therefore equal to unity and, thus,

$$\omega_1 = kp = \bar{x}. \qquad \dots(9)$$

This confirms that the mean and the first moment are identical quantities. The reader should perform all calculations in detail on a numerical case, such as e.g. $(q+p)^7$ i.e. $k=7$. The common factor that can be put out of the brackets is then equal to $kp=7p$. The final results will be $\omega_1 = kp = 7p(q+p)^6 = 7p$.

The calculation (by the same procedure) of the second moment ω_2, i.e.

$$\omega_2 = \sum_{X=0}^{X=k} = X^2P_X, \qquad \dots(10)$$

is much more complicated. It will not be described here but in the Appendix III. The result is quite simple:

$$\omega_2 = kp[(k-1)p+1], \qquad \dots(11)$$

$$\omega_2 = (kp)^2 - kp^2 + kp.$$

The relation (II, 8) by means of which we can calculate the second moment of X with respect to $\omega_1 = kp$ is

$$\mu_2 = \omega_2 - \omega_1^2,$$

$$\mu_2 = (kp)^2 - kp^2 + kp - (kp)^2,$$

$$\mu_2 = kp(1-p),$$

$$\mu_2 = kpq \ . \qquad ...(12)$$

This is an important formula; it will be constantly used in the sequel.

Important Remark

Note that in Sections 1, 2, 3 and 4 of Chapter I, the symbols designating variable quantities such as X, X_0, kp, V etc. are assumed to represent integers. In certain numerical problems, this point (as in the problem treated in Appendix I) must be treated appropriately. In theoretical deductions however, starting with Section 5, these variable quantities become larger and larger so that it does not matter whether they are integers or not. And it is always permissible to assume that their numerical values are such that the derived quantities, involved in the analysis, are also integers. Thus, for instance, when $p=q=\dfrac{1}{2}$ we can assume that k is a (large) even number.

Section 5. Asymptotic Expressions

The calculations with various expressions based on the binomial expansions would be almost impossible when the number of trials is large. This domain of mathematics has been totally transformed by the discovery by James Stirling (18th century) of his famous formula that expresses factorials in terms of exponentials. Stirling's formula is:

$$k! = \left(\frac{k}{e}\right)^k \sqrt{2\pi k} \qquad ...(13)$$

in which e is the transcendental number defined by the converging series:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.71828 \ .$$

e is the base of the so-called "natural" logarithms.

The transformation of expressions such as (5) is far from being a simple operation. It involves various expansions (Taylor's expansions) accompanied by delicate evaluations of the orders of magnitude. The remarkably simple results to which leads this serious mathematical operation is:

$$P_V = \frac{1}{\sqrt{2\pi}\sqrt{kpq}} e^{-\frac{V^2}{2kpq}} . \qquad ...(14)$$

The reader is reminded that V is the deviation of the variable from its mean kp which, without making a significant error, can be rounded to an integer value. In conformity with (12), P_V can also be written:

$$P_V = \frac{1}{\sqrt{2\pi}\sqrt{\mu_2}} e^{-\frac{V^2}{2\mu_2}} . \qquad ...(15)$$

In all applications of these formulae which will be made in the sequel, we will not be interested in the probability of one specific value of V but in the probability that a deviation will lie between certain narrow limits V' and V'' $(V'' > V')$. Let us write:

$$V'' = V' + \Delta V . \qquad ...(16)$$

The law of total probability indicates that the probability that V will take any of the values $P_{V'}, P_{V'+1}, P_{V'+2}, \cdots P_{V'+\Delta V}$ is equal to the sum of these values. If ΔV is small, then these values are not very different from each other and it may be assumed that their sum is equivalent to:

$$\Delta P_V = \Delta V \, P_m , \qquad ...(17)$$

P_m being the probability of a value located in the middle of ΔV or, in any case, close to:

$$V_m \approx \frac{1}{2}(V'' + V'). \qquad ...(18)$$

Thus ΔP_V denotes the probability that the variable V will be located somewhere inside the interval ΔV. The fact that P_m must be multiplied by the "length"* of the interval

* Note that here the term "length" (of an interval) means the number of discrete values V located between V' and V''.

ΔV suggests that it be called the "probability density" in the interval ΔV.

Let us now re-write (16) under the form:

$$\Delta P_V = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{V}{\sqrt{\mu_2}}\right)^2} \frac{\Delta V}{\sqrt{\mu_2}} . \qquad \text{...(19)}$$

It shows that the probability that a deviation V will fall into an interval ΔV is a function of the ratios of V and ΔV to $\sqrt{\mu_2}$. The quantities

$$v = \frac{V}{\sqrt{\mu_2}}, \qquad \Delta v = \frac{\Delta V}{\sqrt{\mu_2}} \qquad \text{...(20)}$$

are termed *reduced* values of V and ΔV, respectively. The reduced values are thus obtained by using as unit the square root of the second moment. So that, finally:

$$\Delta P_v = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \Delta v . \qquad \text{...(21)}$$

While the theory of Bernoulli trials starts with an essentially integer number X, in the asymptotic expressions based on Stirling's formula all variables are no longer integers but should be considered as of a discrete nature.

Numerical Examples

A few numerical applications will give an idea of the orders of magnitude of various quantities encountered in preceding sections. As a first example of the Bernoulli trials let us treat the case where $k = 10000$, $p = q = \frac{1}{2}$ and calculate the probability that the deviation V from the most probable value $kp = 5000$ will fall into the interval between $V' = 45$ and $V'' = 55$.

The centre of the interval $\Delta V = 55-45 = 10$ is at $V_m = \frac{1}{2}(55+45) = 50$. The second moment μ_2 is equal to:

$$\mu_2 = kpq = \frac{10000}{2 \times 2} = 2500 \, ,$$

$$\sqrt{\mu_2} = 50 \, .$$

The reduced value v_m of V_m is therefore:

$$v_m = \frac{V_m}{\sqrt{\mu_2}} = \frac{50}{50} = 1 \, .$$

and

$$\frac{v_m^2}{2} = 0.5, \qquad \Delta v = \frac{\Delta V}{\sqrt{\mu_2}} = \frac{10}{50} = 0.2 \, .$$

The relation (21) is here

$$\Delta P_v = \frac{1}{\sqrt{2\pi}} \, e^{-0.5} \times 0.2 \, .$$

There are tables for the expression $\frac{1}{\sqrt{2\pi}} \, e^{-\frac{v^2}{2}}$ as a function $\phi(v)$. We shall find in these tables that: $\phi(v_m = 1) = 0.24$, hence:

$$\Delta P_v = 0.24 \times 0.2 = 0.048.$$

Thus the probability that in a throw of 10000 cards the deviation from 5000 will fall in the interval between $V = 45$ and $V = 55$ is of the order of 4.8 percent.

For $v = 0$ (central interval, between $V = -5$ to $V = +5$), $\phi(0) = 0.40$ so that the probability is of the order of $0.40 \times 0.2 = 0.08$, i.e. 8 percent.

It is interesting to notice that this probability is decreasing very fast with increasing v. When $v = 3$, then $\phi(3) = 0.0044$ and therefore

$$\Delta P_v = 0.0044 \times 0.2 = 0.00088 = 0.088 \text{ percent.}$$

At $v = 4$, $\phi(4) = 0.0020$.

The function:

$$\phi(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$$

is often called simply the "*normal function*". More appropriately, it should be termed "***probability density function (pdf) of the normal variate***". The probability that v will fall into Δv is

$$\Delta P_v = \phi(v)\Delta v \,. \qquad ...(22)$$

Table of the Function $\phi(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$

v	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.3989	.3989	.3989	.3988	.3986	.3984	.3982	.3980	.3977	.3973
0.1	.3970	.3965	.3961	.3956	.3951	.3945	.3939	.3932	.3925	.3918
0.2	.3910	.3902	.3894	.3885	.3876	.3867	.3857	.3847	.3836	.3825
0.3	.3814	.3802	.3790	.3778	.3765	.3752	.3739	.3725	.3712	.3697
0.4	.3683	.3668	.3653	.3637	.3621	.3605	.3589	.3572	.3555	.3538
0.5	.3521	.3503	.3485	.3467	.3448	.3429	.3410	.3391	.3372	.3352
0.6	.3332	.3312	.3292	.3271	.3251	.3230	.3209	.3187	.3166	.3144
0.7	.3123	.3101	.3079	.3056	.3034	.3011	.2989	.2966	.2943	.2920
0.8	.2897	.2874	.2850	.2827	.2803	.2780	.2756	.2732	.2709	.2685
0.9	.2661	.2637	.2613	.2589	.2565	.2541	.2516	.2492	.2468	.2444
1.0	.2420	.2396	.2371	.2347	.2323	.2299	.2275	.2251	.2227	.2203
1.1	.2179	.2155	.2131	.2107	.2083	.2059	.2036	.2012	.1989	.1965
1.2	.1942	.1919	.1895	.1872	.1849	.1826	.1804	.1781	.1758	.1736
1.3	.1714	.1691	.1669	.1647	.1626	.1604	.1582	.1561	.1539	.1518
1.4	.1497	.1476	.1456	.1435	.1415	.1394	.1374	.1354	.1334	.1315
1.5	.1295	.1276	.1257	.1238	.1219	.1200	.1182	.1163	.1145	.1127
1.6	.1109	.1092	.1074	.1057	.1040	.1023	.1006	.0989	.0973	.0957
1.7	.0940	.0925	.0909	.0893	.0878	.0863	.0848	.0833	.0818	.0804
1.8	.0790	.0775	.0761	.0748	.0734	.0721	.0707	.0694	.0681	.0669
1.9	.0656	.0644	.0632	.0620	.0608	.0596	.0584	.0573	.0562	.0551
2.0	.0540	.0529	.0519	.0508	.0498	.0488	.0478	.0468	.0459	.0449
2.1	.0440	.0431	.0422	.0413	.0404	.0395	.0387	.0379	.0371	.0363
2.2	.0355	.0347	.0339	.0332	.0325	.0317	.0310	.0303	.0297	.0290
2.3	.0283	.0277	.0270	.0264	.0258	.0252	.0246	.0241	.0235	.0229
2.4	.0224	.0219	.0213	.0208	.0203	.0198	.0194	.0189	.0184	.0180
2.5	.0175	.0171	.0167	.0163	.0158	.0154	.0151	.0147	.0143	.0139
2.6	.0136	.0132	.0129	.0126	.0122	.0119	.0116	.0113	.0110	.0107
2.7	.0104	.0101	.0099	.0096	.0093	.0091	.0088	.0086	.0084	.0081
2.8	.0079	.0077	.0075	.0073	.0071	.0069	.0067	.0065	.0063	.0061
2.9	.0060	.0058	.0056	.0055	.0053	.0051	.0050	.0048	.0047	.0046
3.0	.0044	.0043	.0042	.0040	.0039	.0038	.0037	.0036	.0035	.0034
3.1	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026	.0025	.0025
3.2	.0024	.0023	.0022	.0022	.0021	.0020	.0020	.0019	.0018	.0018
3.3	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014	.0013	.0013
3.4	.0012	.0012	.0012	.0011	.0011	.0010	.0010	.0010	.0009	.0009
3.5	.0009	.0008	.0008	.0008	.0008	.0007	.0007	.0007	.0007	.0006
3.6	.0006	.0006	.0006	.0005	.0005	.0005	.0005	.0005	.0005	.0004
3.7	.0004	.0004	.0004	.0004	.0004	.0004	.0003	.0003	.0003	.0003
3.8	.0003	.0003	.0003	.0003	.0003	.0002	.0002	.0002	.0002	.0002
3.9	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0001	.0001

$$v = 0.8435$$

$$\left.\begin{aligned}\phi(0.84) &= 0.3251\\ \phi(0.85) &= 0.3230\end{aligned}\right\} \quad \delta_\phi = 0.0024 \qquad \frac{35}{100}\times 21 \approx 7$$

$$\phi(0.8435) = 0.3251 - 0.0007 = 0.3244$$

Chapter II
Basic Theory of Random Errors

Section 1. Introduction. Hagen's Theory

There are in the history of errors in observations three particularly important dates.

1755: publication by Thomas Simpson of a "Letter" in Phil. Trans. Roy. Soc., vol 49, part I (pp. 82-93). In this letter, the author recommends that when a measurement is repeated several times, *all* results should be taken into account and not only those which seem to be "good". He also recommends that the *mean* of all measurements be explicitly recognized as the "best approximation" to the measured quantity. Obviously, Simpson had a "statistically structured" mind.

1809: publication by F. Gauss of "Theoria Motus", Hamburg. (English modern translation: Dover Pub.) In this book, one of the most important and famous in the history of science, Gauss treats the problem of errors in a very particular manner that is not founded directly on the binomial theory, but on the role the mean plays in large samples of repeated observations.

1837: publication of "Grundzüge der Wahrscheinlichkeitsrechnung" by G.H.L. Hagen, Berlin 1837 (2nd edition in 1867). Although various authors have expressed, well before Hagen, the idea that an accidental error results from a combination of a large number of very small errors, the great merit of Hagen is that he gave this idea a clear and rigorous mathematical form. Thus it became possible to apply to the study of random errors the methods of mathematical analysis that constitute the basis of the calculus of probability.

Hagen was certainly an outstanding engineer and mathematician. His book must have had a serious impact on the scientific spheres of his time. Perhaps Hagen's work was partially eclipsed by his illustrious contemporary, Gauss.

Before we start the analysis of Hagen's theory, it is appropriate to make the following general remarks.

a) A sample of "*repeated*" measurements (or measurements that are collected by a process which is equivalent to "repetition" *e.g.* formation of loops in geodesy) must be of the highest possible quality. This implies that all scientific knowledge available at the date of the measurements has been fully put into action. Here, the personality of the observer(s) plays a predominant role and, as such, cannot be intrinsically evaluated. The goodness of a metrological operation is the result of an enormous number of partial efforts and endeavours.

b) Although Hagen does not express it explicitly, the structure of his ideas shows that he accepts the concept of "true value" of the measured quantity. This concept is more fully analysed in Appendix V. Here we shall simply adhere to a sort of general consensus that such a concept can be used in the discussions that follow.

Let us assume that the repeated measurements of a stable physical quantity, yielded a set of N results $m_i (i=1, 2, \cdots N)$. If these numbers are represented by points on an axis (origin $m = 0$), a simple inspection will show that these points m_i will form a dense cluster in a clearly visible region. The density of points will decrease with the distance from the cluster.

The probability that an additional point will be close to the cluster is larger than the probability that it will be far from it. There is therefore a relation between the magnitude of a result and its probability of occurence.

Hagen's theory is based on the following fundamental assumptions:

1. Every measurement is disturbed by a very large number of small errors termed *elementary* errors.
2. All elementary errors are of the same magnitude.
3. Every elementary error has the same chance to be positive as to be negative.

One readily notices that all theorems concerning elementary errors and their combinations will be deducible from those established for Bernoulli trials. Suppose now that each card of the pack of k cards bears the symbol $+\frac{\epsilon}{2}$ on one side and $-\frac{\epsilon}{2}$ on the other side. In a set containing X plusses, the deviation V is equal to $V = X - kp$ and the sum H of all symbols $\pm\frac{\epsilon}{2}$ will be equal to:

$$H = \left(+\frac{\epsilon}{2}\right) X + \left(-\frac{\epsilon}{2}\right) (k-X) .$$

As $X = V+kp$, we have

$$H = \epsilon\left[V+\frac{k}{2}(2p-1)\right]$$

and, because $p = \frac{1}{2}$, this gives the simple relation

$$H = \epsilon V \qquad ...(23)$$

by means of which the expression ΔP_H will be deduced from that of ΔP_V .

This is readily done as follows. The exponent of e in (19) is put in the form

$$\left(\frac{V}{\sqrt{\mu_2}}\right)^2 = \frac{(\epsilon V)^2}{(\epsilon\sqrt{\mu_2})^2} = \frac{H^2}{\epsilon^2\mu_2}$$

so that it is easy to notice that the denominator $\epsilon^2\mu_2 = \epsilon^2 kpq$ is the second moment of $H = \epsilon V$. This moment will, in all further expressions, be designated by the symbol S^2 (capital letter):

$$S^2 = \epsilon^2\mu_2 = \epsilon^2 kpq . \qquad ...(24)$$

If $p = q = \frac{1}{2}$,

$$S^2 = \frac{\epsilon^2 k}{4}, \qquad S = \frac{\epsilon\sqrt{k}}{2} \qquad ...(25)$$

and, therefore,

$$P_H = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{H^2}{2S^2}}\, \frac{1}{S}; \quad \Delta P_H = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{H^2}{2S^2}}\, \frac{\Delta H}{S} . \qquad ...(26)$$

Hagen's assumptions automatically lead to our examining what happens when k tends toward ∞ and therefore ϵ tends toward 0. If such hypotheses have to lead to experimentally meaningful results, then it is necessary to postulate that the following double condition must be satisfied:

$$\begin{matrix} k \to \infty \\ \epsilon \to 0 \end{matrix} \left\{ \begin{matrix} \lim H = \text{finite}. \\ \lim \epsilon^2 k = \text{finite}. \end{matrix} \right.$$

The symbols for these limits are:

$$\lim H = x , \qquad ...(27)$$

$$\lim \frac{\epsilon^2 k}{4} = \sigma^2 \text{ (variance)} . \qquad ...(28)$$

The variable x must now be treated as *continuous* and so is also the function $f(x)$ the form of which is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} . \qquad ...(29)$$

By similarity with the function which defines the variate V, it is termed the "*probability density function*" (pdf) in the vicinity of the value x. The probability dP_x that x will fall into dx is therefore equal to

$$dP_x = f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx . \qquad ...(30)$$

In literature, $f(x)$ is often called simply the "normal curve" or "Gaussian curve". Similarly to the transformation of V into v, the variate x is transformed into the reduced variate λ by the relation

$$\lambda = \frac{x}{\sigma} . \qquad ...(31)$$

Then also:

$$d\lambda = \frac{dx}{\sigma} , \qquad ...(32)$$

and (30) becomes

$$dP_\lambda = \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} d\lambda . \qquad ...(33)$$

The numerical values of the expression

$$\psi(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}} \qquad ...(34)$$

are tabulated*. The central value $\psi(0)$ is $\psi(0) = 0.3989$. If, for instance $\Delta x = \frac{\sigma}{10}$, then $\Delta\lambda = 0.1$ and therefore

$$dP_\lambda = 0.3989 \times 0.1 = 0.03989 .$$

Thus the probability that x will fall into the central interval Δx, Δx being equal to one tenth of the standard deviation σ, is of the order of 4 percent.

Section 2. Fitting of a Normal Curve

Let us now go back to the set of N numbers m_i obtained by measuring repeatedly a certain physical quantity m_0. There are cases where it is legitimate to consider m_0 not only as really existing but as known; for instance in geodesy m_i may mean the deviation of the sum of the three separately measured angles of a triangle from the true value $m_0 = 180°$. If the deviations $(m_i - 180)$ conform to the Hagen theory then we can write

$$x_i = m_i - m_0 \qquad ...(35)$$

and therefore the value x_i should conform to the function $f(x)$ as it is defined in (30).

If the whole set of m_i's is available, the calculations can be carried out without difficulty as follows.

The value of the variance σ^2 is directly computed from the individual results m_i:

$$\sigma^2 = \frac{1}{N} \cdot \sum_i (m_i - m_0)^2 . \qquad ...(36)$$

This operation presents no difficulty when modern computational machinery is available. Once σ^2 is calculated, the operator can choose a convenient fraction of σ as the classification interval Δx. For very large N, it is convenient to use $\Delta x = \frac{\sigma}{10}$ or $\Delta x = \frac{\sigma}{20}$. For not very large N, $\frac{\sigma}{5}$ is frequently used. The sample is now "classified"

* The table of the normal function was given in Chapter I, Section 5; the variable v is identical to the variable λ.

(see Fig. 1 and Fig. 2). Its central interval extends from $\frac{-\Delta x}{2}$ to $\frac{+\Delta x}{2}$, (centre C_0 at $x_0 = 0$), then, to the right, from $\frac{+\Delta x}{2}$ to $\frac{+3\Delta x}{2}$ (centre C_1 at $x_1 = +1$), from $\frac{+3\Delta x}{2}$ to $\frac{+5\Delta x}{2}$ (centre C_2 at $x_2 = +2\Delta x$) etc. The same operation is then performed towards negative values. Sometimes both wings (on medium size samples, say $N \approx 1000$) terminate abruptly and neatly at $x = 4\sigma$ or $x = 5\sigma$ and at the same distance from the origin. However, sometimes the wings present certain "problems", the handling of which cannot be made according to stringent and precise rules. The opinion of the observer-experimenter is then the most important factor and must be taken into account.

If m_0 is not known, it is legitimate to use the mean $\overline{m}$ as an estimate for m_0. The mean $\overline{m}$ is computed by the formula:

$$\overline{m} = \frac{1}{N} \cdot \sum_i m_i, \quad i = 1, 2, 3, \ldots N. \qquad \ldots(37)$$

It is shown in Appendix V that the formula for the variance is slightly different from (36) and is:

$$\sigma^2 = \frac{1}{N-1} \cdot \sum_i (m_i - \overline{m})^2 . \qquad \ldots(38)$$

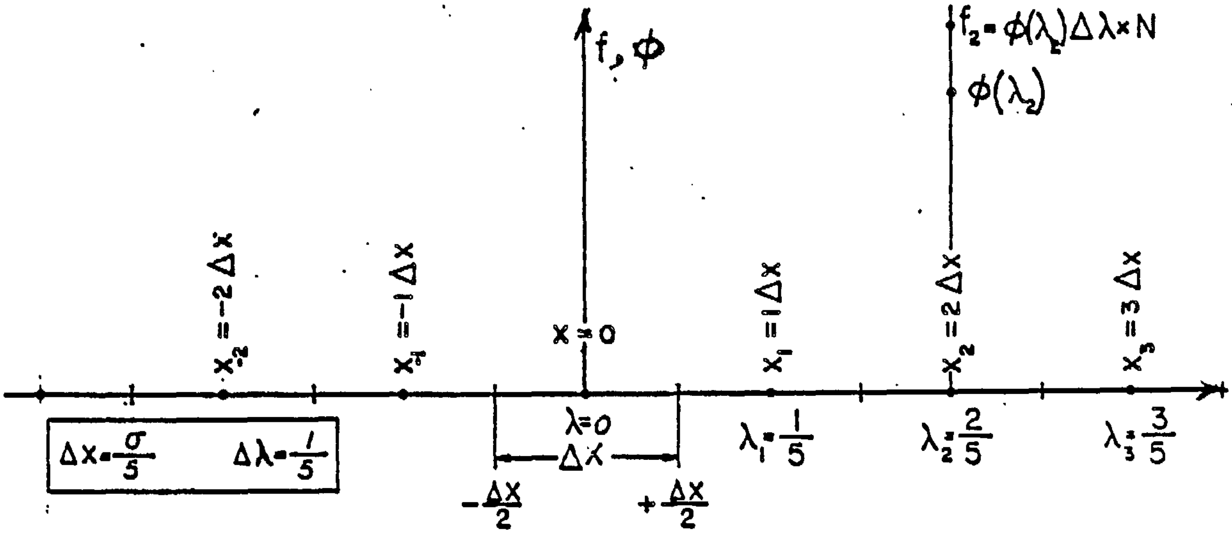


Fig. 1. Horizontal axis prepared for a classified sample.

Once all the λ's are calculated, the values of the corresponding $\phi(\lambda)$ are found in the table of the Normal Function. For instance, the probability $\Delta P_{\lambda}(\lambda_1)$ is expressed by the product

$$\Delta P_{\lambda}(\lambda_1) \;=\; \phi(\lambda_1)\Delta\lambda_1 \;=\; \phi(\lambda_1)\times\frac{1}{5}\,. \qquad ...(39)$$

The final step is the calculation of the so-called "theoretical" class frequency:

$$f(x_1) \;=\; f(\lambda_1) \;=\; \phi(\lambda_1)\times\frac{1}{5}\times N\,. \qquad ...(40)$$

We shall now treat those cases where the whole set of observations m_i is not available. This often happens in literature when a sample is already classified by means of a certain interval Δm but the classification of which is not referred to the central interval (centered on $\overline{m}$). This is represented in Fig. 2 which shows $(k+1)$ intervals of classification numbered from $j = 0$ to $j = \pm k$,

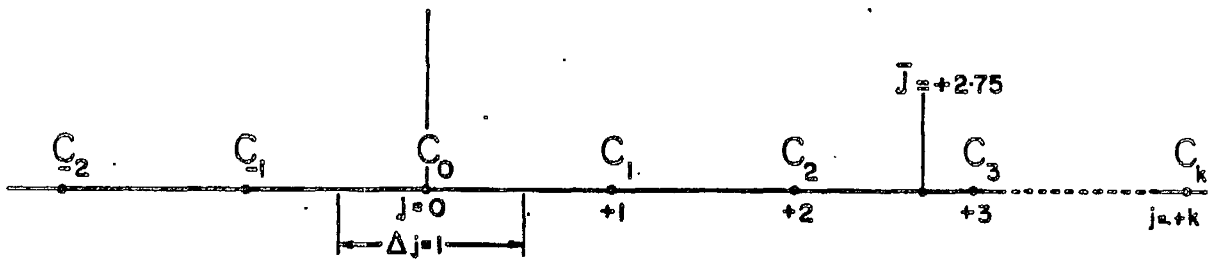


Fig. 2. Use of the "rank" j as variable.

j being the rank of the interval and F_j being the class frequency.

It is, of course, perfectly possible to calculate the value of the mean $\bar{j} = \frac{1}{N} \sum_j jF_j$ and the variance $\sigma^2 = \frac{1}{N-1} \sum_j (j-\bar{j})^2 F_j$. It is obvious that the position of $\bar{m}$ will not coincide with the centre of an interval. Let us assume that it is located as shown in Fig. 2, *i.e.* at a point corresponding to $\bar{j} = 2.75$. The distances of all points $C_1, C_2, \cdots$ from $\bar{j}$ are calculated as follows:

C_0	distant from $\bar{j}$ by	$0-2.75$	$= -2.75$
C_1	distant from $\bar{j}$ by	$1-2.75$	$= -1.75$
C_2	distant from $\bar{j}$ by	$2-2.75$	$= -0.75$
C_3	distant from $\bar{j}$ by	$3-2.75$	$= +0.25$
C_4	distant from $\bar{j}$ by	$4-2.75$	$= +1.25$
...	"	...	

Each of the numbers in the last column must be now divided by σ to give the λ's of the centres C_j. If the observed class frequencies F_j are of proper structure (*i.e.* suggest that the distribution could be normal) the calculation of theoretical class frequencies is carried out as in (40) with $\Delta\lambda = \frac{\Delta j}{\sigma}$ $(\Delta j = 1)$ and N equal to $N = \sum_j F_j$.

It is to be noted that the axis of symmetry of the curve passes through $\bar{m}$ but the central ordinate cannot be calculated and the theoretical points are not placed symmetrically with respect to the central ordinate.

The example that follows is taken from the measurements of electrical energy by means of domestic meters.

There are 144 nominally "identical" meters registering (in series) the electrical energy consumed over a one year period and measured against a group of "monitors" considered as yielding the true value of consumed energy. It is assumed that the deviations of the ordinary meters are distributed quasi-normally and thus in conformity with a normal curve.

The 144 numbers obtained are actually the percentages that indicate the deviations of a meter with respect to the monitors. They are classified by means of an interval equal to 0.1 percent. The individual elements are all listed in Table I A.

The classification intervals are numbered from $j = -9$ to $j = +25$ as shown in Table IB, F_j being the observed class frequency. The calculation of the mean $\bar{j}$ and the variance s^2 leads to the values

$$\bar{j} = +4.3681 \quad \text{(axis of symmetry)},$$

$$s^2 = 21.5909, \quad s = 4.6466 \; .$$

Table I - A

$Y_{1,1}$ =	−0.01	$Y_{2,1}$ =	+0.72	$Y_{3,1}$ =	+0.62	$Y_{4,1}$ =	+1.24
	−0.20		+0.56		+0.60		+1.40
	+0.12		−0.95		+0.09		+0.72
	+0.39		+0.83		+0.76		+1.15
	+0.07		+0.24		+0.14		+0.80
	−0.17		−0.40		−0.72		+0.62
	+0.16		+0.44		+0.73		+0.64
	−0.01		−0.91		+1.06		+1.02
$Y_{1,9}$ =	+0.54	$Y_{2,9}$ =	−0.49	$Y_{3,9}$ =	+0.87	$Y_{4,9}$ =	+0.95
$Y_{1,10}$ =	+0.31	$Y_{2,10}$ =	−0.13	$Y_{3,10}$ =	+0.05	$Y_{4,10}$ =	+0.70
	+0.30		+0.27		−0.29		+0.64
	+0.11		+0.43		−0.21		+1.20
	+0.65		+0.34		−0.38		+0.57
	+0.36		+0.07		+1.17		+0.89
	+0.58		−0.07		+0.60		+0.75
	+0.56		+0.40		−0.14		+0.92
	+0.65		+0.53		−0.18		+0.87
$Y_{1,18}$ =	+0.47	$Y_{2,18}$ =	+0.05	$Y_{3,18}$ =	+0.24	$Y_{4,18}$ =	+0.81
$Y_{1,19}$ =	+0.32	$Y_{2,19}$ =	+0.26	$Y_{3,19}$ =	+0.39	$Y_{4,19}$ =	+0.51
	+0.11		+0.69		−0.07		+0.60
	+0.07		+0.98		+0.68		+0.98
	+0.65		+0.55		−0.18		+0.44
	+0.72		+0.83		+0.59		+0.72
	+0.69		+0.30		−0.03		+0.25
	+0.68		+0.62		+0.98		+0.92
	+0.62		+0.31		+0.31		+0.81
$Y_{1,27}$ =	+0.35	$Y_{2,27}$ =	+0.42	$Y_{3,27}$ =	+0.03	$Y_{4,27}$ =	+1.20
$Y_{1,28}$ =	+0.22	$Y_{2,28}$ =	+0.50	$Y_{3,28}$ =	+0.27	$Y_{4,28}$ =	+0.27
	+0.46		+0.66		+0.40		+2.50
	+0.36		+0.30		+0.03		+0.59
	+0.05		+0.77		+0.28		+0.45
	+0.42		−0.03		+0.15		+0.47
	+0.99		+0.50		−0.43		+0.56
	+0.40		+0.04		+0.56		+1.65
	+0.40		+0.20		+0.57		+0.88
$Y_{1,36}$ =	+0.63	$Y_{2,36}$ =	+0.32	$Y_{3,36}$ =	−0.08	$Y_{4,36}$ =	+0.45

Table I - B

j	F_j		f_j	
-9	2		0.20	
-8	0		0.36	
-7	1	→ 4	0.62	→ 3.83
-6	0		1.03	
-5	1		1.62	
-4	3	→ 4	2.44	→ 5.96
-3	1		3.52	
-2	5		4.83	
-1	5		6.34	
0	9		7.95	
+1	9		9.51	
+2	7		10.86	
+3	14		11.84	
+4	15		12.32	
+5	10		12.25	
+6	20		11.62	
+7	13		10.53	
+8	8		9.11	
+9	7		7.52	
+10	5		5.93	
+11	2	→ 6	4.46	→ 7.67
+12	4		3.21	
+13	0		2.20	
+14	1		1.44	
+15	0		0.90	
+16	0		0.54	
+17	1		0.31	
+18	0		0.17	
+19	0	→ 3	0.09	→ 5.72
+20	0		0.04	
+21	0		0.02	
+22	0		0.01	
+23	0		0.00	
+24	0		0.00	
+25	1		0.00	
	144		143.80	

For the disposition of the calculations follow the pattern of Ex. 4 (Screws Fabricating Machine)

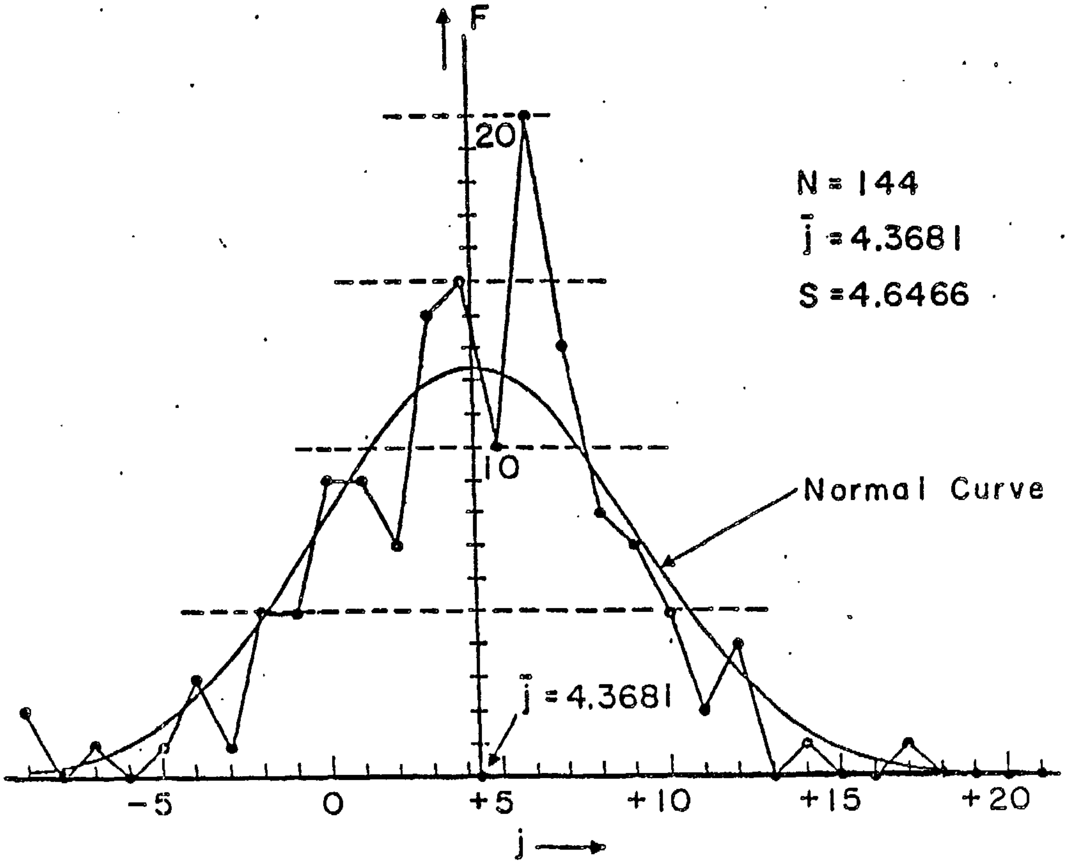


Fig. 3. Sample of measurements performed on a group of electrical energy meters.

Note that the dots on the j-axis represent the centres of the intervals: eg. in the interval $j = 7$ are located all elements the values of which are between 0.65 and 0.75.

Section 3. Basic Integrals Related to the Normal Function

The fundamental integral from which all other integrals are derived is:

$$J_n = \int_0^\infty z^n e^{-z^2} dz \,. \qquad \text{...(41)}$$

The important feature of J_n is that it can be expressed by means of an integral of lower order, namely J_{n-2}. Thus, if the expressions for J_0 and J_1 can be directly established, all other integrals can be formed step by step by a recurrence formula.

Only J_1 possesses a simple primitive function. It is

$$F(z) = -\frac{1}{2} e^{-z^2},$$

as it can easily be checked by differentiating $F(z)$. Hence

$$J_1 = \left| -\frac{1}{2} e^{-z^2} \right|_0^\infty = +\frac{1}{2} \,.$$

The calculation of the expression for J_0 is not simple: it is based on the calculation of a double integral by using polar coordinates. This operation will not be described here and the expression for J_0 will be deduced by the following indirect method which is, in its essence, based on the theory of binomial expansion as it has been treated in Chapter I, Section IV.

Let us refer to the equation (30),

$$f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx \,.$$

It is obvious that the summation of $f(x)dx$ for all values of x (from $-\infty$ to $+\infty$) must lead to the total probability equal to "unity" (*i.e.* to be equivalent to "certitude"):

$$\int_{-\infty}^{+\infty} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2\sigma^2}} dx = 1 \,.$$

The change of variable

$$\frac{x}{\sigma\sqrt{2}} = z, \quad x = z\sigma\sqrt{2}, \quad dx = \sigma\sqrt{2}dz$$

leads to

$$J_0 = \int_0^\infty e^{-z^2} dz = \frac{\sqrt{\pi}}{2} .$$

The above mentioned recurrence formula is deduced from the following differential relation:

$$d\left(z^{n-1} e^{-z^2}\right) = (n-1)z^{n-2}e^{-z^2}dz - 2z^n e^{-z^2}dz.$$

Integrating between 0 and ∞ we obtain

$$\left|z^{n-1}e^{-z^2}\right|_0^\infty = (n-1)\int_0^\infty z^{n-2}e^{-z^2}dz - 2\int_0^\infty z^n e^{-z^2}dz .$$

The left-hand term is equal to zero and thus we have

$$0 = (n-1)J_{n-2} - 2J_n ,$$

$$J_n = \frac{n-1}{2} J_{n-2} . \qquad \text{...(42)}$$

Table II Basic Integrals

$$J_0 = \int_0^\infty e^{-z^2}dz = \frac{\sqrt{\pi}}{2} \qquad J_1 = \int_0^\infty ze^{-z^2}dz = \frac{1}{2}$$

$$J_2 = \int_0^\infty z^2e^{-z^2}dz = \frac{\sqrt{\pi}}{4} \qquad J_3 = \int_0^\infty z^3e^{-z^2}dz = \frac{1}{2}$$

$$J_4 = \int_0^\infty z^4e^{-z^2}dz = 3\frac{\sqrt{\pi}}{8} \qquad J_5 = \int_0^\infty z^5e^{-z^2}dz = 1$$

By means of these formulae, and those which can be further established, it is possible to form the integrated expressions for various forms of moments μ_n of the normal variate.

The moment μ_n is defined by the integral

$$\mu_n = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^n e^{-\frac{x^2}{2\sigma^2}} dx \, . \qquad \text{...(43)}$$

The curve $f(x)$ is symmetrical with respect to the y-axis so that *when n is odd,* the integral from $-\infty$ to 0 is equal, but of the opposite sign, to the integral from 0 to $+\infty$. Hence, when n is odd, all moments μ_n are equal to zero: $\mu_1 = \mu_3 = \mu_5 = \cdots = 0.$

To express the values of μ_n (n even) in terms of the basic integrals of Table II, use the change of variable

$$x = z\sigma\sqrt{2}, \qquad dx = \sigma\sqrt{2}dz.$$

This will lead to the results given in the Table III below.

In this table are also listed the so-called "absolute" moments ν_n. The formulae for these moments are the same as for μ_n except that the variable x is replaced by its absolute value $|x|$:

$$\nu_n = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} |x|^n e^{-\frac{|x|^2}{2\sigma^2}} d|x| \, .$$

Because of the symmetry of the curve with respect to the vertical axis this expression is equivalent to

$$\nu_n = \frac{2}{\sigma\sqrt{2\pi}} \int_{0}^{+\infty} x^n e^{-\frac{x^2}{2\sigma^2}} dx \, . \qquad \text{...(44A)}$$

Table III

Moments of the Normal Variate

$\mu_0 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}} dx = 1$	$\nu_0 = \mu_0 = 1$
$\mu_1 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^1 e^{-\frac{x^2}{2\sigma^2}} dx = 0$	$\nu_1 = \frac{2}{\sigma\sqrt{2\pi}} \int_{0}^{+\infty} x^1 e^{-\frac{x^2}{2\sigma^2}} dx = \frac{2}{\sqrt{2\pi}}\sigma$
$\mu_2 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx = \sigma^2$	$\nu_2 = \mu_2 = \sigma^2$
$\mu_3 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^3 e^{-\frac{x^2}{2\sigma^2}} dx = 0$	$\nu_3 = \frac{2}{\sigma\sqrt{2\pi}} \int_{0}^{+\infty} x^3 e^{-\frac{x^2}{2\sigma^2}} dx = \frac{4}{\sqrt{2\pi}}\sigma^3$
$\mu_4 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^4 e^{-\frac{x^2}{2\sigma^2}} dx = 3\sigma^4$	$\nu_4 = \mu_4 = 3\sigma^4$
$\mu_5 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^5 e^{-\frac{x^2}{2\sigma^2}} dx = 0$	$\nu_5 = \frac{2}{\sigma\sqrt{2\pi}} \int_{0}^{+\infty} x^5 e^{-\frac{x^2}{2\sigma^2}} dx = \frac{16}{\sqrt{2\pi}}\sigma^5$

It is obvious that, when n is even, $\nu_n \equiv \mu_n$. But each ν_n in which n *is odd,* has its own specific value.

As all expressions for ν_n are obtained by transforming ν_n into J_n, there must be a recurrence relation that expresses ν_n in terms of ν_{n-2}. This relation is

$$\nu_n = (n-1)\sigma^2 \nu_{n-2} .$$

Its demonstration is very simple and is performed by the same substitution as in all previous calculations, *i.e.*

$$\frac{x}{\sigma\sqrt{2}} = z .$$

It leads to the equation

$$\nu_n = \frac{2^{\frac{n}{2}+1}}{\sqrt{\pi}}\sigma^n \int_0^\infty e^{-z^2} z^n \, dz = \frac{2^{\frac{n}{2}+1}}{\sqrt{\pi}}\sigma^n J_n . \qquad \text{...(44B)}$$

If n is replaced by $n-2$, this becomes

$$\nu_{n-2} = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \sigma^{n-2} J_{n-2},$$

and therefore,

$$\nu_{n-2} = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \sigma^{n-2} \cdot \frac{2J_n}{n-1} = \frac{2^{\frac{n}{2}+1}}{\sqrt{\pi}}\sigma^{n-2} \cdot \frac{J_n}{n-1} .$$

Finally

$$\frac{\nu_n}{\nu_{n-2}} = \sigma^2(n-1) . \qquad \text{...(45)}$$

The notion of moment can be further generalized by considering n as a non-integer, in particular as a fractional number. This has already been partly done and described in the author's previous studies and publications. However, it is only a small incursion into a vast domain of mathematical investigations.

Chapter III
Mixtures and Dichotomy

Section 1. Mixture of Centered Normal Populations

Let $f_1(x)$ and $f_2(x)$ denote the pdf's of two normal variates:

$$f_1(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}} , \qquad f_2(x) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_2^2}} .$$

The probability that a value x of the first variate will fall into an interval Δx is equal to $f_1(x)\Delta x$ and the probability that the *same* x, but belonging to the second variate, will fall into *the same* Δx is $f_2(x)\Delta x$.

Consider now a large sample of N_1 elements the distribution of which conforms to the pdf $f_1(x)$ and a large sample of N_2 the distribution of which conforms to the pdf $f_2(x)$. If these two samples are mixed i.e. combined into one single sample of $N = N_1 + N_2$ elements, the sample N_1 will introduce into Δx a number of elements equal to $N_1 f_1(x)\Delta x$. The sample N_2 will introduce $N_2 f_2(x)\Delta x$. The total population in Δx will therefore be equal to

$$N_1 f_1(x)\Delta x + N_2 f_2(x)\Delta x .$$

It is always permissible to assume that (in theory) there is a pdf for the mixture the size of which is of course, $N = N_1 + N_2$. Let $\psi(x)$ denote this pdf. It is now possible to write the equation

$$N\psi(x)\Delta x = N_1 f_1(x)\Delta x + N_2 f_2(x)\Delta x . \qquad ...(46)$$

This equation is called "*equation of mixture*". If the sizes N_1, N_2 are replaced by the "proportional sizes"

$$p_1 = \frac{N_1}{N} , \qquad p_2 = \frac{N_2}{N} ,$$

the equation takes the following final form

$$\psi(x)\Delta x = \frac{p_1}{\sigma_1\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}} \Delta x + \frac{p_2}{\sigma_2\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_2^2}} \Delta x . \qquad ...(47)$$

This equation, in which Δx is replaced by the differential dx, is directly suitable for the calculation of the expressions for the moments of the variable x. The expressions for the moments in terms of $\psi(x)$ are thus expressed in terms of the moments of the components, $f_1(x)$ and $f_2(x)$.

Before calculating the expressions for the moments, let us first notice that by putting $x = 0$ we obtain the expression for the central ordinate $\psi(0)$:

$$\psi(0) = \frac{1}{\sqrt{2\pi}} \left(\frac{p_1}{\sigma_1} + \frac{p_2}{\sigma_2} \right) . \qquad ...(48)$$

The expressions for the successive moments by means of the equation

$$\mu_n = \int_{-\infty}^{+\infty} x^n \psi(x) dx = p_1 \int_{-\infty}^{+\infty} x^n f_1(x) dx + p_2 \int_{-\infty}^{+\infty} x^n f_2(x) dx \qquad ...(49a)$$

are directly obtainable from Table III. Thus we have:

$$\begin{aligned}
n &= 0 , & \mu_0 &= p_1 + p_2 = 1 . \\
n_2 &= 2 , & \mu_2 &= p_1\sigma^2 + p_2\sigma_2^2 . \\
n_4 &= 4 , & \mu_4 &= 3p_1\sigma_1^4 + 3p_2\sigma_2^4 . \\
n &= 1, 3, 5 \cdots & \mu_1 &= \mu_3 = \mu_5 \cdots = 0 .
\end{aligned} \qquad ...(49b)$$

All moments of odd orders are equal to zero.

If the parameters p_1, σ_1, p_2, σ_2 are not given explicitly but the components are represented by "observed" diagrams (i.e. class frequencies) the calculation of the mixture moments can be made in two different ways:

a) by mixing the populations of the diagrams and calculating the moments by means of the mixture class frequencies,

b) by fitting into each component a normal function and using the resulting parameters of the functions.

There will be a difference between a) and b). Its magnitude will depend on how well the normal curves represent the corresponding diagrams. If the diagrams are not sufficiently normal the operation is meaningless.

As an illustration, the calculations will be performed on the following numerical data:

$$\left.\begin{matrix} N_1 = 300, & p_1 = \frac{1}{3}, & \sigma_1 = 3 & \sigma_1^2 = 9 \\ N_2 = 600, & p_2 = \frac{2}{3}, & \sigma_2 = 6 & \sigma_2^2 = 36 \end{matrix}\right\} \Delta j = 1.$$

The computed values for the μ_2, $\psi(0)$ and F_0 are

$$\mu_2 = \left(\frac{1}{3}3^2 + \frac{2}{3}6^2\right) = 27,$$

$$\psi_0 = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{3\times 3} + \frac{2}{3\times 6}\right) = 0.0886,$$

$$F_0 = (300 + 600)\times 0.0886 = 79.74 .$$

The variance of the normal curve that fits into the mixture is equal to $s^2 = \mu_2 = 27$. Hence the central class frequency of the normal curve is

$$F'_0 = \frac{1}{\sqrt{2\pi}}\cdot\frac{N}{\sqrt{\mu_2}} = 0.3989\cdot\frac{900}{\sqrt{27}} = 69.06 .$$

The leptokurtosis is therefore of the order of 15 percent. It is due to the fact that the variances are unequal. If the variances of the components are equal to each other it is easy to show that the mixture is also normal, with the same variance. This property of mixtures has been known for some time. A new and ingenious demonstration of this property is due to the National Research Council's mathematician N.T. Gridgeman.

The main practical interest of the theory of mixtures resides in the fact that this theory forms the base of the operation which is termed "dichotomy" and which is the inverse of that of mixture: calculation of the parameters of the components from the observed class frequencies. It is presented in the sequel under the form of a numerical operation.

A Numerical Example of Dichotomy.

A sample of $N = 2000$ elements is assumed to be a mixture of two sub-samples of 1000 elements each:

$$\left.\begin{array}{l} N_1 = 1000 \\ N_2 = 1000 \end{array}\right\} \quad p_1 = p_2 = \frac{1}{2} \cdot$$

j	F_j	j	F_j
0	179.5 = F_0		
±1	174.9	±11	9.5
2	160.9	12	5.6
3	141.9	13	3.2
4	118.5	14	1.8
5	94.1	15	1.0
6	71.2	16	0.7
7	51.5	17	0.4
8	35.7	18	0.2
9	23.7	19	0.1
±10	15.2	±20	0.0
			1999.7

The sample N, is distributed into 41 classes, from $j = -20$ to $j = +20$. Here the variable is the rank j and therefore $\Delta j = 1$ (omitted from equations).

The equation (49) can be put under the form:

$$N\psi(0) = \frac{N}{\sqrt{2\pi}}\left(\frac{p_1}{\sigma_1} + \frac{p_2}{\sigma_2}\right) = F_0 \,,$$

$$\frac{2000}{\sqrt{2\pi}} \cdot \frac{1}{2} \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} \right) = 179.5 \ .$$

Simplified, it becomes

$$\frac{\sigma_1 + \sigma_2}{\sigma_1 \sigma_2} = 0.45 \ .$$

To use the relation (48b) it is necessary to calculate the second moment μ_2 of the variable j as it is presented in the Table. Direct calculation by the formula

$$\mu_2 = \frac{1}{2000} \sum j^2 F$$

gives

$$\mu_2 = 20.5$$

so that by (48):

$$\sigma_1^2 + \sigma_2^2 = 2\mu_2 = 41.0.$$

Thus the values of σ_1 and σ_2 are the solution of the system of equations

$$\left. \begin{aligned} a) \quad & \frac{\sigma_1 + \sigma_2}{\sigma_1 \sigma_2} = 0.45 \\ b) \quad & \sigma_1^2 + \sigma_2^2 = 41.00 \end{aligned} \right\}$$

Solution:

1) square the equation a) and eliminate the term $(\sigma_1^2 + \sigma_2^2)$ to obtain

$$\frac{41 + 2(\sigma_1 \sigma_2)}{(\sigma_1 \sigma_2)^2} = 0.2025.$$

2) Consider $(\sigma_1 \sigma_2)$ as an unknown: $\sigma_1 \sigma_2 = Q$. Solve the equation

$$0.2025 Q^2 - 2Q - 41 = 0.$$

The useful solution is: $Q = 20$.

3) From a): $\sigma_1 + \sigma_2 = 9$ so that σ_1 and σ_2 are the solution of the equation $s^2 - 9s + 20 = 0$. These solutions are $s' = \sigma_1 = 5$ and $s'' = \sigma_2 = 4$. The mixture's components are represented in Fig 4.

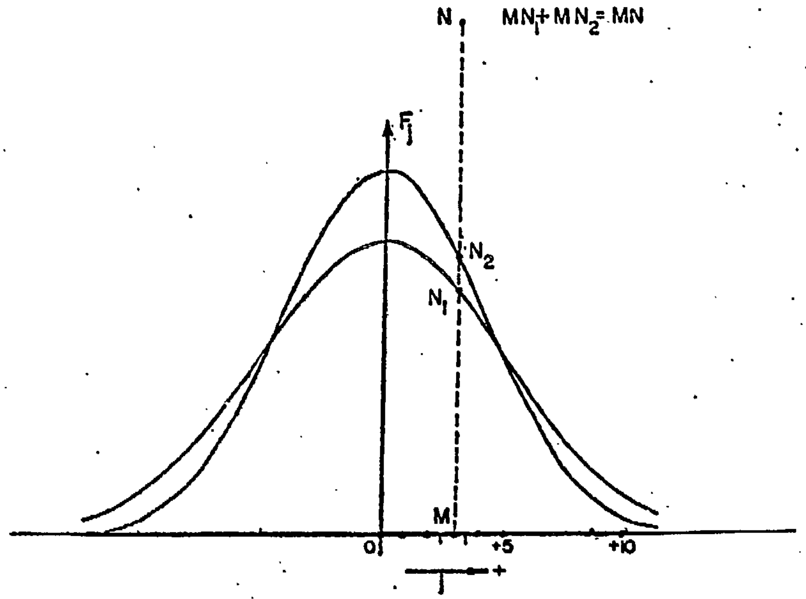


Fig. 4 Components of the mixture of two coaxial normal populations.

Section 2. Mixture of Decentered Normal Populations

If the centre of the variate is not at the origin of the axes ($x = 0$) but at a distance a from the origin, then the expression for the ordinate y is

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}} \qquad ..(50)$$

so that the moment of n^{th} order of x with respect to the origin $x = 0$ is

$$\omega_n = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^n e^{-\frac{(x-a)^2}{2\sigma^2}} dx \; .$$

All ω-moments can be calculated by means of the change of variable

$$(x-a) = u \; , \quad dx = du.$$

Thus

$$\omega_n = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (a+u)^n e^{-\frac{u^2}{2\sigma^2}} du. \qquad ...(51)$$

Calculation of a few first moments:

$$\omega_0 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{u^2}{2\sigma^2}} du ,$$

$$\underline{\omega_0 = 1.} \qquad ...(52a)$$

$$\omega_1 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (a+u) e^{-\frac{u^2}{2\sigma^2}} du,$$

$$\omega_1 = \frac{a}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{u^2}{2\sigma^2}} du + \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} u e^{-\frac{u^2}{2\sigma^2}} du .$$

$$\underline{\omega_1 = a+0 = a} . \qquad ...(52b)$$

$$\omega_2 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (a+u)^2 e^{-\frac{u^2}{2\sigma^2}} du ,$$

$$\omega_2 = \frac{1}{\sigma\sqrt{2\pi}} \left\{ a^2 \int_{-\infty}^{+\infty} e^{-\frac{u^2}{2\sigma^2}} du + 2a \int_{-\infty}^{+\infty} u e^{-\frac{u^2}{2\sigma^2}} du + \int_{-\infty}^{+\infty} u^2 e^{-\frac{u^2}{2\sigma^2}} du \right\},$$

$$\underline{\omega_2 = a^2 + 0 + \sigma^2 = \sigma^2 + a^2 .} \qquad ..(52c)$$

Further moments:

$$\omega_3 = 3a\sigma^2 + a^3, \qquad ...(52d)$$

$$\omega_4 = 3\sigma^4 + 6a\sigma^2 + a^4, \qquad ...(52e)$$

$$\omega_5 = 15a\sigma^4 + 10a^3\sigma^2 + a^5 . \qquad ...(52f)$$

The importance of these expressions for ω-moments has been shown at the turn of the century by K. Pearson who was the first to analyse skew distributions which are found in biology and which may be considered as mixtures of decentered normal populations.

Let us assume with Pearson, that a skew curve represents a mixture of two normal distributions the parameters of which are

Curve 1: Prop. size p_1, variance σ_1^2, position a_1 .

Curve 2: Prop. size p_2, variance σ_2^2, position a_2 .

The moments with respect to the origin of the mixture are ω_0, ω_1, ω_2, ω_3, ω_4, ω_5. Using the expressions (52) we form Pearson's system:

$$\begin{aligned}
&p_1 && + p_2 && = \omega_0 = 1 \\
&p_1 a_1 && + p_2 a_2 && = \omega_1 \\
&p_1(\sigma_1^2 + a_1^2) && + p_2(\sigma_2^2 + a_2^2) && = \omega_2 \qquad \ldots(53a) \\
&p_1(3a_1\sigma_1^2 + a_1^3) && + p_2(3a_2\sigma_2^2 + a_2^3) && = \omega_3 \\
&p_1(3\sigma_1^4 + 6a_1^2\sigma_1^2 + a_1^4) && + p_2(3\sigma_2^4 + 6a_2^2\sigma_2^2 + a_2^4) && = \omega_4 \\
&p_1(15a_1\sigma_1^4 + 10a_1^3\sigma_1^2 + a_1^5) && + p_2(15a_2\sigma_2^4 + 10a_2^3\sigma_2^2 + a_2^5) && = \omega_5 \ .
\end{aligned}$$

The solution of this system by a rigorous algebraic method would be a practically impossible undertaking. What made it possible is Pearson's idea to calculate the moments with respect to the mean $\bar{y}$ instead of the origin. Hence the set of symbols $\omega_0 \cdots \omega_5$ is replaced by the set $\mu_0 \cdots \mu_5$ in which $\mu_1 = 0$. Also, of course, the symbols a_1 and a_2 refer now to the distances from the mean $\bar{y}$ and are therefore replaced by m_1 and m_2 respectively. The system Pearson actually uses in his memoir is therefore:

$$\begin{aligned}
&p_1 && + p_2 && = \mu_0 = 1 \\
&p_1 m_1 && + p_2 m_2 && = \mu_1 = 0 \\
&p_1(\sigma_1^2 + m_1^2) && + p_2(\sigma_2^2 + m_2^2) && = \mu_2 \qquad \ldots(53b) \\
&p_1(3m_1\sigma_1^2 + m_1^3) && + p_2(3m_2\sigma_2^2 + m_2^3) && = \mu_3 \\
&p_1(3\sigma_1^4 + 6m_1^2\sigma_1^2 + m_1^4) && + p_2(3\sigma_2^4 + 6m_2^2\sigma_2^2 + m_2^4) && = \mu_4 \\
&p_1(15m_1\sigma_1^4 + 10m_1^3\sigma_1^2 + m_1^5) && + p_2(15m_2\sigma_2^4 + 10m_2^3\sigma_2^2 + m_2^5) && = \mu_5 \ .
\end{aligned}$$

The solution of this system with respect to the six unknowns p_1, p_2, m_1, m_2, σ_1, σ_2 requires a lot of ingenuity and therefore Pearson's accomplishment must be considered as an outstanding mathematical success. The detailed examination of Pearson's memoir is strongly recommended and is highly rewarding.

As in the preceeding section a case of dichotomy will be described briefly under the form of a numerical operation.

A Numerical Example of Pearson's Equations

The class frequencies considered as "observed" are presented in the adjoining table. The diagram clearly indicates that the distribution is skew and triangular.

j	F_j	j	F_j
-12	0	+12	0
-11	6	+11	6
-10	6	+10	13
-9	19	+9	32
-8	25	+8	38
-7	25	+7	57
-6	37	+6	51
-5	31	+5	70
-4	37	+4	75
-3	44	+3	64
-2	38	+2	57
-1	57	+1	64
0	51		
			903

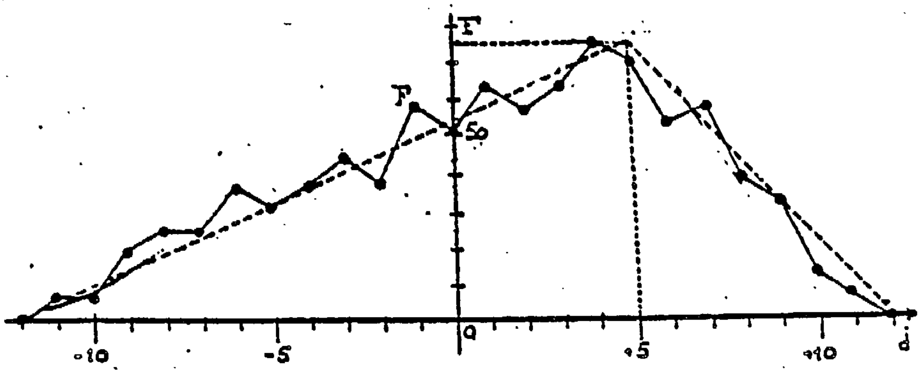


Fig. 5 Triangular Distribution

The calculation of the mean and of the six moments leads to the following results:

$$\bar{j} = +1.1639$$

$$\mu_0 = 1 \qquad \mu_3 = -41.5599,$$

$$\mu_1 = 0 \qquad \mu_4 = +1501.0062,$$

$$\mu_2 = +25.7672 \qquad \mu_5 = -5583.1893.$$

Pearson's system of equations is therefore:

$$p_1 + p_2 = \mu_0 = 1$$

$$p_1 m_1 + p_2 m_2 = \mu_1 = 0$$

$$p_1(\sigma_1^2 + m_1^2) + p_2(\sigma_2^2 + m_2^2) = \mu_2 = +25.7672$$

$$p_1(3m_1\sigma_1^2 + m_1^3) + p_2(3m_2\sigma_2^2 + m_2^3) = \mu_3 = -41.5599$$

$$p_1(3\sigma_1^4 + 6m_1^2\sigma_1^2 + m_1^4) + p_2(3\sigma_2^4 + 6m_2^2\sigma_2^2 + m_2^4) = \mu_4 = +1501.0062$$

$$p_1(15m_1\sigma_1^4 + 10m_1^3\sigma_1^2 + m_1^5) + p_2(15m_2\sigma_2^4 + 10m_2^3\sigma_2^2 + m_2^5) = \mu_5 = -5583.1893.$$

The solution of this system is accomplished by successively eliminating the unknowns. It finally leads to one equation of ninth degree which must be solved numerically. When examining Pearson's memoir one is impressed by the author's skill and ingenuity in handling complicated algebraic expressions and his confidence in the final success. The solution of the final nonic with the author's primitive calculating hand operated machine must also be considered as an outstanding achievement. The parameters of the component populations are:

$$C_1 \begin{cases} m_1 = -4.996 \\ p_1 = 0.406 \\ \sigma_1 = 3.179 \end{cases} \qquad C_2 \begin{cases} m_2 = 3.419 \\ p_2 = 0.594 \\ \sigma_2 = 2.777 \end{cases}$$

$$a_1 = -3.833 \qquad a_2 = +4.582$$
$$N_1 = 366.618\,. \qquad N_2 = 536.382\,.$$

The curves C_1, C_2 and their mixture are represented in Fig. 6.

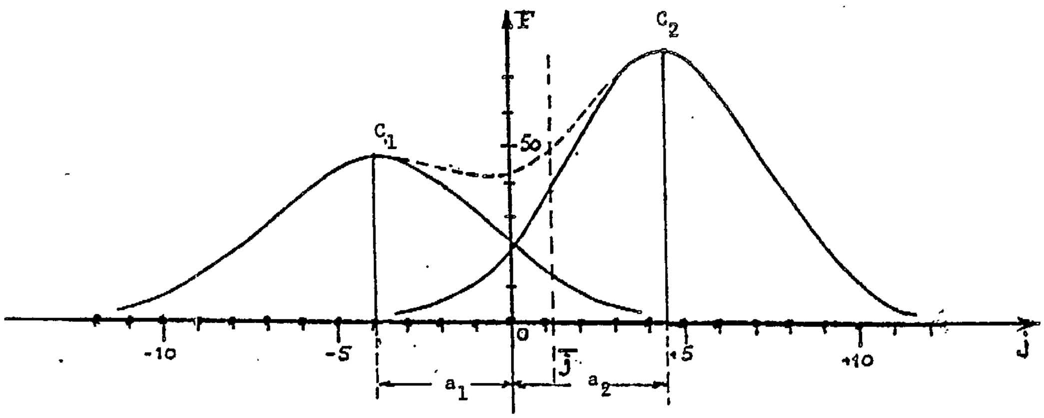


Fig. 6 Dichotomy of the sample in Fig. 5

$a_1 = -3.833,\ a_2 = +4.582,\ j = +1.164$

Chapter IV
Theory of Least Squares and Systems of Equations

Section 1. Compound Probability of a Sample

This section is a natural extension of Section 2, Chapter III. Let us call m_1, m_2, m_3 the values obtained by measuring a fixed quantity and x_1, x_2, x_3 the corresponding deviations from the mean. Without affecting the generality of the reasonings we may assume that the three values x_1, x_2, x_3 belong to the operations of the same metrological quality *i.e.* have the same variance σ^2. The question may arise what is the compound probability dP_C that another set of three measurements will yield values that will fall in the same intervals dx_1, dx_2, dx_3. The reader can easily establish by means of the theorem of compound probability that the answer is:

$$dP_C = dP_{x_1,x_2,x_3} = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^3 e^{-\left[\frac{x_1^2}{2\sigma^2}+\frac{x_2^2}{2\sigma^2}+\frac{x_3^2}{2\sigma^2}\right]} dx_1\ dx_2\ dx_3$$

$$dP_C = \frac{1}{\sigma^3(\sqrt{2\pi})^3} e^{-\frac{1}{2\sigma^2}\left(x_1^2 + x_2^2 + x_3^2\right)} (dx)^3$$

This formula can be readily generalized for samples of any number of values in $m_i (i=1, 2, \cdots N)$ so that it is possible to write

$$dP_C = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^N e^{-\frac{1}{2\sigma^2}\sum x_i^2} (dx)^N .$$

Naturally, dP_C will be a maximum when $\sum x_i^2$ will be a minimum. One must bear in mind that $x_i = m_i - \overline{m}$ so that it is legitimate to raise the question whether the sum of squares $\sum x_i^2$ is really the smallest when the variable ξ in the expression of the function

$$Q(\xi) = (m_1-\xi)^2 + (m_2-\xi)^2 + \cdots + (m_N-\xi)^2 \qquad ...(54)$$

takes on the value $\xi = \overline{m}$. To answer this question consider the derivative of $Q(\xi)$ with respect to ξ, *i.e.*

$$\frac{dQ(x)}{d\xi} = -2\left[(m_1-\xi) + (m_2-\xi) + \cdots + (m_N-\xi)\right]$$

$$= -2\left[m_1 + m_2 + \cdots + m_N - N\xi\right] = -2\left(\sum m_i - N\xi\right) .$$

This expression, equated to zero, yields

$$\sum m_i - N\xi = 0 \qquad i.e.\ \ \xi = \frac{1}{N}\sum_i m_i = \overline{m}. \qquad \text{...(55)}$$

This is one of the most important theorems of the whole calculus of probability: *if a set of N points m_i is given on an axis and a mobile point ξ, the sum of squares of the distances $(m_i-\xi)$, i.e. $Q(\xi) = \sum_i (m_i-\xi)^2$ is a minimum when ξ coincides with $\overline{m}$.* It is to be noted that this theorem is valid in all cases, *i.e.* is independent of the positions of the points m_i. If m_i represent the results of N measurements of a fixed quantity and if it is assumed that these m_i conform to a binomial - normal theory, then the mean $\overline{m}$ is the value that renders a maximum the compound probability dP_C of the whole sample. This is the reason why the mean $\overline{m}$ is termed the *most probable* value of the measured quantity. It is a remarkable coincidence that the constant and generalized use of the "mean" in science and in everyday life is perfectly justified by the properties of the normal distribution. But it also raises the question whether the normal law is absolutely and universally valid? There is no simple and clear-cut answer to this important mathematical and philosophical question.

A few remarks concerning the mean:

1st A set of N values m_i as they are defined above, *i.e.* as results of N measurements of an unknown quantity X,

$$X = m_i \qquad i = 1, 2, \cdots N$$

constitutes a system of what in the sequel will be termed "equations of condition". An equation of condition is not an equation in a strict mathematical sense but only a *symbolic representation of the outcome of a physical measurement.* Furthermore, the relation

$$X = \overline{m}$$

is termed a "*solution*" of the above system by the "*method of least squares*".

2^{nd} It has been already mentioned in the preceeding sections that the mean of the square of $(m_i - \overline{m})$ should be taken under the form (generally called Bessel's Formula):

$$s^2 = \frac{1}{N-1}\sum_i (m_i - \overline{m})^2.$$

This is due to the fact that by forming the set of N differences $(m_i - \overline{m})$ we lose one degree of freedom, *i.e.* reduce by one unit the number of independent measurements.

However, such a justification of the fact that $(N-1)$ is substituted to N, is not a real mathematical demonstration. The latter is given in Appendix V.

Let us now consider the question how, in practice, large samples are formed. In all preceeding cases we have assumed that the values m_i are the result of repeated measurements of a given stable physical quantity, *e.g.* the length of a gauge. In the literature the term "fixed quantity" is frequently used for this purpose. What is the actual meaning of the term "fixed"? The objective of the present work is not to get lost in endless philosophical discussions. Perhaps, in order to make a long story short, it would be appropriate to simply replace this term by that of "statistically stable". On a macroscopic scale nothing is perfectly stable but we easily distinguish between the height of a cloud and the height of a mountain. The position of the top of a mountain fluctuates in appearance according to the atmospheric conditions during the measurement operations and oscillates about a certain average position: we call it statistically stable. In the sequel such quantities will be simply termed "stable".

The second question is what is the meaning of the term "*repeated*" measurements? In certain cases this meaning is clear, *e.g.* when an observer (using a goniometer) measures 100 times a stable azimuth. But, when a geodesist surveys a region containing a hundred triangles, he actually determines the sum of internal angles of the triangles in the net but he does *not* measure the angles of the *same unique* triangle. However, as theoretically the sum of internal angles in all triangles is 180°, the hundred values obtained by the surveyor may legitimately be considered as 100 repeated measurements of one single quantity. Various other, more complicated, cases will be treated in the following sections. It will be always possible to show there that they can be reduced to the simple case of "repeated" measurements.

Section 2. Formation of Systems of Equations

In all previous cases the unknown quantity is assumed to be measured directly as *e.g.* an angle is measured by means of a theodolite. Very often, in practice, measurements are performed on combinations of several unknown quantities and the values of individual components are deduced from the solutions of more or less complicated systems of simultaneous equations termed "equations of condition". The calibration of working mass standards presents a typical case of such operations and will be used as a convenient example.

There are three types of operations that are performed on mass standards:

Type I: a working standard (*e.g.* a kilogram) is calibrated against a high quality standard *e.g.* against a "National Standard" the value of which may have been deduced from direct comparisons against the Prototype Kilogram of the International Bureau of Weights and Measures. Such calibrations are generally performed many times and the results are treated by methods described in preceeding sections.

Type II: a typical example of this type, leading to a system of simultaneous equations of conditions, is the calibration of multiple and submultiples of the kilogram. A set of weights may contain weights of the following values: 1) multiples (in kg): 1, 2, 3, 5, 10, 20, 30, 50, 100 etc. 2) submultiples (in kg): 0.1, 0.2, 0.3, 0.5, 0.01, 0.02 etc. Every Type II calibration is characterized by the fact that it contains a *reference mass* the value M of which is indicated by an equation termed *equation of definition.* Its value is thus assumed to be known with an accuracy that is superior to that of other weights of the set. Weights are denoted by symbols such as (5), (2), (0.1) etc.

If the weights are well adjusted to their nominal values, the comparisons (on a double pan balance) can be carried out without the help of additional masses. Very often, however, small additional masses are necessary, *e.g.* of the order of a milligram.

Let us consider, for instance, the following set of weights:

$$(10) = M;\ (5),\ (2),\ (2'),\ (1),\ (1')\ . \qquad ...(56)$$

With these weights it is possible to make ten comparisons and thus to obtain ten equations of condition. There are five unknowns so that the number of equations exceeds that of unknowns by $10 - 5 = 5$ units. It is, of course, possible to repeat or to omit some of the weighings and thus to increase (or to decrease) the number of equations of condition with respect to the number of unknowns. High quality weighings are long and tedious operations so that an observer is reluctant to increase their number. In fact, "repetitions" are used only if they are really justified.

The system containing all possible equations may be called "basic system"; other systems may be simply called "modified" systems: in Table IV, B is presented a system deduced from the basic system A by omitting the equations marked with an asterisk (4, 5, 12) and using twice the equations marked with a circle (9, 10, 11). Note that the subscripts in the modified system are totally independent of those in the basic system.

Table IV

Masses: (10) = M, (5), (3), (2), (1), (1′).

A. Basic System	B. Modified System	
$+(5)+(3)+(2)+(1)-(1') = m_1 + M$	$+(5)+(3)+(2)+(1)-(1') = m_1 + M$	
$+(5)+(3)+(2)-(1)+(1') = m_2 + M$	$+(5)+(3)+(2)-(1)+(1') = m_2 + M$	
$+(5)+(3)+(1)+(1') = m_3 + M$	$+(5)+(3)+(1)+(1') = m_3 + M$	
$+(5)+(3)+(2) = m_4 + M$ *	$+(5)-(3)-(2)+(1)-(1') = m_4$	
$+(5)-(3)-(2) = m_5$ *	$+(5)-(3)-(2)-(1)+(1') = m_5$	
$+(5)-(3)-(2)+(1)-(1') = m_6$	$+(5)-(3)-(1)-(1') = m_6$	...(57a,b)
$+(5)-(3)-(2)-(1)+(1') = m_7$	$+(3)-(2)-(1) = m_7$	
$+(5)-(3)-(1)-(1') = m_8$	$+(3)-(2)-(1) = m_8$	
$+(3)-(2)-(1) = m_9$ ⊙	$+(3)-(2)-(1') = m_9$	
$+(3)-(2)-(1') = m_{10}$ ⊙	$+(3)-(2)-(1') = m_{10}$	
$+(2)-(1)-(1') = m_{11}$ ⊙	$+(2)-(1)-(1') = m_{11}$	
$+(1)-(1') = m_{12}$ *	$+(2)-(1)-(1') = m_{12}$	

Type III: This type differs from type II by the fact that it does not contain a separate defining mass M. A *combination* of masses is postulated to have a known mass and this constitutes the necessary equation of definition.

An example of type III is the set of weights (8), (4), (2), (1), (1′), (1″) in which the equation of definition is

$$M = (8) + (4) + (2) + (1) + (1') = (16) = 16 \text{ ounces} .$$

It leads to the following system of equations of condition:

Equations of Condition

$$
\begin{array}{rrrrrl}
+(4) & +(2) & +(1) & +(1') & & = m_1 + (8) \\
+(4) & +(2) & +(1) & & +(1'') & = m_2 + (8) \\
+(4) & +(2) & & +(1') & +(1'') & = m_3 + (8) \\
+(4) & -(2) & -(1) & -(1') & & = m_4 \\
+(4) & -(2) & -(1) & & -(1'') & = m_5 \\
+(4) & -(2) & & -(1') & -(1'') & = m_6 \\
 & +(2) & -(1) & -(1') & & = m_7 \\
 & +(2) & -(1) & & -(1'') & = m_8 \\
 & +(2) & & -(1') & -(1'') & = m_9
\end{array}
\qquad \ldots(57c)
$$

It is to be noted that, in this system, the lowest mass put on a pan is equal to 2 ounces. This is due to the fact that the balance has no longer the appropriate sensitivity when charged with masses of the order of one ounce. This system is used for the masses expressed in pounds and ounces.

Since the universal acceptance of the (metric) SI units, the set of weights which became the most useful, and which is likely to remain the most used, is the set (5), (2), (2′), (1), (1′). It can be used as well for calibrating the submultiples as the multiples of the kilogram. Its system of equations is:

$$
\begin{array}{rrrrrl}
+(5) & -(2) & -(2') & -(1) & & = m_1 \\
+(5) & -(2) & -(2') & & -(1') & = m_2 \\
 & +(2) & -(2') & +(1) & -(1') & = m_3 \\
 & +(2) & -(2') & -(1) & +(1') & = m_4 \\
 & +(2) & -(2') & & & = m_5 \\
 & +(2) & & -(1) & -(1') & = m_6 \\
 & & +(2') & -(1) & -(1') & = m_7 \\
 & & & +(1) & -(1') & = m_8
\end{array}
\qquad \ldots(57d)
$$

The equation of definition can be either

1) of type II, *e.g.* (1′) = M, M designating a mass of nominal value 1 kg. The unknowns are the multiples of the kg.

2) of type III, *e.g.* (5) + (2) + (2′) + (1) = M being a mass also of nominal values 1 kg but the unknowns are now expressed in submultiples of the kg: here they are expressed in a unit equal to 100g, *i.e.* "hectogram".

Even if one of the unknowns, say (1′) is eliminated between the equations of condition and one of the equations of definition, the number of equations (8) remains larger than that of unknowns (4). Therefore, from a rigorous mathematical standpoint, the system has no real solution.

How the method of least squares can lead us to the most satisfactory set of approximate solutions is described in the next section.

Section 3. Solution of Linear Systems

Suppose we have a length standard the value of which at 0° C is not known and the coefficients of temperature of which (linear α and quadratic β) must be determined. All temperatures t will be considered as known without error but length measurements will be considered as affected by random errors (Hagen's type). The classical form of the dilatation equation, l_0 being the length at 0°C and l_t at t°C, is

$$l_0 + \alpha t + \beta t^2 = l_t \; . \qquad \text{...(58)}$$

However, for a general study of similar problems, this equation is given a more general form:

$$x + yt + zt^2 = m \qquad \text{...(59)}$$

in which x, y, z are the unknowns and m is the observed quantity. We can also treat *systems* of such equations, each of which is of the form:

$$a_i x + b_i y + c_i z = m_i \qquad i = 1, 2, \cdots N \qquad \text{...(60)}$$

where a_i, b_i, c_i are known numerical coefficients and m_i are the observed quantities. It is to this type of systems of equations, in which N may become quite large (in any case $N \geq 4$), that the method of least squares will now be applied. It is, however, necessary to underline very strongly the fact that the relation such as (59) is not a real rigorous mathematical equation but only a symbolic representation of the result of a measurement. In all rigor, we must write:

$$a_i x + b_i y + c_i z - m_i = v_i, \qquad i = 1, 2, \cdots N. \qquad \text{...(61)}$$

The second term v_i being due only to the presence of random errors, is in general very small. Let us also note that *any* set of *three* equations chosen arbitrarily in the system (60) has a specific set of solutions. In practice, these solutions present no particular interest.

The solutions which are deduced from the totality of *all available equations of condition* are termed "*adjusted*" solutions. The compensation is made by means of the "*method of least squares*". It is described as follows.

If there are for instance, $N = 12$ measurements, the condition of "least squares" takes the form:

$$S = v_1^2 + v_2^2 + v_3^2 + \cdots + v_{12}^2 = \sum_{i=1}^{i=12} v_i^2 = min. \qquad \text{...(62)}$$

Replacing v_i by their expressions (61) we obtain for S:

$$\begin{aligned} S = & \left(a_1 x + b_1 y + c_1 z - m_1\right)^2 + \\ & + \left(a_2 x + b_2 y + c_2 z - m_2\right)^2 + \\ & + \left(a_3 x + b_3 y + c_3 z - m_3\right)^2 + \\ & + \cdots + \\ & + \left(a_{12} x + b_{12} y + c_{12} z - m_{12}\right)^2 . \end{aligned}$$

The condition S = min is equivalent to the equations:

$$\frac{dS}{dx} = 0, \qquad \frac{dS}{dy} = 0, \qquad \frac{dS}{dz} = 0 . \qquad ...(63)$$

These equations are termed *normal equations.* They form the following system, linear in x, y, z:

$$\begin{aligned} (aa)x + (ab)y + (ac)z &= (am) , \\ (ba)x + (bb)y + (bc)z &= (bm), \\ (ca)x + (cb)y + (cc)z &= (cm) , \end{aligned} \qquad ...(64)$$

with

$$\begin{aligned} (aa) &= a_1^2 + a_2^2 + a_3^2 + \cdots + a_{12}^2 , \\ (bb) &= b_1^2 + b_2^2 + b_3^2 + \cdots + b_{12}^2 , \\ (cc) &= c_1^2 + c_2^2 + c_3^2 + \cdots + c_{12}^2 , \\ (ab) &= a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_{12} b_{12} , \\ (ac) &= a_1 c_1 + a_2 c_2 + a_3 c_3 + \cdots + a_{12} c_{12} , \\ &\cdots\cdots \\ (am) &= a_1 m_1 + a_2 m_2 + a_3 m_3 + \cdots + a_{12} m_{12} . \end{aligned} \qquad ...(65)$$

The system of normal equations (64) has a set of 3 well defined solutions x, y, z if the following condition is satisfied

$$\Delta = \begin{vmatrix} (aa) & (ab) & (ac) \\ (ba) & (bb) & (bc) \\ (ca) & (cb) & (cc) \end{vmatrix} \neq 0. \qquad ...(66)$$

In many systems (particularly in the domain of masses) Δ is equal to zero. The calculations that follow do not apply to such cases. This point is examined at the end of the present section.

Here the calculation is continued as follows.

$$\begin{cases} a_1 = 1 & a_2 = 1 & a_3 = 1 & \dots & a_{12} = 1 , \\ b_1 = t_1 , & b_2 = t_2 , & b_3 = t_3 , & \dots & b_{12} = t_{12} , \\ c_1 = t_1^2 , & c_2 = t_2^2 , & c_3 = t_3^2 , & \dots & c_{12} = t_{12}^2 , \end{cases}$$

so that

$$(aa) = 12 , \qquad (ab) = \sum t , \qquad (ac) = \sum t^2 ,$$

$$(ba) = \sum t , \qquad (bb) = \sum t^2 , \qquad (bc) = \sum t^3 ,$$

$$(ca) = \sum t^2 , \qquad (cb) = \sum t^3 , \qquad (cc) = \sum t^4 ,$$

$$(am) = \sum m , \qquad (bm) = \sum mt , \qquad (cm) = \sum mt^2 .$$

As the determinant Δ is clearly not equal to zero, the normal equations are

$$x \cdot 12 + y\sum t + z\sum t^2 = \sum m$$

$$x\sum t + y\sum t^2 + z\sum t^3 = \sum mt$$

$$x\sum t^2 + y\sum t^3 + z\sum t^4 = \sum mt^2 .$$

The solution of this system *i.e.* x, y, z are the *adjusted* values of the unknowns obtained by the method of least squares. As the latter is based on the fundamental properties of the normal curve, the adjusted solutions are also rightly termed *the most probable values.*

The substitution of x, y, z into the equations (61) will yield the numerical values for v_i:

$$v_i = (a_i x + b_i y + c_i z - m_i) . \qquad \dots(67)$$

These values can now be considered as "deviations". They play the same role as the deviations of individual (repeated) observations from their mean $\overline{m}$. The variance is computed by dividing the sum of squares $\sum_i v_i^2$ by the number of degrees of freedom. In (38) this number was equal to the number of observed m_i's reduced by one unit (from

139 to 138). Here it must be reduced by three units because there are three normal equations the effect of each of which is to reduce by one unit the number of independent equations of conditions *i.e.* of degrees of freedom. Hence

$$\sigma_m^2 = \frac{1}{(12-3)}\left[v_1^2 + v_2^2 + v_3^2 + \cdots + v_{12}^2\right] . \qquad \text{...(68)}$$

The subscript m indicates that σ_m^2 is the variance of the deviations of the observed quantities m_i and not of the deviations related to any specific unknown. The procedures for establishing the variances on specific unknowns are treated individually in various cases found in practice.

Let us now study the algebraic methods for solving normal equations. The method that is the most used in practice is that of determinants. Applied to (64) this method leads to the following expressions of x, y, z:

...(69)

$$x = \frac{+1}{\Delta}\begin{vmatrix}(am) & (ab) & (ac)\\ (bm) & (bb) & (bc)\\ (cm) & (cb) & (cc)\end{vmatrix}, \quad y = \frac{-1}{\Delta}\begin{vmatrix}(aa) & (am) & (ac)\\ (ba) & (bm) & (bc)\\ (ca) & (cm) & (cc)\end{vmatrix}, \quad z = \frac{+1}{\Delta}\begin{vmatrix}(aa) & (ab) & (am)\\ (ba) & (bb) & (bm)\\ (ca) & (cb) & (cm)\end{vmatrix}.$$

The determinants are calculated by classical development procedures as given in algebra. The calculations are not difficult but they become more informative when they are combined with the method of undetermined coefficients. This method is described as follows.

The first equation of (64) is multiplied by λ, the second by μ, the third by ν. This gives:

$$\begin{aligned}\lambda(aa)x + \lambda(ab)y + \lambda(ac)z &= \lambda(am),\\ \mu(ba)x + \mu(bb)y + \mu(bc)z &= \mu(bm),\\ \nu(ca)x + \nu(cb)y + \nu(cc)z &= \nu(cm) .\end{aligned} \qquad \text{...(70a)}$$

The summation of all terms leads to

$$x\Big[\lambda(aa) + \mu(ba) + \nu(ca)\Big] +$$

$$+ y\Big[\lambda(ab) + \mu(bb) + \nu(cb)\Big] +$$

$$+ z\Big[\lambda(ac) + \mu(bc) + \nu(cc)\Big] = \lambda(am) + \mu(bm) + \nu(cm) .$$

It is always possible to select λ, μ, ν so as to make the coefficient of x equal to 1 and those of y and z equal to zero. The resulting system in which λ, μ, ν are the unknowns is:

$$\begin{aligned} (aa)\lambda + (ab)\mu + (ac)\nu &= 1 , \\ (ba)\lambda + (bb)\mu + (bc)\nu &= 0 , \\ (ca)\lambda + (cb)\mu + (cc)\nu &= 0 . \end{aligned} \qquad \text{...(70b)}$$

Its solutions λ, μ, ν, (always finite as $\Delta \neq 0$) will lead to the value of x:

$$x = \lambda(am) + \mu(bm) + \nu(cm) . \qquad \text{...(71)}$$

If now in (70a) we designate the unknowns by λ', μ', ν' and shift 1 to the second equation, we obtain the system

$$\begin{aligned} (aa)\lambda' + (ab)\mu' + (ac)\nu' &= 0 , \\ (ba)\lambda' + (bb)\mu' + (bc)\nu' &= 1 , \\ (ca)\lambda' + (cb)\mu' + (cc)\nu' &= 0 . \end{aligned} \qquad \text{...(72)}$$

It will lead to an equation for y. Finally, using the symbols λ'', μ'', ν'', we form an equation for z. The compensated values of x, y and z are therefore

$$\begin{aligned} x &= \lambda(am) + \mu(bm) + \nu(cm) , \\ y &= \lambda'(am) + \mu'(bm) + \nu'(cm) , \\ z &= \lambda''(am) + \mu''(bm) + \nu''(cm) . \end{aligned} \qquad \text{...(73)}$$

It is still possible to go somewhat farther in these transformations by noticing that (73) can be put under the form

$$\begin{aligned} x &= \sum_i m_i(\lambda a_i + \mu b_i + \nu c_i) &&= \sum_i m_i \alpha_i \ , \\ y &= \sum_i m_i(\lambda' a_i + \mu' b_i + \nu' c_i) &&= \sum_i m_i \beta_i \ , \\ z &= \sum_i m_i(\lambda'' a_i + \mu'' b_i + \nu'' c_i) &&= \sum_i m_i \gamma_i \ , \end{aligned}$$

where

$$\begin{aligned} \alpha_i &= \lambda a_i + \mu b_i + \nu c_i \ , \\ \beta_i &= \lambda' a_i + \mu' b_i + \nu' c_i \ , \\ \gamma_i &= \lambda'' a_i + \mu'' b_i + \nu'' c_i \ . \end{aligned}$$

Hence finally the expressions of x, y, z in terms of observed quantities become:

$$\begin{aligned} x &= \sum_i m_i \alpha_i = (m\alpha) \ , \\ y &= \sum_i m_i \beta_i = (m\beta) \ , \\ z &= \sum_i m_i \gamma_i = (m\gamma) \ . \end{aligned} \qquad \text{...(74)}$$

The theory presented in this section has been used as the fundamental tool not only in the calibration of mass standards but also other metrological domains. It has remained in a stable form for more than a century until the publication of an article in "Metrologia" by M. Grabe in 1978 (Metrologia 14:143). In this article the author shows that the system such as (57a) and (57b) can be significantly modified so as to make it easier to calculate and to improve the accuracy of the results. He also points out that such suggestions have been made and published by well known authors (Lenk, Kohlrausch) in the thirties but, for some strange reasons, have never been adopted by mass metrologists. Grabe's article has been thoroughly analyzed at the N.R.C. and followed by an article in Metrologia by Dr. M. Zuker, N.R.C. biomathematician.

As a conclusion of the present section we shall simply establish the correspondence between the symbols of this section and the symbols specifically used in the calibration of masses.

The first equation of (57d) is now written as follows:

$$(+1)(5) + (-1)(2) + (-1)(2') + (-1)(1) + (0)(1') = m_1 \, .$$

Hence, $a_1 = +1$, $b_1 = -1$, $c_1 = -1$, $d_1 = -1$, $e_1 = 0$, $m_1 = m_1$, and all equations of the system are treated in the same manner. The calculation of the normal equation can be made in two different but equivalent ways:

a) either the substitution $(1') = m$ is made into the equations of condition and then the normal equations (64) are established, or

b) the normal equations are established first and then the substitution $(1') = M$ is made. The procedure a) yields:

$$\begin{array}{lllll}
+(5) & -(2) & -(2') & -(1) & = m_1 \\
+(5) & -(2) & -(2') & & = m_2 + M \\
 & +(2) & -(2') & +(1) & = m_3 + M \\
 & +(2) & -(2') & -(1) & = m_4 - M \\
 & +(2) & -(2') & & = m_5 \\
 & +(2) & & -(1) & = m_6 + M \\
 & & +(2') & -(1) & = m_7 + M \\
 & & & +(1) & = m_8 + M \, .
\end{array} \qquad \ldots(75)$$

and the system of *four* normal equations takes the form

$$\begin{array}{lllll}
2(5) & -2(2) & -2(2') & -(1) & = N_1 + M \\
-2(5) & +6(2) & -(2') & & = N_2 \\
-2(5) & -(2) & +6(2') & & = N_3 \\
-(5) & & & +6(1) & = N_4 + M \, .
\end{array} \qquad \ldots(76)$$

in which $N_1 \ldots N_4$ are linear functions of m_i.

The procedure b) will yield the following system of *five* normal equations:

$$
\begin{array}{rrrrrl}
2(5) & -2(2) & -2(2') & -(1) & -(1') & = N_1 \\
-2(5) & +6(2) & -(2') & & & = N_2 \\
-2(5) & -(2) & +6(2') & & & = N_3 \\
-(5) & & & +6(1) & -(1') & = N_4 \\
-(5) & & & -(1) & +6(1') & = N_5 \, .
\end{array} \qquad \ldots(77)
$$

As the determinant Δ is equal to zero it indicates that one of the equations is redundant and that therefore there exists a linear relation between the N's. This relation is

$$5N_1 + 2N_2 + 2N_3 + N_4 + N_5 = 0 \, .$$

If, for instance, we discard the last equation (N_5) and put $(1') = M$ we find the system above (76).

Section 4. Orthogonal Systems

The main objective of this section is to show why the "basic" system of the linear equations of condition (57A) can be advantageously replaced by the "modified" system (57B). For this purpose let us form, in each of these two systems, one of the coefficients $(\alpha \; \beta)$ which figure in the normal equations, and in which $\alpha \neq \beta$, for instance, $(bc) = b_1c_1 + b_2c_2 + \cdots + b_{12}c_{12}$.

System A b	System A c	System B b	System B c
+1	+1	+1	+1
+1	+1	+1	+1
+1	0	+1	0
+1	+1	−1	−1
−1	−1	−1	−1
−1	−1	−1	0
−1	−1	+1	−1
−1	0	+1	−1
+1	−1	+1	−1
+1	−1	+1	−1
0	+1	0	+1
0	0	0	+1
$\sum bc = (bc) = +4$		$\sum bc = (bc) = 0$	

The specific property of this modified system is that *all* coefficients in which $\alpha \neq \beta$ are equal to zero. This is the reason why the system is termed orthogonal. The values of the coefficients form $(\alpha\alpha)$ are:

$$(aa) = 6 \,,$$
$$(bb) = (cc) = (dd) = (ee) = 10 \,.$$

The system of normal equations is therefore:

$$\begin{aligned}
6(5) \qquad\qquad\qquad\qquad\qquad\qquad &= (am) + 3M \\
10(3) \qquad\qquad\qquad\qquad &= (bm) + 3M \\
10(2) \qquad\qquad\qquad &= (cm) + 2M \\
10(1) \qquad\qquad &= (dm) + M \\
10(1') \qquad &= (em) + M \,.
\end{aligned}$$

The symbols (am), (bm) ... *etc.* are oftem replaced by the symbols N_1, N_2, ... N_5.

The solutions are obtained without effort since each equation may be solved independently, as demonstrated below:

$$(5) = \frac{1}{6}\left[(m_1 + m_2 + m_3 + m_4 + m_5 + m_6) + 3M\right] ,$$

$$(3) = \frac{1}{10}\left[(m_1 + m_2 + m_3 - m_4 - m_5 - m_6 + m_7 + m_8 + m_9 + m_{10}) + 3M\right] ,$$

$$(2) = \frac{1}{10}\left[(m_1 + m_2 - m_4 - m_5 - m_7 - m_8 - m_9 - m_{10} + m_{11} + m_{12}) + 2M\right] ,$$

$$(1) = \frac{1}{10}\left[(m_1 - m_2 + m_3 + m_4 - m_5 - m_6 - m_7 - m_8 - m_{11} - m_{12}) + M\right] ,$$

$$(1') = \frac{1}{10}\left[(-m_1 + m_2 + m_3 - m_4 + m_5 - m_6 - m_9 - m_{10} - m_{11} - m_{12}) + M\right] .$$

The group variance σ_m^2 is calculated in the usual manner and the theorem of propagation of variance leads to the following values of the variances of the individual unknowns:

$$\sigma_{(5)}^2 = \left(\frac{1}{6}\right)^2 (1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2)\sigma_m^2 = \frac{1}{6}\,\sigma_m^2 = 0.167\sigma_m^2 ,$$

$$\sigma_{(3)}^2 = \sigma_{(2)}^2 = \sigma_{(1)}^2 = \sigma_{(1')} = \left(\frac{1}{10}\right)^2 \sigma_m^2 = 0.1\sigma_m^2 .$$

The most important property of orthogonal systems resides in the fact that all unknowns are totally independent of each other. Thus if a "permanent" group of masses is formed, the variance of the group is exclusively equal to the sum of variances of the components. This is due to the fact that, in orthogonal systems, all so-called "covariances" are equal to zero, while in ordinary systems, their numerical values must be taken into account. This point is beyond the level of this course and the reader is referred to treatises based on the matrix calculus.

The reader who is interested in orthogonal systems will find in the Bibliography all information concerning this highly interesting subject. Recently, Dr. G. Chapman, from the Division of Physics of NRCC, has applied the methods based on the use of orthogonal systems to the calibration of angles. He also greatly simplified the methods for transforming the non-orthogonal systems into orthogonal ones. His contribution completes that of Grabe and Zuker.

Taking into account the amount of information accumulated in all metrological laboratories of the world in the course of the last century (based on ordinary non-orthogonal methods) it is likely that progress due to the introduction of new procedures will be rather slow. It is nevertheless obvious that orthogonal systems will finally eliminate all non-orthogonal ones in all those domains of metrology in which they can be applied.

Chapter V
Combination of Variates

Section 1. Summation of Normal Variates

This section must start with a strong emphasis on the difference between the meaning of the term "mixture" and that of the term "sum". Suppose that we have two variates X and Y represented by large samples of sizes N_1 and N_2, respectively. The "mixture" is represented by the sample of size $N_1 + N_2$ obtained by simply pooling the populations of the components.

The "sum" of the variates X and Y is denoted by the symbol $(X + Y)$ and is represented by the sample of $N_1 \times N_2$ elements. It is constituted by all sums that can be formed by adding each element of X to each element of Y.

In general, capital letters are used to designate variates and the corresponding small letters to designate individual elements. Thus we can write*

$$Z = X + Y ,$$

$$z = x + y .$$

The probability density functions (pdf) will be designated by the symbols $\Phi_1(x)$ and $\Phi_2(y)$.

Let z_0 be a certain arbitrary value of z. Hence,

$$z_0 = x + y , \quad y = z_0 - x , \quad dy = dz_0 - dx.$$

The probability that x will fall into dx is

$$dP_x = \Phi_1(x)dx$$

* In some treatises on higher statistics published since the mid-century, the "summation of variates" belongs to a more general type of operation termed "convolution of functions". A narrower practical point of view does not require such an extension of the theory.

and the probability that y will fall into dy is therefore

$$\Phi_2(y)\ dy = \Phi_2(z_0 - x)(dz_0 - dx)\ .$$

The compound probability of x and $(z_0 - x)$ is

$$\Phi_1(x)dx \cdot \Phi_2(z_0 - x)(dz_0 - dx) \qquad ...(78)$$

and is a function of x. Its expression should be formed for all possible values of x. If $[x]$ designates the domain of integration in x, the probability dP_{z_0} that z will fall into dz_0 is equal to the integral

$$dP_{z_0} = \int_{[x]} \Phi_1(x)\ \Phi_2(z_0 - x)\ dx\ dz_0$$

as, obviously the term in $(dx)^2$ can be omitted. The final formula is

$$dP_{z_0} = dz_0 \int_{[x]} \Phi_1(x)\ \Phi_2(z_0 - x)dx\ . \qquad ...(79)$$

Concerning the domain of integration $[x]$ some remarks are necessary. The extreme limits between which X and Y are comprised are not necessarily infinite: X may exist only between x' and x'' while Y may exist only between y' and y''. It is thus necessary to remember the following rule: *the integration domain $[x]$ extends to all values of x which satisfy the relation $z_0 = x + y$, under the formal condition that y be located in its own range, i.e.* between y' and y''.

Numerical Example

Suppose that X and Y range both between 0 and $+\infty$ and that we have $z = 5$. Obviously we can select $x = 2$ so that y can be equal to 3 (which is of course in its range). We can write

$$x = 5 - y$$

and consider the values of x from 2 upwards: when $x = 3$ then $y = 2$; when $x = 4$, then $y = 1$; when $x = 5$ then $y = 0$. It is not possible to go beyond $x = 5$ because $y = 0$ is the lowest value y can take. Thus $x = 5$ is the highest limit of integration.

Similarly, we find that, for $x = 2$, $y = 3$; for $x = 1$, $y = 4$; for $x = 0$, $y = 5$. Hence $x = 0$ is the lowest limit x can take. Thus $[x]$ extends from $x = 0$ to $x = 5$ and

$$dP_{z_0=5} = dz_0 \int_0^5 \Phi_1(x)\ \Phi_2(5-x)dx\ .$$

This expression is generalized for any possible value of z and takes the form:

$$dP_z = dz \int_0^z \Phi_1(x)\ \Phi_2(z-x)\ dx \qquad ...(80)$$

in which the subscript of z is not necessary and is therefore omitted.

In the summation of two *normal* variates *i.e.* when Φ_1 and Φ_2 represent the expressions

$$\Phi_1(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}},$$

$$\Phi_2(y) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_2^2}},$$

the limits in the integrals are $-\infty$ and $+\infty$. The integrations can be performed by well known methods*, and lead to the formula:

$$dP_z = \frac{1}{\sqrt{\sigma_x^2 + \sigma_y^2}\sqrt{2\pi}} e^{-\frac{z^2}{2(\sigma_x^2+\sigma_y^2)}} dz. \qquad ...(81)$$

This result is often presented under the form of a theorem termed *theorem of addition of variances: the summation of two normal variates leads to a normal variate the variance of which is equal to the sum of the variances of the components.*

Another fundamental transformation is the multiplication of a variate (x) by a constant factor α:

* A detailed presentation of this integration is given in Section 3 of this Chapter

$$Z = \alpha X$$

$$z = \alpha x \,, \qquad dz = \alpha dx \,, \qquad \sigma_z^2 = \alpha^2 \sigma_x^2 \,.$$

If in the expression of dP_x for a normal variate, *i.e.*

$$dP_x = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}} dx \,,$$

the variable x is replaced by the variable z, dP_z takes the form

$$dP_z = \frac{1}{\sigma_z \sqrt{2\pi}} e^{-\frac{z^2}{2\sigma_z^2}} \qquad \text{...(82)}$$

in which

$$\sigma_z^2 = \alpha^2 \sigma_x^2 \,, \qquad \sigma_z = \alpha \sigma_x \,. \qquad \text{...(83)}$$

Combining this with (81), we obtain, for any linear combination of two normal variates

$$Z = \alpha X + \beta Y \,, \qquad \text{...(84)}$$

the following expression for σ_z^2:

$$\sigma_z^2 = \alpha^2 \sigma_x^2 + \beta^2 \sigma_y^2 \,. \qquad \text{...(85)}$$

This is one of the most important relations constantly applied in the analysis of linear equations as they appear e.g. in the calibration of mass standards. The reader should refer to the end of Ex. 15 and to the final part of Ex. 16.

The above expressions for $Z = X + Y$ and $Z = \alpha X$ must be modified when the components are decentered, *i.e.* not centered on $x = 0$. The calculations with decentered expressions are somewhat more complicated but present no specific difficulty and constitute excellent exercises. Their results are extremely simple.

For a linear form:

$$1) \quad X\ (centre\ a_x) + Y(centre\ a_y) = Z(centre\ c = a_x + a_y) \qquad ...(86)$$

$$2) \quad \alpha X(centre\ a_x) = Z(centre\ c = \alpha a_x) \qquad ...(87)$$

$$\alpha X(centre\ a_x) + \beta Y(centre\ a_y) = Z(centre\ c = \alpha a_x + \beta a_y) \qquad ...(88)$$

The expression for variances are not affected by the presence of centres:

$$Z = \alpha X(a_x) \qquad \sigma_z^2 = \alpha^2 \sigma_x^2 ,$$

$$Z(c) = \alpha X(a_x) + \beta Y(a_y) \qquad \sigma_z^2 = \alpha^2 \sigma_x^2 + \beta^2 \sigma_y^2 .$$

To avoid any misunderstanding the above calculations are summarized as follows:

Components of the sum:

X: element x, centre a_x, variance σ_x^2, coefficient α,

Y: element y, centre a_y, variance σ_y^2, coefficient β,

Variate sum:

$Z = \alpha X + \beta Y$

Z: element z, centre c, variance σ_z^2.

Probability dP :

$$dP_z = \frac{1}{\sigma_z \sqrt{2\pi}} e^{-\frac{(z-c)^2}{2\sigma^2}} dz \quad \text{in which} \begin{cases} c = \alpha a_x + \beta a_y \\ \sigma_z^2 = \alpha^2 \sigma_x^2 + \beta^2 \sigma_y^2 \end{cases}$$

Section 2. Expected Values

There are good reasons for having a universally adopted symbol for denoting the mean value of a variable quantity. Because of the historical origins of the calculus of probability, this symbol is $E(\ )$, the initial of "expectation" (in French, "espérance"). The symbol between parentheses designates the quantity the mean value of which is considered.

Here are some of the quantities previously considered expressed by means of the symbol E:

$$
\begin{aligned}
E(x) &= \bar{x} = \omega_1(x) , \\
E(x-\bar{x}) &= \mu_1(x) = 0 , \\
E(x^2) &= \omega_2(x) , \\
E\left[(x-\bar{x})^2\right] &= \mu_2(x) = \sigma^2 , \quad etc.
\end{aligned}
\qquad \text{...(91)}
$$

It must be strongly underlined that when the symbol E is used in combinations of independent variates X and Y as e.g. $E(x+y)$, the symbol $(x+y)$ designates the sum of any element x of X and any element y of Y. Thus the process of summation of variates is considered in a convolutional sense, *i.e.* each element of one variate is associated with each element of the other variate. In the operation below the addition is performed on the following variates:

variate X, represented by m elements $x_i(i = 1, 2, \cdots m)$ the probability of x_i being p_i.

variate Y, represented by m elements $y_j(j = 1, 2, \cdots n)$ the probability of y_j being p_j.

The variate $(X+Y)$ can thus take $m \times n$ values the compound probability p_{ij} of each of them being equal to

$$p_{ij} = p_{ji} = p_i p_j = p_j p_i .$$

By the theorem of compound probability:

$$E(X+Y) = E(x+y) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}(x_i + y_j) = \sum_i \sum_j p_i p_j (x_i + y_j) ,$$

$$= \sum_i \sum_j p_i p_j x_i + \sum_i \sum_j p_i p_j y_j ,$$

$$= \sum_j p_j \sum_i p_i x_i + \sum_i p_i \sum_j p_j y_j .$$

As $\sum_j p_j = 1$ and $\sum_i p_i = 1$, we have

$$\sum(x+y) = \sum_i p_i x_i + \sum_j p_j y_j ,$$

$$\sum(x+y) = E(x) + E(y) . \qquad ...(92)$$

Expressed in symbolic notation this is:

$$E(X+Y) = E(X) + E(Y) .$$

Hence we have the important theorem: *the expected value of the sum of two independent variates is equal to the sum of their respective expected values.*

In a similar way may be calculated the expected value of the product (XY):

$$E(XY) = E(\overline{xy}) = \sum_i \sum_j p_i p_j x_i y_j .$$

By two successive summations

$$E(xy) = \sum_i p_i x_i E(y) = E(y) \sum_i p_i x_i ,$$

$$= E(x) \cdot E(y) . \qquad ...(93)$$

Here the theorem is: *the expected value of a product of two independent variates is equal to the product of their respective expected values.*

These theorems lead to very interesting conclusions when they are applied to two *normal* variates X and Y both distributed about their mean values $\bar{x} = 0$ and $\bar{y} = 0$, respectively. Then

$$E(x+y) = E(x) + E(y) = 0,$$

$$E(xy) = E(x)E(y) = 0\ .$$

A very important expectation is of the form

$$E\left[(X+Y)^2\right] = E\left[(x+y)^2\right].$$

The sum $x+y$ is of course considered in the convolutional sense: any element x is added to any element y. Now,

$$(x+y)^2 = x^2 + 2xy + y^2$$

and, hence,

$$E\left[(x+y)^2\right] = E(x^2) + E(y^2) + 2E(x)E(y)\ .$$

As the last term is equal to zero,

$$E\left[(X+Y)^2\right] = E(x^2) + E(y^2)\ . \qquad ...(94)$$

This equation is equivalent to the theorem of addition of variances. If we write $Z = X + Y$,

$$E(z^2) = E(x^2) + E(y^2)\ ,$$

or, what is the same,

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2\ . \qquad ...(95)$$

The above theorems may be established by means of expressions in which the variables are continuous. As it has been said previously such operations are not always simple. However, they should be performed by a reader who is interested in the mathematical aspects of the theory.

A very important point, particularly for observational techniques, is the distribution of the mean of a small sample drawn from the population of a normal (or quasi-normal) variate $X(\mu, \sigma^2)$. Let us for instance, consider a sample consisting of five elements x_i (i = 1, 2, $\cdots$ 5). Instead of treating them as drawn from the same X let us assume the existence of five variates having *identical* populations each population supplying one single element to the sample. Thus the sum

$$S = \sum_i x_i = x_1 + x_2 + x_3 + x_4 + x_5 = 5\bar{x} \, .$$

can be treated as an element created by the summation of of five individual (but identical to each other) variates. It is easy to establish that according to the above described theorems,

$$E(\bar{x}) = \mu \, , \qquad \text{...(96)}$$

i.e. that $\bar{x}$ has the same centre as all other variates. The variance of $\bar{x}$ is calculated as follows.

According to the theorem of summation, the variance of the sum S is equal to $n\sigma^2$. Its standard deviation is $\sigma\sqrt{n}$. As S is formed on n terms, the standard deviation on each term, *i.e.* on $\bar{x}$, is equal to $\frac{\sigma\sqrt{n}}{n} = \frac{\sigma}{\sqrt{n}}$. Those observers who perform repeated measurements (e.g. geodesists) but group them into small samples, are very familiar with the expression $\frac{\sigma}{\sqrt{n}}$. They follow the simple rule that if the samples are all of the same size n then

$$\textit{std. dev. of the sample mean} = \frac{\sigma}{\textit{square root of } n} \qquad \text{...(97)}$$

Section 3. Integration of the Equation of Summation

The fact that "variance" (or, what is the same, "standard deviation") occupies such an important place in statistics is actually relatively new. In most of the older treatises, *i.e.* in the nineteenth century and the first half of the twentieth century, the Normal Law was used under the form

$$f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}$$

The parameter h is then termed "precision" and, in terms of the standard deviation σ, is equal to

$$h = \frac{1}{\sigma\sqrt{2}} .$$

Obviously certain algebraic calculations should be of a simpler form when h is used instead of σ. Such is the case when the calculations concerning convolutions are performed in detail.

So far as the limits of the integrations are concerned, the fact that they are always equal to $\pm\infty$ removes the complications that are treated when formula (79) is described.

The formula (79) applied to the component variates

$$f_1(x) = \frac{h_1}{\sqrt{\pi}} e^{-h_1^2 x^2} , \quad f_2(y) = \frac{h_2}{\sqrt{\pi}} e^{-h_2^2 y^2}$$

gives

$$dP_{Z_0} = dz_0 \frac{h_1 h_2}{\pi} \int_{-\infty}^{+\infty} e^{-\left[(h_1^2 + h_2^2)x^2 - 2h_2^2 z_0 x + h_2^2 z_0^2\right]} dx$$

This integral is of the type that can be integrated by means of the identity

$$a\theta^2 + b\theta + c = a\left(\theta + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} .$$

The substitution

$$y = \sqrt{a}\left(\theta + \frac{b}{2a}\right), \qquad dy = d\theta\sqrt{a} ,$$

introduced into the integral

$$G = \int_{-\infty}^{+\infty} e^{-(a\theta^2 + b\theta + c)} d\theta$$

gives

$$G = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} e^{-y^2 - r} dy , \qquad r = \frac{4ac - b^2}{4a}$$

and therefore

$$G = \frac{e^{-r}}{\sqrt{a}} \int_{-\infty}^{+\infty} e^{-y^2} dy$$

By (43)

$$G = \frac{e^{-r}}{\sqrt{a}} \sqrt{\pi}$$

The expressions of a, b and c in terms of the symbols h_1, h_2 and z_0 are

$$a = h_1^2 + h_2^2 \ ; \quad b = -2h_2^2 z_0 \ ; \quad c = h_2^2 z_0^2 \ ;$$

so that

$$r = \frac{4(h_1^2 + h_2^2)h_2^2 z_0^2 - 4h_2^4 z_0^2}{4(h_1^2 + h_2^2)} = \frac{h_1^2 h_2^2 z_0^2}{h_1^2 + h_2^2} ,$$

$$\sqrt{a} = \sqrt{h_1^2 + h_2^2}$$

and, finally

$$G = \frac{\sqrt{\pi}}{\sqrt{h_1^2 + h_2^2}} e^{-\frac{h_1^2 h_2^2}{h_1^2 + h_2^2} z_0^2} ,$$

so that (79) becomes

$$dP_{z_0} = dz_0 \frac{h_1 h_2}{\pi} \qquad G = \frac{1}{\sqrt{\pi}} \frac{h_1 h_2}{\sqrt{h_1^2 + h_2^2}} e^{-\left(\frac{h_1 h_2}{\sqrt{h_1^2 + h_2^2}}\right)^2 z_0^2} dz_0 .$$

A simple replacement of h_1 and h_2 by the symbols σ_1 and σ_2 will lead directly to the expression (95).

Summary: if the "precisions" of the components are h_1 and h_2, and the "precision" of the variate Z obtained by summations is designated by H, then

$$H = \frac{h_1 h_2}{\sqrt{h_1^2 + h_2^2}} ,$$

or

$$\frac{1}{H^2} = \frac{1}{h_1^2} + \frac{1}{h_2^2} .$$

Expressed in terms of variances, this relation, in conformity with (95), takes the form

$$S^2 = \sigma_1^2 + \sigma_2^2 ,$$

in which

$$S = \frac{1}{H\sqrt{2}} .$$

Chapter VI
Euler's Functions and Variates

Section 1. Gamma and Beta Functions

The **Gamma function** $\Gamma(n)$ is defined by the definite integral

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx \; . \qquad ...(98)$$

The variable takes on therefore only positive values ($x \geq 0$) and it can be shown that the integral is convergent only when n is positive.

$\Gamma(n)$ satisfies the fundamental relation

$$\Gamma(n) = (n-1)\Gamma(n-1) \qquad ...(99)$$

provided, of course, that $(n-1)$ is also positive. This relation is demonstrated by means of the method of "integration by parts". To apply this method, it is convenient to put the integrand under the form

$$(x^{n-1})\cdot(e^{-x}dx) \; .$$

which suggests that we consider the following partial functions:

$$u = x^{n-1} \quad \text{and} \quad v = (-e^{-x}) \; .$$

As $dv = e^{-x}dx$, the integrand in (98) is of the form udv. According to the method of integration by parts, the formula

$$d(uv) = udv + vdu$$

leads here to the relation

$$d\left[(x^{n-1})\cdot(-e^{-x})\right] = (x^{n-1})\cdot(e^{-x}dx) + (-e^{-x})\left[(n-1)x^{n-2}dx\right] .$$

Integrating both sides, we obtain:

$$\left| -x^{n-1}\, e^{-x} \right|_0^\infty = \int_0^\infty e^{-x}\, x^{n-1} dx \;-\; (n-1)\int_0^\infty e^{-x}\, x^{(n-1)-1} dx \;.$$

As the left-hand term is equal to zero, this relation reduces to

$$\int_0^\infty e^{-x}\, x^{n-1} dx \;=\; \Gamma(n) \;=\; (n-1)\int_0^\infty e^{-x}\, x^{(n-1)-1} dx$$

i.e. to (99):

$$\Gamma(n) \;=\; (n-1)\Gamma(n-1) \;.$$

So far as n is concerned, two cases must be examined

a) n integer, *e.g.* $n=4$. Then

$$\Gamma(4) \;=\; 3\Gamma(3) \;=\; 3\times 2\Gamma(2) \;=\; 3\times 2\times 1\times \Gamma(1) \;.$$

As $\Gamma(1) = 1$, therefore $\Gamma(4) = 3!$, and, in general, for any integer n we have

$$\Gamma(n) \;=\; (n-1)! \qquad \text{...(100)}$$

b) n not integer, *e.g.* $n=4.25$. Then

$$\Gamma(4.25) \;=\; 3.25\times 2.25\times 1.25\Gamma(0.25)$$

and the value of $\Gamma(0.25)$ must be determined by a numerical integration.

The function $\Gamma(n)$ is tabulated between $n=1$ and $n=2$ so that, in practice, we can calculate $\Gamma(4.25)$ by the relation

$$\Gamma(4.25) \;=\; 3.25\times 2.25\times 1.25\Gamma(1.25) \;. \qquad \text{...(102)}$$

Table V

Values of $\Gamma(n)$, $1 \leq n \leq 2$

n	Γ	n	Γ	n	Γ	n	Γ
1.00	1.00000	1.25	.90640	1.50	.88623	1.75	.91906
1.01	.99433	1.26	.90440	1.51	.88659	1.76	.92137
1.02	.98884	1.27	.90250	1.52	.88704	1.77	.92376
1.03	.98355	1.28	.90072	1.53	.88757	1.78	.92623
1.04	.97844	1.29	.89904	1.54	.88818	1.79	.92877
1.05	.97350	1.30	.89747	1.55	.88887	1.80	.93138
1.06	.96874	1.31	.89600	1.56	.88964	1.81	.93408
1.07	.96415	1.32	.89464	1.57	.89049	1.82	.93685
1.08	.95973	1.33	.89338	1.58	.89142	1.83	.93969
1.09	.95546	1.34	.89222	1.59	.89243	1.84	.94261
1.10	.95135	1.35	.89115	1.60	.89352	1.85	.94561
1.11	.94739	1.36	.89018	1.61	.89468	1.86	.94869
1.12	.94359	1.37	.88931	1.62	.89592	1.87	.95184
1.13	.93993	1.38	.88854	1.63	.89724	1.88	.95507
1.14	.93642	1.39	.88785	1.64	.89864	1.89	.95838
1.15	.93304	1.40	.88726	1.65	.90012	1.90	.96177
1.16	.92980	1.41	.88676	1.66	.90167	1.91	.96523
1.17	.92670	1.42	.88636	1.67	.90330	1.92	.96878
1.18	.92373	1.43	.88604	1.68	.90500	1.93	.97240
1.19	.92088	1.44	.88580	1.69	.90678	1.94	.97610
1.20	.91817	1.45	.88565	1.70	.90864	1.95	.97988
1.21	.91558	1.46	.88560	1.71	.91057	1.96	.98374
1.22	.91311	1.47	.88563	1.72	.91258	1.97	.98763
1.23	.91075	1.48	.88575	1.73	.91466	1.98	.99171
1.24	.90852	1.49	.88595	1.74	.91683	1.99	.99581
						2.00	1.00000

If the value such as $\Gamma(0.25)$ is required, it can be deduced, as follows, from the relation $\Gamma(1.25) = 0.25\Gamma(0.25)$: $\dfrac{\Gamma(1.25)}{0.25} = 4\times 0.90640 = 3.61560$.

The substitution $x = y^2$ leads to the frequently used form:

$$\Gamma(n) = 2\int_0^\infty e^{-y^2}\, y^{2n-1}\, dy \; . \qquad \ldots(104)$$

This shows that (Table II, J_0):

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-y^2}\, dy = \sqrt{\pi} . \qquad ...(105)$$

Another frequently used form is obtained by the substitution $x = ay$:

$$\Gamma(n) = a^n \int_0^\infty e^{-ay}\, y^{n-1}\, dy . \qquad ...(106)$$

The **Beta Function B** is defined by the integral

$$B(l,m) = \int_0^1 x^{l-1}\, (1-x)^{m-1}\, dx . \qquad ...(107)$$

This integral is convergent only for positive values of l and m: $l > 0$, $m > 0$. It is easy to see immediately that for $l = m = 1$,

$$B(1,1) = \int_0^1 dx = 1 . \qquad ...(108)$$

The integral $B(l,m)$ can take several forms as shown in the sequel. In particular, the substitution a) $x = 1-y$ shows the symmetry in l and m:

$$B(l,m) = B(m,l) .$$

Here are a few of the currently used forms of B:

a) $x = 1-y$: $B = \int_0^1 y^{m-1}\, (1-y)^{l-1}\, dy$ (Symmetry)

b) $x = \sin^2\theta$: $B = \int_0^{\frac{\pi}{2}} \sin^{2l-1}\theta\, \cos^{2m-1}\theta\, d\theta$

c) $x = \dfrac{1}{1+y}$: $B = \int_0^\infty \dfrac{y^{m-1}}{(1+y)^{l+m}}\, dy$ ...(109)

d) In *c*):

$y = \dfrac{1}{z}$: $B = \int_0^\infty \dfrac{z^{l-1}}{(1+z)^{l+m}}\, dz$

e) In *c*): Decompose B as follows:

$$B = \int_0^1 \frac{y^{m-1}}{(1+y)^{l+m}}\, dy + \int_1^\infty \frac{y^{m-1}}{(1+y)^{l+m}}\, dy \ .$$

Then, in the second integral, put $y = \dfrac{1}{z}$ to obtain

$$B = \int_0^1 \frac{y^{m-1}}{(1+y)^{l+m}}\, dy + \int_0^1 \frac{z^{l-1}}{(1+z)^{l+m}}\, dz \ .$$

The integrals can be grouped in one single integral by replacing y and z by the same symbol, say x:

$$B = \int_0^1 \frac{x^{m-1} + x^{l-1}}{(1+x)^{l+m}} \ . \qquad ...(110)$$

This form confirms that B is symmetrical in l and m. Note that the form b) leads to

$$B(½\ ,\ ½) = 2\int_0^{\frac{\pi}{2}} d\theta = \pi \ . \qquad ...(111)$$

Beta and Gamma functions are interconnected by the relation

$$B(l,m) = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)} \ . \qquad ...(112)$$

This relation can be proven by means of a double integral but there is still another method, due to Jacobi, which is very direct and elegant. It starts with the relation (106) in which we replace a by $(1+y)$ and n by $l+m$. Hence,

$$\frac{\Gamma(l+m)}{(1+y)^{l+m}} = \int_0^\infty e^{-(1+y)x}\, x^{l+m-1}\, dx \; .$$

Now both sides are multiplied by $y^{m-1}dy$ and integrated between 0 and ∞:

$$\Gamma(l+m)\int_0^\infty \frac{y^{m+1}}{(1+y)^{l+m}}\, dy = \int_0^\infty y^{m-1}\, dy \int_0^\infty e^{-x-xy}\, x^{l+m-1}\, dx \; .$$

The order of integrating is arranged as follows:

$$\Gamma(l+m)B(l,m) = \int_0^\infty x^{l+m-1}\, e^{-x}\, dx \int_0^\infty y^{m-1}\, e^{-xy}\, dy \; .$$

This shows that the integral in y is equal to

$$\int_0^\infty e^{-xy}\, y^{m-1}\, dy = \frac{\Gamma(m)}{x^n} \; .$$

and that therefore

$$\Gamma(l+m)\, B(l,m) = \Gamma(m)\int_0^\infty x^{l-1}\, e^{-x}\, dx = \Gamma(m)\Gamma(l) \; ,$$

which is equivalent to (112), *i.e.*:

$$B(l,m) = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)} \; .$$

Section 2. Gamma and Beta Variates

A very important event in the calculus of probability was the discovery in 1875 by Helmert (who was a geodesist) that there is a link between the Gaussian normal function and Euler's functions. It was provoked by the fact that Helmert had the idea to investigate the properties of the exponent in the normal function, *i.e.* the properties of the variate u defined as being equal to

$$u = \frac{x^2}{2\sigma^2} . \qquad \text{...(114)}$$

If the symbol u is introduced into the expression of dP_x, the result takes the form

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{e^{-u} \, u^{-1/2}}{2\sqrt{\pi}} du .$$

Here an important point must be taken into account: when x varies from $-\infty$ to $+\infty$ but u varies only from 0 to $+\infty$, each single value of u corresponds to two equal values $(+x)^2$ and $(-x)^2$. Hence, in the formula for dP_u, the right-hand term must be doubled:

$$dP_u = \frac{e^{-u} \, u^{-1/2}}{\sqrt{\pi}} du .$$

We can notice that, on the one hand, $-1/2 = 1/2 - 1$ and, on the other hand, that $\sqrt{\pi} = \Gamma(1/2)$. Hence, dP_u can be given the form

$$dP_u = \frac{e^{-u} \, u^{1/2 - 1}}{\Gamma(1/2)} du .$$

Now, the right-hand term can be treated as a special case of a function $f(u)$ the general form of which is

$$f(u) = \frac{e^{-u} \, u^{n-1}}{\Gamma(n)} . \qquad \text{...(115)}$$

The function $f(u)$ is considered as the pdf of a *variate u termed "Gamma Variate with parameter n."* It must be always borne in mind that u ranges from 0 to $+\infty$ and n is always positive. Note also that the expression for $f(u)$ is automatically normalized as

$$\int_0^\infty f(u) du = \frac{1}{\Gamma(n)} \int_0^\infty e^{-u} \, u^{n-1} \, du = \frac{\Gamma(n)}{\Gamma(n)} = 1 .$$

The first and the second moments of u are easily calculated as follows:

First Moment

$$\omega_1 = E(u) = \bar{u} = \frac{1}{\Gamma(n)}\int_0^\infty ue^{-u}\, u^{n-1}\, du \ ,$$

$$= \frac{1}{\Gamma(n)}\int_0^\infty e^{-u}\, u^{(n+1)-1}\, du = \frac{\Gamma(n+1)}{\Gamma(n)} = n \ . \qquad ...(116)$$

Second Moment

$$\omega_2 = \frac{1}{\Gamma(n)}\int_0^\infty u^2\, e^{-u}\, u^{n-1}\, du = \frac{1}{\Gamma(n)}\int_0^\infty e^{-u}\, u^{(n+2)-1}\, du = (n+1)n \ . \qquad ...(117)$$

The second moment μ_2 about the mean, *i.e.* the variance, is

$$\mu_2 = \omega_2 - \omega_1^2 = (n+1)n - n^2 \ ,$$

$$\mu_2 = n \ . \qquad ...(118)$$

It must be noted that the moments (about the mean) of odd orders are not equal to zero. For instance

$$\mu_3 = 2n \ .$$

Note: In the literature, the Gamma variate is often denoted by the symbol $\gamma(n)$, this symbol being used adjectively.

The outline of the fundamental properties of Euler's second variate, namely the "Beta variate", follows the same pattern as that of the Gamma variate. The Beta variate will play an important role only after other variates have been introduced, in particular after the role of the Gamma variate had been described in the theory of chi-square. There are two kinds of Beta variates termed the $\beta_1(l,m)$ variate and the $\beta_2(l,m)$ variate.

Beta variate of the first kind. The pdf of this variate is

$$f(x,l,m) = \frac{x^{l-1}(1-x)^{m-1}}{B(l,m)} \ . \qquad ...(119)$$

The parameters l, m are both positive, and x is a continuous variable ranging from 0 to +1. $B(l,m)$ is the normalizing factor:

$$\frac{1}{B(l,m)} \int_0^1 x^{l-1} (1-x)^{m-1} \, dx = 1 .$$

The calculation of the moments ω_1 and ω_2 presents no difficulty:

First Moment:

$$\omega_1 = E(x) = \bar{x} = \frac{1}{B(l,m)} \int_0^1 x^{(l+1)-1} (1-x)^{m-1} \, dx ,$$

$$\omega_1 = \frac{B(l+1,m)}{B(l,m)} ,$$

$$\omega_1 = \frac{\Gamma(l+1)\,\Gamma(m)}{\Gamma(l+m+1)} \cdot \frac{\Gamma(l+m)}{\Gamma(l)\,\Gamma(m)} = \frac{l\Gamma(l)\,\Gamma(l+m)}{(l+m)\Gamma(l+m)\,\Gamma(l)} ,$$

$$\omega_1 = \bar{x} = \frac{l}{l+m} . \qquad \text{...(120)}$$

Second Moment:

$$\omega_2 = \frac{B(l+2,m)}{B(l,m)} = \frac{\Gamma(l+2)\,\Gamma(l+m)}{\Gamma(l+m+2)\,\Gamma(l)} = \frac{l(l+1)}{(l+m)(l+m+1)} . \qquad \text{...(121)}$$

Variance:

$$\mu_2 = \omega_2 - \omega_1^2 = \frac{lm}{(l+m)^2(l+m+1)} . \qquad \text{...(122)}$$

It must be strongly underlined that when operating with individual values of the pdf of the β_1-variate, the symbols l and m cannot be considered as interchangeable. Their positions should be always strictly controlled, *i.e.* the order in the symbol $f(u,l,m)$ should correspond to the order in the right-hand term.

Beta variate of the second kind. The pdf of this variate has a form that is significantly different from that of the first kind:

$$f(x,l,m) = \frac{x^{l-1}}{B(l,m)(1+x)^{l+m}} . \qquad \text{...(123)}$$

As in the β_1-variate, l and m are positive and the range of x is 0 to $+\infty$. The order of parameters in the right-hand term should correspond to the order in the symbol

$f(x,l,m)$. It is easy to check that the expression (123) is normalized.

The calculation of moments is done as for other variates:

First moment:

$$\omega_1 = \frac{1}{B(l,m)}\int_0^\infty \frac{x^{(l+1)-1}}{(1+x)^{(l+1)+(m-1)}}\,dx = \frac{B(l+1,m-1)}{B(l,m)} .$$

If all B are expressed in terms of the Gamma function, the expression for ω_1 becomes

$$\omega_1 = \frac{l}{m-1} . \qquad \text{...(124)}$$

It must be underlined that this calculation is valid only if $m>1$. If such is not the case, the integrals will not be convergent. The same remark applies also to the calculation of ω_2 but it is now the condition $m>2$ that must be satisfied.

Second moment:

$$\omega_2 = \frac{1}{B(l,m)}\int_0^\infty \frac{x^{l+1}}{(1+x)^{l+m}}\,dx = \frac{B(l+2,m-2)}{B(l,m)} ,$$

$$\omega_2 = \frac{l(l+1)}{(m-1)(m-2)} . \qquad \text{...(125)}$$

Variance:

$$\sigma^2 = \frac{l(l+1)}{(m-1)(m-2)} - \frac{l^2}{(m-1)^2} = \frac{l(l+m-1)}{(m-1)^2(m-2)} . \qquad \text{...(126)}$$

Taking into account the existence of a relationship between the normal variate and the γ-variate, it is easy to forsee that the summation of γ-variates will also be a fundamental operation. It is treated in the following section. As for the Gamma variate, the symbols $\beta_1(l,m)$ and $\beta_2(l,m)$ are used adjectively.

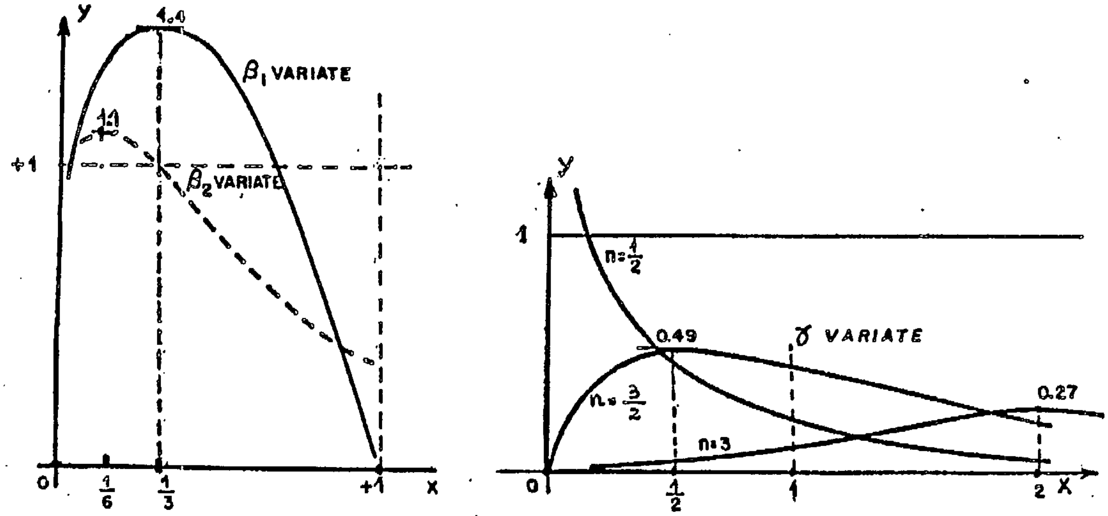


Fig. 7. Gamma and Beta Variates

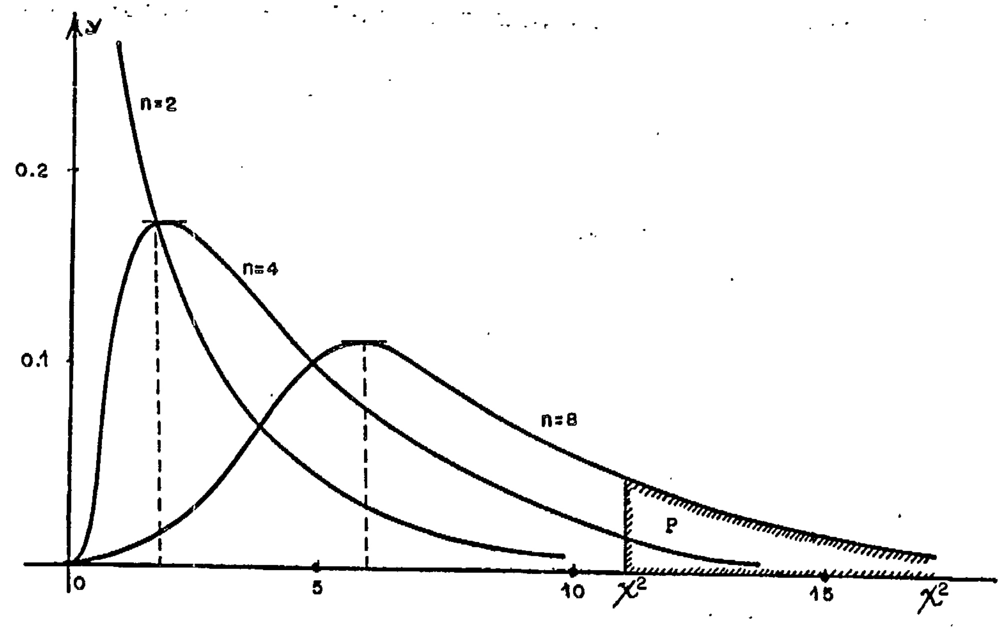


Fig. 8. Chi-square Variate

Section 3. Summation of Gamma Variates

Let U and V be two Gamma variates with parameters a and b, respectively. Their pdf's are therefore

$$\psi_1(u) = \frac{e^{-u}\ u^{a-1}}{\Gamma(a)} ,$$

$$\psi_2(v) = \frac{e^{-v}\, v^{b-1}}{\Gamma(b)} .$$

The variate $Z = U + V$ resulting from the summation of U and V will be calculated by means of an integral expression that is analogous to (80). We can therefore write

$$dP_{z_0} = dz_0 \int_0^{z_0} \psi_1(u)\, \psi_2(z_0 - u)\, du ,$$

$$dP_{z_0} = \frac{dz_0}{\Gamma(a)\Gamma(b)} \int_0^{z_0} e^{-u}\, u^{a-1} \cdot e^{-(z_0-u)} (z_0 - u)^{b-1}\, du ,$$

$$dP_{z_0} = \frac{e^{-z_0} dz_0}{\Gamma(a)\Gamma(b)} \int_0^{z_0} u^{a-1} (z_0 - u)^{b-1}\, du .$$

The form of the integral suggests that it can be reduced to a Beta function with parameters a and b. Actually this can be done by means of the substitution

$$u = z_0 t .$$

It leads to the following expression of dP_z in which the subscript 0 in z_0 is omitted:

$$dP_z = \frac{e^{-z} dz}{\Gamma(a)\Gamma(b)} z^{(a+b)-1} \int_0^1 t^{a-1} (1-t)^{b-1} dt .$$

The definite integral is, by definition, equal to the Beta function $B(a,b)$. By applying to the latter the transformation (112), we obtain

$$dP_z = \frac{e^{-z} z^{(a+b)-1}}{\Gamma(a+b)} dz .$$

Hence the following theorem: the sum Z of two Gamma variates X and Y (with parameters a and b, respectively) is a Gamma variate the parameter c of which is equal to the sum $(a+b)$.

$$dP_z = \frac{e^{-z} z^{c-1}}{\Gamma(c)} dz , \quad \text{with} \quad c = (a+b) . \qquad \ldots (127)$$

This can also be expressed using the symbols for the means:

$$\bar{z} = E(z) = E(x) + E(y) = \bar{x}+\bar{y} = a+b = c \ .$$

If we have a set of n identical Gamma variates and constitute a sample of n elements by drawing one element from each variate, the sum of this sample's elements will be an element of a Gamma variate the parameter of which is equal to na. Such an operation is totally equivalent to that of drawing n elements from the population of one single variate of parameter a. Hence, we can write

$$z = a + a + \cdots + a = na \ ,$$

$$\frac{\bar{z}}{n} = \frac{na}{n} = a \ . \qquad \ldots(128)$$

The distribution of the mean of the sample is thus identical to the distribution of the population from which the sample has been drawn.

The process of summation is readily generalized for any number n of Gamma variates:

$$W = U_1 + U_2 + \cdots + U_n \ .$$

The expression for dP_w is a simple extension of (125) and takes the form

$$dP_w = \frac{e^{-w} w^{(a_1+a_2+\ldots+a_n)-1}}{\Gamma(a_1+a_2+\ldots+a_n)} \ . \qquad \ldots(129)$$

Let us now consider a set of n identical normal variates (*i.e.* all centered on zero and having the same variance σ^2). If an exponent u_i is drawn from each variate, the sum

$$w = u_1 + u_2 + \cdots + u_n = \frac{n}{2\sigma^2}\sum_i x_i^2$$

will be a Gamma variate with the parameter $n \times ½$.

The probability dP_w that w will fall into an interval dw is thus equal to

$$dP_w = \frac{e^{-w} w^{(\frac{n}{2}-1)}}{\Gamma(\frac{n}{2})} \ . \qquad \ldots(130)$$

Suppose now that a set of n elements u_i ($i=1, 2, \cdots n$) is considered as a *reference* set to which all subsequent similar sets will be referred. Designating by w_0 the reference set, we obtain for the probability dP_{w_0} the expression

$$dP_{w_0} = \frac{e^{-w_0} \, w_0^{(\frac{n}{2}-1)}}{\Gamma(\frac{n}{2})} dw_0 \; . \qquad ...(131)$$

Having calculated the numerical value of w_0, we can calculate the total probability that any other value w will be larger than w_0 by the integral

$$P(w > w_0) = \int_{w_0}^{\infty} \frac{e^{-w} \, w^{(\frac{n}{2}-1)}}{\Gamma(\frac{n}{2})} dw \; . \qquad ...(132)$$

Similarly, it is possible to calculate $P(w < w_0)$ by the integral

$$P(w < w_0) = \int_{0}^{w_0} \frac{e^{-w} \, w^{(\frac{n}{2}-1)}}{\Gamma(\frac{n}{2})} dw \; . \qquad ...(133)$$

These integrals will play an important role in the presentation of the theory of Pearson's chi-square test. However, some subtle transformations must still be performed before the above developed theory becomes directly applicable to observed data. These transformations, due to the genius of Pearson and Fisher, lead to the famous "chi-square" test so frequently applied in many domains, sometimes to cases which are only loosely connected with the normal distribution.

Section 4. Pearson's Chi-square Variate

Karl Pearson, two decades after Helmert and totally ignoring Helmert's work, undertook to investigate the properties of the exponent in the normal function and its relation to the Euler's Gamma variate. Instead of considering the whole exponent $(\frac{x^2}{2\sigma^2})$ as a new variate, he defined the new variate χ by the formula

$$\chi^2 = \frac{x^2}{\sigma^2} \ .$$

Hence,

$$\chi^2 = 2u, \quad u = \frac{\chi^2}{2} \ ,$$

and all formulae and relations expressed in terms of u may be rewritten in terms of χ^2. Thus, for instance, the number of identical normal variates being denoted by N, the formula (130) becomes

$$dP_{(\frac{\chi^2}{2})} = \frac{1}{\Gamma(\frac{N}{2})} e^{-\frac{\chi^2}{2}} \left(\frac{\chi^2}{2}\right)^{\frac{N}{2}-1} d\left(\frac{\chi^2}{2}\right)$$

$$dP_{\chi^2} = \frac{1}{2^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{N}{2}-1} d(\chi^2) \qquad \text{...(134)}$$

In the most general case, we may apply the chi-square theory to a set of N normal variates in which all variates are different from each other. Each component is, however, centered on zero but has its specific variance σ_i^2, $i=1, 2, \cdots N$. Each variate yields its specific chi-square χ_i^2:

$$\chi_i^2 = \frac{x_i^2}{\sigma_i^2}$$

and, according to the theorem of addition of gamma variates, the sum

$$\chi^2 = \sum_{i=1}^{N} \chi_i^2 = \sum_{i=1}^{N} \frac{x_i^2}{\sigma_i^2}$$

will be a variate the probability dP_{χ^2} of which is equal to the expression (analogous to (134))

$$dP_{\chi^2} = \frac{1}{2^{\frac{N}{2}}\Gamma(\frac{N}{2})} e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{N}{2}-1} d(\chi^2) \qquad \text{...(135)}$$

From this expression, it is possible to compute (numerically) the values of the probabilities (132) and (133) with respect to a certain selected reference value χ_0^2.

As here all normal variates are centered on $x=0$ or (what is the same) on a known centre, the number of degrees of freedom is $\nu=N$.

The reader must always bear in mind that, instead of considering a set of N independent variates of the same mean and the same variance, it is equivalent to consider one single variate from the population of which N elements are drawn at random.

In the applications that will be made in Section 5, the variates have all different centres and different variances but in certain cases these parameters are considered as known so that the formula (135) is directly applicable. If there is any loss of degrees of freedom, it is due to causes which must be appropriately evidenced.

Such would be the case if for example the normal variate were represented by a population x_i with a ***known variance*** σ^2 but with an ***unknown mean*** μ. The sum of squares

$$\sum_i (x_i - \bar{x})^2$$

would be a chi-square variate but with a number of freedoms ν equal to $(n-1)$. The proof of this fact is based on a method that uses the so-called linear orthogonal transformations.

The impact of χ^2 theory on the analysis of large samples of observations became considerable when Pearson realized that it can be applied to "class-variates" that are formed in the process of repeated sampling. In this process, each sample produces a completely independent set of parameters but, when samples are *large* and of *high quality,* it can be reasonably assumed that they conform to certain plausible conditions which lead to less stringent but still useful conclusions.

The first condition that can thus be assumed is that in repeated sampling all samples have the same mean μ and the same variance σ^2. If all samples are distributed in the same number $(2k+1)$ of classes (ranging from $j=-k$ to $j=+k$), the probability that an element will fall into the j^{th} interval is also practically constant and equal to

$$p_j = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{j^2}{2\sigma^2}} \Delta j \,, \quad (\Delta j = 1) \,. \qquad ...(136)$$

The theoretical class frequency in the class j is therefore

$$f_j = Np_j \ . \qquad ...(137)$$

Let us designate by F_j the observed class frequency. In repeated sampling, F_j will become a variate and its value will be oscillating about f_j in a manner which we may assume to be very close to normality, with $\bar{F}_j = f_j$. In accord with a well-known theorem, the variance σ_j^2 of F_j is equal to

$$\sigma_j^2 = E\left[(F_j - Np_j)^2\right] = Np_j q_j \ , \quad (q_j = 1 - p_j) \ . \qquad ...(138)$$

This expression would be adequate if the size N of the sample was constant. However, the constancy of N would imply that the variates F_j are *not independent* of each other. In order to remove this constraint, we must assume that N is also a variate in itself and that it oscillates in an approximately "normal" way, about a mean value $\bar{N}$. Thus the variate which must be examined is not $(F - Np_j)$ but

$$(F - \bar{N}p_j) \qquad ...(139)$$

and therefore instead of (138), we must analyse

$$\sigma_j^2 = E\left[(F_j - \bar{N}p_j)^2\right] \ . \qquad ...(140)$$

This is done by means of the identity

$$(F_j - \bar{N}p_j) = (F_j - Np_j) + (Np_j - \bar{N}p_j) \ .$$

Its square is

$$(F_j - \bar{N}p_j)^2 = (F_j - Np_j)^2 + p_j^2(N - \bar{N})^2 + 2(F_j - Np_j)(Np_j - \bar{N}p_j) \ .$$

The expectation of the left-hand term is, according to (140), equal to σ_j^2. The expectations of the three terms in the right-hand side are calculated as follows.

A) $E\left[(F_j - Np_j)^2\right]$. By analogy with what had been done in Bernoulli Trials, this expectation is calculated in two steps: first with respect to F_j, then with respect to N. These steps give:

1) $E_{(F_j)}\left[(F_j - Np_j)^2\right] = Np_j q_j$.

2) $E_{(N)}(Np_j q_j) = \overline{N}p_j q_j$.

B) $E\left[p_j^2(N-\overline{N})^2\right] = p_j^2\sigma_N^2$.

C) The term is considered as practically equal to zero. While in A) and B) all components are positive, in C) they are positive or negative. As the positive terms are as probable as the negative ones, the total sum is likely to be close to zero.

The combination of A), B) and C) leads to

$$\sigma_j^2 = \overline{N}p_j(1-p_j) + p_j^2\sigma_N^2 . \qquad ...(141)$$

This is an important relation which expresses all class variances σ_j^2 in terms of the parameters of the samples. Here will be used again the fact that the sample's size N is an independent variate. Thus, according to the theorem of addition of variances:

$$\sigma_N^2 = \sum_j \sigma_j^2$$

and, by (141)

$$\sigma_N^2 = \overline{N}\sum_j (p_j q_j) + \sigma_N^2 \sum_j p_j^2 .$$

As $\sum p_j = 1$, this relation can also take the form

$$\sigma_N^2 \sum_j (p_j - p_j^2) = \overline{N}\sum_j p_j q_j .$$

Finally, the division of both sides by $p_j(1-p_j)$ yields the remarkably simple relation

$$\sigma_N^2 = \overline{N} . \qquad ...(142)$$

If this is substituted into (141) we obtain another remarkable relation

$$\sigma_j^2 = \overline{N}p_j = f_j . \qquad ...(143)$$

This formula has an enormous impact on the theory of χ^2 and its practical applications. Now, in each class j, the component of χ^2 can be calculated by a formula deducible from (134) by simply replacing σ_j^2 by f_j:

$$\chi^2 = \sum_j \frac{(F_j - f_j)^2}{f_j} \quad \ldots(144)$$

This is probably one of the most important formulae in the whole domain of Statistics. It forms the base of the celebrated "Pearson's Chi-square Test" which is used in all those domains of human activity where "hypotheses" are formulated and tested. It is a powerful tool which, as all powerful tools, should be used very cautiously. As it has been underlined above, it involves some conditions that are not stringent but only reasonable and have no precise limits. There are, for instance, in practice, certain operations in which the notion of "repeated observations" has no real meaning. In such cases, the result of a Chi-square test can be considered only as leading to simple "suggestions".

Section 5. Analysis of Large Samples

From the standpoint of the validity of the conclusions drawn from the Chi-square test those scientific activities in which "repeated observations" play a major role are particularly privileged. They operate with large aggregates of observations, generally performed in the best possible conditions of scientific control and stability.

In major metrological operations (in geodesy, gravimetry *etc*.), "chi-square" is used to test the hypotheses concerning the distribution of large samples of observations *i.e.* the fitting into these samples of the normal curve or the curves which derive from the normal law and which are termed "modulated normal".

In order to clarify the ideas about χ^2 let us summarise its main characteristics:

a) In the expression (144) the right-hand term contains only class frequencies. First, F_j, *i.e.* the observed frequencies directly resulting from the classification of the sample elements x_i. Second, f_j, *i.e.* the theoretical frequencies as they result from the well-known process of fitting into the sample an appropriate curve. This constitutes the ***hypothesis to be tested, viz***: that there are good reasons to believe that the sample can be considered as drawn from a universe distributed in accord with the function defining the curve.

b) The numerical value obtained for χ^2 is obviously capable of informing us about the adequacy of the chosen curve: a small χ^2 shows that the observed frequencies are close to the theoretical frequencies, in other words that the curve is appropriate. A large χ^2 would indicate that the choice is not correct. Our opinion would be, however, exclusively "qualitative" and not "quantitative".

c) A particularly important property of χ^2 is to be a variate. Thus, for each value of χ_0^2 it is possible to calculate the chance that it will fall into an interval $d\chi_0^2$ and, in repeated sampling, the total chance P to obtain a χ^2 which is larger than χ_0^2 or, on the contrary, which is smaller than χ_0^2.

So far as the sample size N is concerned, let us remind the reader what has already been said above:

d) In high quality repeated samples, the fact that they are "large" authorizes us to assume that all such samples have the same mean and the same variance. Although these parameters are calculated from the sample elements, they are treated as independent and their calculation does not constitute a constraint.

e) The only operation which establishes a constraint is the calculation of $\bar{N}$ which, in its turn, is used to calculate σ_N^2. This leads to the conclusion that the number of degrees of freedom may be reduced by one unit, *i.e.* that it can be considered as equal to

$$\nu = (2k+1)-1 \ . \qquad ...(145)$$

In the table of χ^2, the first column is that of ν, *i.e.* the number of degrees of freedom. This number is given either by (145) or is further reduced by 1 or 2 units according to our opinion on the nature and quality of the sample.

Table VI

Values of Chi-square with Probability P of Being Exceeded

ν	P							
	0.99	0.95	0.50	0.30	0.20	0.10	0.05	0.01
1	0.0002	0.004	0.46	1.07	1.64	2.71	3.84	6.64
2	0.020	0.103	1.39	2.41	3.22	4.60	5.99	9.21
3	0.115	0.35	2.37	3.66	4.64	6.25	7.82	11.34
4	0.30	0.71	3.36	4.88	5.99	7.78	9.49	13.28
5	0.55	1.14	4.35	6.06	7.29	9.24	11.07	15.09
6	0.87	1.64	5.35	7.23	8.56	10.64	12.59	16.81
7	1.24	2.17	6.35	8.38	9.80	12.02	14.07	18.48
8	1.65	2.73	7.34	9.52	11.03	13.36	15.51	20.09
9	2.09	3.32	8.34	10.66	12.24	14.68	16.92	21.67
10	2.56	3.94	9.34	11.78	13.44	15.99	18.31	23.21
11	3.05	4.58	10.34	12.90	14.63	17.28	19.68	24.72
12	3.57	5.23	11.34	14.01	15.81	18.55	21.03	26.22
13	4.11	5.89	12.34	15.12	16.98	19.81	22.36	27.69
14	4.66	6.57	13.34	16.22	18.15	21.06	23.68	29.14
15	5.23	7.26	14.34	17.32	19.31	22.31	25.00	30.58
16	5.81	7.96	15.34	18.42	20.46	23.54	26.30	32.00
17	6.41	8.67	16.34	19.51	21.62	24.77	27.59	33.41
18	7.02	9.39	17.34	20.60	22.76	25.99	28.87	34.80
19	7.63	10.12	18.34	21.69	23.90	27.20	30.14	36.19
20	8.26	10.85	19.34	22.78	25.04	28.41	31.41	37.57
21	8.90	11.59	20.34	23.86	26.17	29.62	32.67	38.93
22	9.54	12.34	21.34	24.94	27.30	30.81	33.92	40.29
23	10.20	13.09	22.34	26.02	28.43	32.01	35.17	41.64
24	10.86	13.85	23.34	27.10	29.55	33.20	36.42	42.98
25	11.52	14.61	24.34	28.17	30.68	34.38	37.65	44.31
26	12.20	15.38	25.34	29.25	31.80	35.56	38.88	45.64
27	12.88	16.15	26.34	30.32	32.91	36.74	40.11	46.96
28	13.56	16.93	27.34	31.39	34.03	37.92	41.34	48.28
29	14.26	17.71	28.34	32.46	35.14	39.09	42.56	49.59
30	14.95	18.49	29.34	33.53	36.25	40.26	43.77	50.89

To interpret the indication of this table one has to remember the following rule: if for a calculated χ^2, P *is small* (small chance to be exceeded) this means that χ^2 *is large* and that the formulated hypothesis is likely to be *unacceptable*. But if P *is large* (large chance to be exceeded) this means that χ^2 *is small* and that the formulated hypothesis is likely to be *acceptable*.

The problems which often arise on both extreme wings of the observed samples are due to the fact that these wing terminals present strange complicated structures. The common practice is to group some thinly populated classes in which the observed frequency is of the order of at least a few units.

Section 6. Basic Integrals and Moments in terms of Γ

The basic integral

$$J_n = \int_0^\infty z^n \, e^{-z^2} \, dz$$

may be expressed in terms of the Gaussian function by the change of variable

$$z^2 = y \ , \quad z = y^{1/2} \ , \quad dz = 1/2 y^{-1/2} \, dy \ .$$

This leads to

$$J_n = \frac{1}{2}\int_0^\infty e^{-y} \, y^{\frac{n-1}{2}} \, dy \ .$$

To determine the parameter of the Gamma function we use the identity

$$\frac{n-1}{2} = \left(\frac{n-1}{2}+1\right) - 1 = \frac{n+1}{2} - 1$$

so that

$$J_n = \frac{1}{2}\int_0^\infty e^{-y} \, y^{\frac{n-1}{2}} \, dy = \frac{1}{2}\int_0^\infty e^{-y} \, y^{\frac{n+1}{2}-1} \, dy$$

and, finally,

$$J_n = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right) . \qquad ...(146)$$

If now n is replaced by $(n+2)$ and the resulting expression for J_{n+2} is introduced into the recurrence formula, the resulting equation is identical to the equation (99). This point is treated in detail in Ex. 22 (B).

The relation (146) in which n is successively made equal to 0, 1, 2, ... leads to the corresponding, often used, values of Γ:

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} , \quad \Gamma(1) = 1 , \quad \Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2} , \quad \Gamma(2) = 1 , \quad \cdots$$

The transformation $z = \dfrac{x}{\sigma\sqrt{2}}$, which leads to the expressions of the moments μ_n and ν_n in terms of J_n, finally leads to their expressions in terms of Gamma functions:

$$\nu_n = \frac{2^{\frac{n}{2}+1}}{\sqrt{\pi}} \sigma^n J_n ,$$

$$\nu_n = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \sigma^n \Gamma\left(\frac{n+1}{2}\right) . \qquad ...(147)$$

This formula is also valid for μ_n moments but only for even values of n:

$$\mu_n = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^n e^{-\frac{x^2}{2\sigma^2}} dx , \quad (n \text{ even}) .$$

For n odd, $\mu_1 = \mu_3 = \mu_5 = \cdots = 0$. For the numerical values of μ_n and ν_n refer to Table III.

Summary:

$$\left.\begin{array}{l}\nu_n \text{ even and odd} \\ \mu_n \text{ even only}\end{array}\right\} = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \sigma^n \Gamma\left(\frac{n+1}{2}\right)$$

$$\mu_n \text{ odd} \qquad = 0$$

The recurrence formula for ν_n moments has been established in Chapter II, Section 3. by means of the recurrence formula for J_n.

Appendices

Note

Each appendix constitutes an extension of the study of a subject treated in the text: either towards the more elementary theory on which the study is based, or, on the contrary, towards a deeper understanding of the finer points of the subject matter.

Although the appendices can be omitted in the first reading, their contents must be finally completely understood and assimilated.

Appendix I
Existence Theorem for the Most Probable Group

If x designates the number of plusses in the largest group, then obviously it must satisfy the condition:

$$P_{x-1} < P_x > P_{x+1}$$

in which

$$P_x = \frac{k!}{x!(k-x)!} p^x q^{k-x},$$

$$P_{x+1} = \frac{k!}{(x+1)!(k-x-1)!} p^{x+1} q^{k-x-1},$$

$$P_{x-1} = \frac{k!}{(x-1)!(k-x+1)!} p^{x-1} q^{k-x+1}.$$

Now

$$\frac{P_x}{P_{x-1}} = \frac{k-x+1}{x} . \frac{p}{q} > 1$$

and

$$\frac{P_{x+1}}{P_x} = \frac{k-x}{x+1} . \frac{p}{q} < 1 .$$

Hence

$$kp - xp + p > xq ,$$

$$kp - xp - q < xq ,$$

and this can be arranged as follows:

$$kp - xp + p > xq > kp - xp - q\,,$$

$$kp + p > xq + xp > kp - q\,, \qquad xq + xp = x,$$

$$kp + p > x > kp - q\,. \qquad \text{...(I.1)}$$

Let us compute the difference between the first and third term

$$(kp + p) - (kp - q) = kp + p - kp + q = 1\,.$$

The fact that the difference is equal to 1 indicates that between these two numbers exists an integer, unless both numbers are themselves two consecutive integers.

Example:

$$k = 12\,, \qquad p = q = \frac{1}{2}\,.$$

$$kp + p = 12 \times \frac{1}{2} + \frac{1}{2} = 6.5\,.$$

$$kp - q = 12 \times \frac{1}{2} - \frac{1}{2} = 5.5\,.$$

$$6.5 > x > 5.5\,,$$

$$x = 6\,.$$

It can happen that $(kp+p)$ and $(kp-q)$ are both integers (*e.g.* when $k = 13$, $p = q = \frac{1}{2}$). Then, the relation (I.1) must be given a more general form

$$kp + p \geq x \geq kp - q\,.$$

This indicates that

$$7 \geq x \geq 6$$

and that 6 and 7 have the same probability.

Appendix II
Elementary Theory of Moments

Suppose that a variable of nominal value m is represented numerically by a set of N values m_i (i = 1, 2, $\cdots$ N). The mean value defined by this set is equal to

$$\bar{m} = \frac{1}{N} \sum_i m_i \; . \qquad ...(II.1)$$

Similarly, the mean square of m is equal to

$$\overline{(m^2)} = \frac{1}{N} \sum_i m_i^2 \; . \qquad ...(II.2)$$

If the numbers m_i can be distributed into classes (each class containing only numbers that can be treated as identical to each other) and if each class is attributed a rank j (e.g. from j = 1 to j = k) then

$$\bar{m} = \frac{1}{N} \sum_j F_j m_j \; , \qquad ...(II.3)$$

$$\overline{(m^2)} = \frac{1}{N} \sum_j F_j m_j^2 \; , \qquad ...(II.4)$$

F_j designating the number of values in the class of the j^{th} rank. F_j is generally termed "*class frequency*".

All formulae above are, of course, directly applicable to the sets constituted by the values of the variable X as it results from our repeating N times Bernoulli trials. If N is large so that all class frequencies are properly constituted then the expressions (3) and (4) are

$$\bar{X} = \frac{1}{N} \sum_j F_j X_j \qquad ...(II.5)$$

$$\overline{(X^2)} = \frac{1}{N} \sum_j F_j X_j^2 \qquad ...(II.6)$$

Here, the rank j of a class indicates the number of obtained plusses so that, with 12 cards, j takes the values $j = 0, 1, 2, \cdots 12$.

It is to be noted that in the binomial expansion (7a,b) the symbols P_j represent the true theoretical probabilities while the ratio $\frac{F_j}{N}$ is only an approximate value of P_j. Hence the values obtained from (II.5,6) are termed "*estimates*" of the mean $\overline{X}$ and the mean square $\overline{(X^2)}$.

Important point of nomenclature: a variable such as X, i.e. the probability of which is given either by an algebraic formula or deduced from a sufficiently large number N of events, is termed *variate* and then the quantities such as $\overline{X}$ and $\overline{X^2}$ are termed *moments* of the variate X, and denoted by the symbols ω_1 and ω_2 respectively:

$$\omega_1 = \overline{X} = \frac{1}{N}\sum F_j X_j\ , \qquad \omega_2 = \overline{(X^2)} = \frac{1}{N}\sum_j F_j X_j^2\ . \qquad \text{...(II.7)}$$

Instead of computing moments with respect to $X = 0$, we can compute them with respect to the mean $\overline{X}$. These moments are then denoted by the symbols μ_1 and μ_2. The moment μ_1 is equal to zero:

$$\mu_1 = \frac{1}{N}\sum_j (X_j - \overline{X})F_j$$

$$\mu_1 = \frac{1}{N}\sum_j X_j F_j - \frac{1}{N}\overline{X}\sum_j F_j$$

$$\mu_1 = \overline{X} - \overline{X} = 0\ .$$

The moment μ_2 is computed as follows.

$$\mu_2 = \frac{1}{N}\sum_j F_j(X_j - \overline{X})^2 = \frac{1}{N}\sum_j F_j(X_j^2 - 2X_j\overline{X} + \overline{X}^2)$$

$$= \frac{1}{N}\sum_j F_j X_j^2 - \frac{1}{N}\cdot 2\overline{X}\sum_j F_j X_j + \frac{1}{N}\overline{X}^2\sum_j F_j$$

$$= \omega_2 - 2\overline{X}\cdot\overline{X} + \overline{X}^2 = \omega_2 - \overline{X}^2 = \omega_2 - \omega_1^2 \qquad \text{...(II.8)}$$

It is to be noted that all formulae in this Appendix are expressed in terms of the symbols F_j and N. In fact, their form is thus the most general as they can be used for all possible meanings of the variable X_j. This symbol may be applied to a totally erratic aggregate of numbers or to a set of values obtained by means of a device closely and clearly related to the notion of probability (cards, dice, wheel of fortune, lottery etc.). Now, two cases can be considered:

First case: N being a moderately large number, all F_j are obtained experimentally by making a device operate N times.

Second case: The ratio $\frac{F_j}{N}$ (according to the Third Axiom) is equal to the probability, in its strict sense, of the variable X_j, this probability being given by the probability function, such as P_j in Bernoulli Trials.

In the First Case all formulae given above may be used as they are and the moments thus obtained are called "estimates" of true moments.

In the Second Case the formulae must be written as follows:

$$\omega_1 = \sum_j P_j X_j \,, \qquad \omega_2 = \sum P_j X_j^2 \,,$$

$$\mu_1 = \sum_j P_j (X_j - \overline{X}) \,, \qquad \text{...(II.9)}$$

$$\mu_2 = \sum_j P_j (X_j - \overline{X})^2 \,.$$

The symbols ω_1, ω_2, μ_1, μ_2 are now termed *exact* moments of the *variate.*

Numerical Example

A die is thrown 450 times. The results are recorded and classified:

X	=	1	2	3	4	5	6
F	=	81	70	92	59	65	83

Calculate: a) the theoretical mean $\overline{X}$ and its actual estimate,

b) the theoretical second moment ω_2 and its actual estimate,

c) deduce the values of the theoretical moment μ_2 and its actual estimate.

Calculation of ω_1 (observed) and ω_1 (theoretical)

X	F(obs)	f(the)	XF	Xf
1	81	75	81	75
2	70	75	140	150
3	92	75	276	225
4	59	75	236	300
5	65	75	325	375
6	83	75	498	450
21	450	450	1556	1575

$$\omega_1(obs.) = \frac{1556}{450} = 3.46$$

$$\omega_1(the.) = \frac{1575}{450} = 3.50$$

X^2	$F(obs)$	$f(the)$	X^2F	X^2f
1	81	75	81	75
4	70	75	280	300
9	92	75	828	675
16	59	75	944	1200
25	65	75	1625	1875
36	83	75	29875	1625
36	83	75	2988	2700
91	450	450	6746	6825

$$\omega_2(obs.) = \frac{6746}{450} = 14.99$$

$$\omega_2(the.) = \frac{6825}{450} = 15.17$$

$$\mu_2(obs.) = \omega_2 - \omega_1^2 = 14.99 - (3.458)^2 = 3.03$$

$$\mu_2(the.) = \omega_2 - \omega_1^2 = 15.17 - (3.5)^2 = 2.92$$

The calculation of $\mu_2(obs)$ may be made directly by using as reference $\overline{X} = \omega_1(obs) = 3.46$

X	$X-\overline{X}$	$(X-\overline{X})^2$	$F\cdot(X-\overline{X})^2$
1	−2.46	6.05	490.05
2	−1.46	2.13	149.10
3	−0.46	0.21	19.32
4	+0.54	0.29	17.11
5	+1.54	2.37	154.05
6	+2.54	6.45	535.32
			1364.98

$$\mu_2 = \frac{1}{N-1}\sum F\cdot(X-\overline{X})^2$$

It is to be noted that here one degree of freedom has been used to calculate $\overline{X}$ so that in this formula the denominator must be equal to $N-1$. Hence

$$\mu_2 = \frac{1}{450-1} \times 1364.98$$

$$\mu_2 = 3.04$$

Appendix III
Calculation of the Second Moments of the Variable X in Bernoulli Trials

According to (10) the expanded expression for ω_2 is:

$$\omega_2 = 0^2 q^k + 1^2\frac{k}{1}pq^{k-1} + 2^2\frac{k(k-1)}{1\cdot 2}p^2q^{k-2} + 3^2\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}p^3q^{k-3} + \cdots$$

$$+ (k-2)^2\frac{k(k-1)(k-2)\ \cdots\ 3}{1\cdot 2\ \cdots\ (k-3)(k-2)}p^{k-2}q^2$$

$$+ (k-1)^2\frac{k(k-1)(k-2)\ \cdots\ 2}{1\cdot 2\ \cdots\ (k-2)(k-1)}p^{k-1}q$$

$$+ k^2\frac{k(k-1)(k-2)\ \cdots\ 1}{1\cdot 2\ \cdots\ (k-1)k}p^k .$$

Simplifying, we find

$$\omega_2 = 1\frac{k}{1}pq^{k-1} + 2\frac{k(k-1)}{1}p^2q^{k-2} + 3\frac{k(k-1)(k-2)}{1\cdot 2}p^3q^{k-3} + \cdots$$

$$+ (k-2)\frac{k(k-1)(k-2)}{1\cdot 2}p^{k-2}q^2 + (k-1)\frac{k(k-1)}{1}p^{k-1}q + k\cdot kp^k .$$

In the right-hand term we can put kp out of the brackets, this gives

$$\omega_2 = kp\left\{1q^{k-1} + 2\frac{k-1}{1}pq^{k-2} + 3\frac{(k-1)(k-2)}{1\cdot 2}p^2q^{k-3} + \cdots\right.$$

$$\left. + (k-2)\frac{(k-1)(k-2)}{1\cdot 2}p^{k-3}q^2 + (k-1)\frac{k-1}{1}p^{k-2}q + k\cdot p^{k-1}\right\},$$

and, identically,

$$\omega_2 = kp\left\{(0+1)q^{k-1} + (1+1)\frac{k-1}{1}pq^{k-2} + (2+1)\frac{(k-1)(k-2)}{1\cdot 2}p^2q^{k-3} + \cdots\right.$$

$$\left. + \Big[(k-3)+1\Big]\frac{(k-1)(k-2)}{1\cdot 2}p^{k-3}q^2 + \Big[(k-2)+1\Big]\frac{k-1}{1}p^{k-2}q + \Big[(k-1)+1\Big]p^{k-1}\right\}$$

so that

$$\omega_2 = kp\Big\{0q^{k-1} + 1\frac{k-1}{1}pq^{k-2} + 2\frac{(k-1)(k-2)}{1\cdot 2}p^2q^{k-3} + \cdots$$

$$+ (k-3)\frac{(k-1)(k-2)}{1\cdot 2}p^{k-3}q^2 + (k-2)\frac{k-1}{1}p^{k-2}q + (k-1)p^{k-1}$$

$$+ q^{k-1} + \frac{k-1}{1}pq^{k-2} + \frac{(k-1)(k-2)}{1\cdot 2}p^2q^{k-3} + \cdots$$

$$+ \frac{(k-1)(k-2)}{1\cdot 2}p^{k-3}q^2 + \frac{k-1}{1}p^{k-2}q + p^{k-1}\Big\}.$$

Inside {} we recognize two sums:

1° the sum of those terms in which the numerical coefficients are 0, 1, 2, $\cdots (k-3)$, $(k-2)$, $(k-1)$; it represents the mean of a binomial variate which takes the values 0, 1, 2, $\cdots (k-1)$ and is therefore equivalent to $(k-1)p$;

2° the remaining terms the sum of which forms the development $(q+p)^{k-1}$ and which is therefore equal to unity. Therefore

$$\omega_2 = kp\Big[(k-1)\cdot p+1\Big]. \qquad \text{...(III.1)}$$

The second moment about $\bar{X} = kp$ is, as in II.8, given by

$$\mu_2 = \omega_2 - \bar{X}^2 = kp\Big[(k-1)\cdot p+1\Big] - (kp)^2,$$

$$\mu_2 = kp\cdot(1-p),$$

$$\mu_2 = kpq. \qquad \text{...(III.2)}$$

An excellent exercise consists in establishing the expressions for ω_1 and ω_2 for a numerical value of k, for instance $k = 7$. Applying the above methods one finds without too much difficulty that:

$$\omega_1 = 7p(p+q)^6 = 7p \quad \text{and} \quad \omega_2 = 7p(6p+1).$$

Appendix IV
Hagen's Derivation of the Normal Law

In order to simplify calculations, the method produced by Hagen will be applied to the expression (5) the second term of which will be treated as a function of V, k being an even constant.

$$f(V) = \frac{k!}{\left(\frac{k}{2}+V\right)!\left(\frac{k}{2}-V\right)!}\left(\frac{1}{2}\right)^k . \qquad \text{...(IV.1)}$$

Since k is a large number, an increment of one unit can be legitimately considered as very small. Hence

$$f(V+1) = \frac{k!}{\left(\frac{k}{2}+V+1\right)!\left(\frac{k}{2}-V-1\right)!}\cdot\left(\frac{1}{2}\right)^k$$

and

$$\frac{f(V+1)}{f(V)} = \frac{\frac{k}{2}-V}{\frac{k}{2}+V+1} .$$

The unit in the denominator can be neglected so that

$$\frac{f(V+1)}{f(V)} = \frac{k-2V}{k+2V} .$$

This relation is transformed by expanding $f(V)$ by means of the Taylor expansion limited to its first term:

$$f(V+\Delta V) = f(V) + \frac{df}{dV}\Delta V .$$

Here $\Delta V = 1$ so that

$$f(V+1) = f(V) + \frac{df}{dV} . \qquad \text{...(IV.2)}$$

Now

$$\frac{f(V+1)}{f(V)} = 1 + \frac{1}{f(V)} \cdot \frac{df}{dV} = \frac{k-2V}{k+2V}$$

and the last term can be transformed as follows.

$$\frac{k-2V}{k+2V} = \frac{(k-2V)^2}{k^2-4V^2} = \frac{k^2-4kV+4V^2}{k^2-4V^2},$$

$$= \frac{1-\frac{4V}{k}+\frac{4V^2}{k^2}}{1-\frac{4V^2}{k^2}}. \qquad \text{...(IV.3)}$$

In this formula all terms containing k^2 can be treated as very small so that

$$1+\frac{1}{f(V)} \cdot \frac{df}{dV} = 1-\frac{4V}{k} \qquad \text{...(IV.4)}$$

and

$$\frac{df}{f(V)} = -\frac{4V}{k}dV.$$

This represents a differential equation the solution of which is well known and has the form

$$\log f(V) = -\frac{4}{k} \cdot \frac{V^2}{2} + \log C$$

or

$$f(V) = Ce^{-\frac{2V^2}{k}}. \qquad \text{...(IV.5)}$$

An important remark must be made here. It substantiates the validity of the transformation from (IV.3) to (IV.4) by emphasizing the fact that $f(V)$ tends to zero exponentially so that all formulae are valid not only when V has a small or moderate value but also when V tends towards $\frac{k}{2}$ as the latter is always very large.

To obtain C it suffices to use the normalization equation

$$\int_{-\infty}^{+\infty} f(V)dV = 1 = \int_{-\infty}^{+\infty} Ce^{-\frac{2V^2}{k}} dV$$

which, combined with the relation

$$\int_{-\infty}^{+\infty} e^{-\frac{2V^2}{k}} dV = \sqrt{\pi k / 2} \ ,$$

leads to

$$f(V) = P_V = \frac{\sqrt{2}}{\sqrt{\pi k}} e^{-\frac{2V^2}{k}} \ .$$

As we already know that $\frac{k}{4} = \sigma^2$, the final form for $f(V)$ becomes

$$f(V) = P_V = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{V^2}{2\sigma^2}} \ . \qquad \text{...(IV.6)}$$

Appendix V
Comments on the Bessel Formula

To establish the validity of this formula it is necessary to refer to the notion of "true value" as this term is used in repeated measurements of a fixed physical quantity. This term is, unfortunately, not very appropriate. In fact, it is misleading. It actually has no relation to the concept of "truth" but designates a hypothetic quantity (here denoted by M) which can only be *described* but not really rigorously *defined*. It is conceived as the limit towards which tends the average value of a collection of individual measurements m_i ($i = 1, 2, \ldots N$) when the size N of the collection tends towards infinity. As the deviations of m_i with respect to M are of the type we consider as random and associated with the conviction that the measurements are highly precise, our mind accepts, as a primary notion, that when N tends towards infinity it finally reaches such a magnitude that further measurements (and their random deviations) cease to produce any meaningful effect.

The deviations with respect to M are in the sequel denoted by the letter ξ so that we have

$$\xi_i = m_i - M$$

$$\overline{\xi} = \overline{m} - M , \quad \left(\overline{m} = \frac{\sum m_i}{N} , \quad \overline{\xi} = \frac{\sum \xi}{N}\right)$$

and therefore,

$$\xi_i - \overline{\xi} = m_i - \overline{m} .$$

Both sides of this equation will be designated by the same symbol:

$$\text{a)} \quad \alpha_i = \xi_i - \overline{\xi}$$

$$\text{b)} \quad \alpha_i = m_i - \overline{m} .$$

Squaring a) we obtain:

$$\alpha_i^2 = \xi_i^2 - 2\xi_i\overline{\xi} + \overline{\xi}^2$$

$$\sum\alpha_i^2 = \sum\xi_i^2 - 2\overline{\xi}\sum\xi_i + N\overline{\xi}^2$$

$$\sum\alpha_i^2 = \sum\xi_i^2 - 2\overline{\xi}\cdot N\overline{\xi} + N\overline{\xi}^2$$

$$\sum\alpha_i^2 = \sum\xi_i^2 - N\overline{\xi}^2$$

The most important step in this calculation concerns the sum $\sum\xi_i^2$. It is to be noted that we have

$$\left(\sum\xi_i\right)^2 = \sum\xi_i^2 + Q ,$$

Q being a polynomial each term of which is a product of the form $\xi_{i'}\cdot\xi_{i''}$ in which i' and i'' take on all values ranging from 1 to N. Taking into account the number of deviations ξ_i, we readily conclude that there must be in Q a quasi-total compensation between the positive and the negative terms and that Q is therefore practically equal to zero: $Q \approx 0$. Hence

$$\sum\xi_i^2 = \left(\sum\xi_i\right)^2 = \left(N\overline{\xi}\right)^2 = N^2\overline{\xi}^2 .$$

The expression for $\sum\alpha_i^2$ becomes

$$\sum\alpha_i^2 = N^2\overline{\xi}^2 - N\overline{\xi}^2 = N\overline{\xi}^2(N-1) .$$

Now, in conformity with its definition, ξ_i is the deviation with respect to the true value M, therefore the variance σ^2 is defined by the expression

$$\sigma^2 = \frac{\sum\xi_i^2}{N} = \frac{N^2\overline{\xi}^2}{N} = N\overline{\xi}^2 .$$

If this is introduced into the expression of $\sum\alpha_i^2$ it leads to $\sum\alpha_i^2 = \sigma^2(N-1)$ *i.e.* to

$$\sigma^2 = \frac{\sum\alpha_i^2}{N-1} .$$

According to b), $\sum\alpha_i^2$ can be computed numerically from the observed values m_i *i.e.* $\sum\alpha_i^2 = \sum\left(m_i - \overline{m}\right)^2$. Hence, finally

$$\sigma^2 = \frac{\sum\left(m_i - \overline{m}\right)^2}{N-1} .$$

Note

The theory above can be generalized for the systems of linear equations with several unknowns. If the number of equations is larger than the number of unknowns, the value analogous to the above $(N-1)$ becomes equal to the difference $(N-k)$ *i.e.* the difference

number of equations – *number of unknowns* .

Appendix VI
Role of the Mean in Samples of Repeated Observations

Gauss solved the problem of the mean long before (and completely independently) Hagen published his theory of elementary errors. His genius has forseen that if a sample of n repeated measurements m_i ($i = 1, 2, \cdots n$) is given, and an estimate m' (for the true value m_0) can be proposed, it must be possible to express (at least approximately) the probability p_i of obtaining m_i, by a function ψ in which the variable is the distance $(m_i - m')$. We must therefore be able to write

$$p_i = \psi(m_i - m') .$$

This hypothesis being accepted, the compound probability of the total sample must become equal to the product

$$P = \psi(m_1 - m')\,\psi(m_2 - m') \cdots \psi(m_n - m') .$$

Furthermore, Gauss has also forseen that it should be possible to attribute to the function ψ a precise algebraic form by postulating that P must be a maximum when $m' = \bar{m}$. Such a condition may be given the following equivalent form:

$$(m_1 - m') + (m_2 - m') + \cdots + (m_n - m') = 0$$

Now the condition $P = \max$ can be formulated by equating to zero the expression for the logarithmic derivative of P:

$$\frac{1}{\psi(m_1 - m')} \cdot \frac{d\psi(m_1 - m')}{dm'} + \frac{1}{\psi(m_2 - m')} \cdot \frac{d\psi(m_2 - m')}{dm'} + \cdots = 0 .$$

This equation takes a simpler form if we write

$$m_i - m' = x_i , \quad dm' = dx_i .$$

This form is

$$\frac{1}{\psi(x_1)} \cdot \frac{d\psi(x_1)}{dx_1} + \frac{1}{\psi(x_2)} \cdot \frac{d\psi(x_2)}{dx_2} + \cdots + \frac{1}{\psi(x_n)} \cdot \frac{d\psi(x_n)}{dx_n} = 0 .$$

All n terms here have the same algebraic structure $F(x)$ so that the final expression for P = max can take the form

$$F(x_1) + F(x_2) + \cdots + F(x_n) = 0 .$$

This form, together with a) constitute the system that can be solved:

$$\begin{cases} F(x_1) + F(x_2) + \cdots + F(x_n) = 0 , \\ x_1 + x_2 + \cdots + x_n = 0 . \end{cases}$$

The only solution for such systems is

$$F(x) = \gamma x , \quad \gamma = \text{constant} .$$

It leads to the differential equation

$$\frac{1}{\psi(x)} \cdot \frac{d\psi(x)}{dx} = \gamma x ,$$

the general solution of which is

$$\log \psi(x) = \gamma \frac{x^2}{2} + C , \quad i.e. \psi(x) = Ce^{\gamma \frac{x^2}{2}} , \quad C = \text{constant} .$$

So far as the values that can be attributed to γ and C, they must conform to the general properties we attribute to random errors. Thus, obviously, γ must be a negative quantity and we can write

$$\gamma = -g^2 .$$

On the other hand, C is deduced from the normalisation condition:

$$\int_{-\infty}^{+\infty} \psi(x) dx = C \int_{-\infty}^{+\infty} e^{-g^2 \frac{x^2}{2}} dx = C \frac{\sqrt{2\pi}}{g} = 1 .$$

Hence,

$$C = \frac{g}{\sqrt{2\pi}}$$

and therefore

$$\psi(x) = \frac{g}{\sqrt{2\pi}} e^{-g^2\frac{x^2}{2}} .$$

The calculation of the variance σ^2 of x presents no difficulty. It is equal to

$$\sigma^2 = \int_{-\infty}^{+\infty} x^2\psi(x)dx = \frac{1}{g^2} .$$

The final expression for $\psi(x)$ is therefore

$$\psi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} .$$

This is the classical expression for the normal function $f(x)$ as it is introduced in Chapter II, Section 1 (29).

The elegance and the conciseness of Gauss' method is outstanding but, of course, this method cannot be compared with Hagen's theory so far as the deep understanding of the nature of random errors is concerned. Some authors treat it as a sort of "justification" of Hagen's method. The amazing fact is that Gauss preceeds Hagen by almost half of a century! A genius does not conform to historical orders.

Appendix VII
Theory of Modulation

In the last two decades, the analysis of large samples of "repeated" observations has been significantly modified by the introduction into the theory of random errors of a new concept *viz* that of "modulation". This concept does not invalidate Hagen's theory, on the contrary, one can say that it adds to this theory a new dimension.

In Hagen's theory, the variance of the error-variate x is defined as being the limit of the expression $\frac{\epsilon^2 k}{4}$ when k tends towards infinity and ϵ towards zero. However, so far as "modulation" is concerned, it is convenient, for calculational purposes (on condition that k is sufficiently large) to use simply the finite form

$$\sigma^2 = \frac{\epsilon^2 k}{4} .$$

Now, each elementary error ϵ is obviously due to a specific "elementary cause" and it is not absolutely necessary to postulate that, at the time when a measurement is performed, *all* k elementary causes must be actually operating. A certain portion of them may temporarily vanish and thus produce no errors; or, what is the same, produce errors equal to zero. The modulation theory is based on the hypothesis that k can be considered as composed of two parts,

$$k = n + z , \qquad \text{...(VII.1)}$$

n designating the number of *non-zero* errors $(\pm \frac{\epsilon}{2})$ and z the number of *zero* errors. As these numbers constantly fluctuate during the measurements, the most fundamental question is what is the probability that, at the time when a certain measurement takes place, the number of, say, non-zero errors will be in the vicinity of a certain n.

Let us assume that this probability can be expressed by a function $\Phi(n)$ such that the number n of non-zero errors that will fall into a small (but finite) interval Δn will be equal to

$$\Phi(n)\Delta n .$$

The function $\Phi(n)$ is termed the "modulation function". All those errors which correspond to a certain n, will be normally distributed with a variance σ_n^2 equal to

$$\sigma_n^2 = \frac{\epsilon^2 n}{4}$$

and the compound probability that, simultaneously, n will fall into Δn and x into Δx, will be given by the product

$$\Phi(n)\Delta n \cdot \frac{1}{\sigma_n\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_n^2}} \Delta x \ .$$

The total probability ΔP_x that x will fall into Δx, whatever the value of n, is

$$\Delta P_x = \frac{2\Delta x}{\epsilon\sqrt{2\pi}} \sum_{n=0}^{k} e^{-\frac{2x^2}{\epsilon^2 n}} \frac{\Phi(n)}{\sqrt{n}} \Delta n \ .$$

Remembering that k is a very large number, we can replace this relation by a practically equivalent form in which both variables (x and n) are considered as continuous:

$$dP_x = \frac{2dx}{\epsilon\sqrt{2\pi}} \int_0^k e^{-\frac{2x^2}{\epsilon^2 n}} \cdot \frac{\Phi(n)}{\sqrt{n}} dn \ . \qquad \text{...(VII.2)}$$

To progress beyond this point, it is necessary to suggest an analytical expression for $\Phi(n)$. This constitutes a new experimental and statistical problem which can be approached only pragmatically *i.e.* by testing various possible mathematical forms. Actually the first form that has been tried, *viz* $\Phi(n) = An$, proved to be acceptable and was readily generalized into

$$\Phi(n) = An^a \ , (A \text{ and } a \text{ constants}) \ .$$

The normalized form of this definition is

$$\Phi(n) = \frac{a+1}{k^{a+1}} n^a \ , \qquad \text{...(VII.3)}$$

as it may be proven as follows. The constant k being a large number, the normalization can be made by means of an integral:

$$\sum_{n=0}^{k} \Phi(n)\Delta n = \int_0^k \Phi(n)dn = A\int_0^k n^a dn = 1 \ ,$$

$$A\left|\frac{n^{a+1}}{a+1}\right|_0^k = 1 \ , \quad A = \frac{a+1}{k^{a+1}} \ .$$

If (VII.3) is introduced into dP_x (VII.2), the latter becomes

$$dP_x = \frac{2(a+1)dx}{\epsilon k^{a+1}\sqrt{2\pi}} \int_0^k e^{-\frac{2x^2}{\epsilon^2 n}} n^{a-\frac{1}{2}} dn \ . \qquad \text{...(VII.4)}$$

The integral cannot be directly calculated in a simple manner but the calculation of the variance of x, denoted here by τ^2, presents no difficulty *. By definition, we have

$$\tau^2 = \frac{2(a+1)}{\epsilon k^{a+1}\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 \, dx \int_0^k e^{-\frac{2x^2}{\epsilon^2 n}} n^{a-\frac{1}{2}} dn \ ,$$

but it can be shown that it is permissible to reverse the order of integrations:

$$\tau^2 = \frac{2(a+1)}{\epsilon k^{a+1}\sqrt{2\pi}} \int_0^k n^{a-\frac{1}{2}} \, dn \int_{-\infty}^{+\infty} x^2 e^{-\frac{2x^2}{\epsilon^2 n}} dx \ .$$

The integral in x is reducible to J_2 (Table II) by the substitution

$$\frac{2x^2}{\epsilon^2 n} = z$$

which leads to

$$\int_{-\infty}^{+\infty} x^2 e^{-\frac{2x^2}{\epsilon^2 n}} dx = \frac{\epsilon^3 n^{\frac{3}{2}} \sqrt{\pi}}{4\sqrt{2}} \ .$$

If this is introduced into (VII.4) and all operations are performed, the final expression for τ^2 becomes

$$\tau^2 = \frac{(a+1)\epsilon^2}{4k^{a+1}} \int_0^k n^{a+1} \, dx = \frac{a+1}{a+2} \frac{\epsilon^2 k}{4} \ . \qquad \text{...(VII.5)}$$

---

* The symbol τ is used instead of the usual σ, the latter being traditionally reserved for the normal variate.

To be able to introduce the symbol τ into the formula for dP_x, it is necessary to perform in (VII.4) the change of variable

$$t = \frac{n}{k} \, . \qquad \text{...(VII.6)}$$

The calculation requires some attention but presents no difficulty and leads to the following expression:

$$dP_x = \frac{2(a+1)dx}{(\epsilon\sqrt{k})\sqrt{2\pi}} \int_0^1 e^{-\frac{2x^2}{(\epsilon^2 k)}\cdot\frac{1}{t}} t^{a-\frac{1}{2}} dt \, . \qquad \text{...(VII.7)}$$

The term $(\epsilon^2 k)$ can now be taken from (VII.5) and introduced into (VII.7) so that the resulting formula will finally contain only the parameters a and τ.

To avoid any misunderstanding, here is this formula presented under the form of the pdf of a modulated variate the parameters of which are a and τ:

$$f(a,\tau,x) = \frac{2(a+1)\sqrt{a+1}}{\tau\sqrt{2\pi}\sqrt{a+2}} \int_0^1 e^{-\frac{x^2(a+1)}{2\tau^2(a+2)}\cdot\frac{1}{t}} t^{a-\frac{1}{2}} dt \, . \qquad \text{...(VII.8)}$$

The definition of the reduced value λ of a modulated variate is identical to that of the normal variate, *viz*

$$\lambda = \frac{x}{\tau}, \quad d\lambda = \frac{\Delta x}{\tau} \, .$$

Hence, as in (31) - (35)

$$\phi(a, \lambda) = \frac{(a+1)\sqrt{a+1}}{\sqrt{2\pi}\sqrt{a+2}} \int_0^1 e^{-\frac{\lambda^2(a+1)}{2(a+2)}\cdot\frac{1}{t}} t^{a-\frac{1}{2}} dt \, . \qquad \text{...(VII.9)}$$

By putting $\lambda = 0$, we obtain the value of the central ordinate

$$y_0 = \phi(a,0) = \frac{(a+1)\sqrt{a+1}}{\sqrt{2\pi}\sqrt{a+2}} \int_0^1 t^{a-\frac{1}{2}} dt,$$

$$= \frac{2a+2}{2a+1} \sqrt{\frac{a+1}{a+2}} \cdot \frac{1}{\sqrt{2\pi}} \, .$$

But the factor $\frac{1}{\sqrt{2\pi}}$ is the value of the central ordinate y'_0 of the pdf of a normal variate. Hence, the ratio

$$\omega(a) \doteq \frac{y_0}{y'_0} = \frac{2a+2}{2a+1}\sqrt{\frac{a+1}{a+2}}$$

indicates the degree of kurtosis of the modulated variate. It is easy to show that $\omega(a)$ is always larger than unity in the useful range of a, *i.e.* when $a \geq 0$. Here are a few numerical values of $\omega(a)$:

$$\omega(0) = 1.41 \; ; \quad \omega(½) = 1.16 \; ; \quad \omega(1) = 1.09 \; ; \quad \omega(2) = 1.04 \; ; \quad \omega(3) = 1.02 \, .$$

The fact that $\omega(a) > 1$ indicates that all modulated distributions are leptokurtic. In practice, when large observed samples are analyzed, the most commonly found values of a are in the region between $a = 0.5$ and 0.2.

The function $\phi(a,\lambda)$ has been tabulated, first by purely numerical methods and finally by expressing the integral in terms of converging series and continuous fractions. All these calculations are presently considered as routine operations and are available, to all observers, at the NRC Computation Centre (Ottawa, Canada). The theory of modulation is treated in a monograph available from the publishers.*

A few tables of the modulated function for the most typical values of a are given at the end of this book. The table for $a = 0.5$ will be used in the analysis of the sample of gravimetric observations described in Exercises 11 and 23.

One immediately notices that at the centre of the diagram the class frequencies are:

$$F_0 = 315 \, , \quad f_0 = 261.50 \, ,$$

so that the leptokurtosis is of the order of 20 percent *i.e.* ω is in the vicinity of 1.2. The calculation of the theoretical class frequencies by means of a table for $a = 0.5$ seems therefore to be a reasonable operation.

* "Theory of random errors and the influence of modulation on their distribution." Verlag K. Wittwer, Stuttgart, Federal Republic of Germany, [3].

Before performing this operation, the reader should go back to Exercises 11 and 23 where the value of χ^2 is calculated to test the normality. The calculation described in Exercise 23 leads to

$$\chi^2 = \sum_j \frac{(F_j - f_j)^2}{f_j} = 54 , \quad \nu = 18 ,$$

which indicates that, in repeated sampling, the probability P of exceeding 54 is extremely small and thus that this P is strongly against the hypothesis.

In the following table is tested the hypothesis that the sample has been drawn from a population which is modulated normal with $a = 0.5$.

Calculation of χ^2 in a Gravimetric Operation

j	F_j (obs)	f_j (theor. $a=0.5$)	χ^2
-12	2	0.33	
-11	3 → 11	0.92 → 7.26	1.92
-10	4	1.89	
-9	2	4.12	
-8	6	8.48	0.73
-7	18	16.46	0.14
-6	21	30.16	2.78
-5	54	52.16	0.06
-4	76	88.10	1.66
-3	132	130.69	0.01
-2	204	188.11	2.14
-1	246	251.02	0.10
0	315	300.56	0.69
+1	251	280.60	3.12
+2	238	220.83	1.34
+3	171	159.24	0.87
+4	104	107.15	0.09
+5	65	67.75	0.11
+6	35	40.37	0.71
+7	21	22.70	0.13
+8	10	12.05	0.35
+9	7	6.04	
+10	4 → 14	2.85 → 10.07	1.53
+11	1	1.27	
+12	2	0.54	

$$\begin{cases} \chi^2 = 18.48 & \nu = 19-1 = 18. \\ P = 0.43 \end{cases}$$

The high value of P shows that the modulated curve ($a = 0.5$) fits very well into the observed diagram and that therefore the hypothesis formulated above is likely to be correct.

The modulated curve is represented in Fig. App. VII. It should be compared with Fig. Ex. 11.

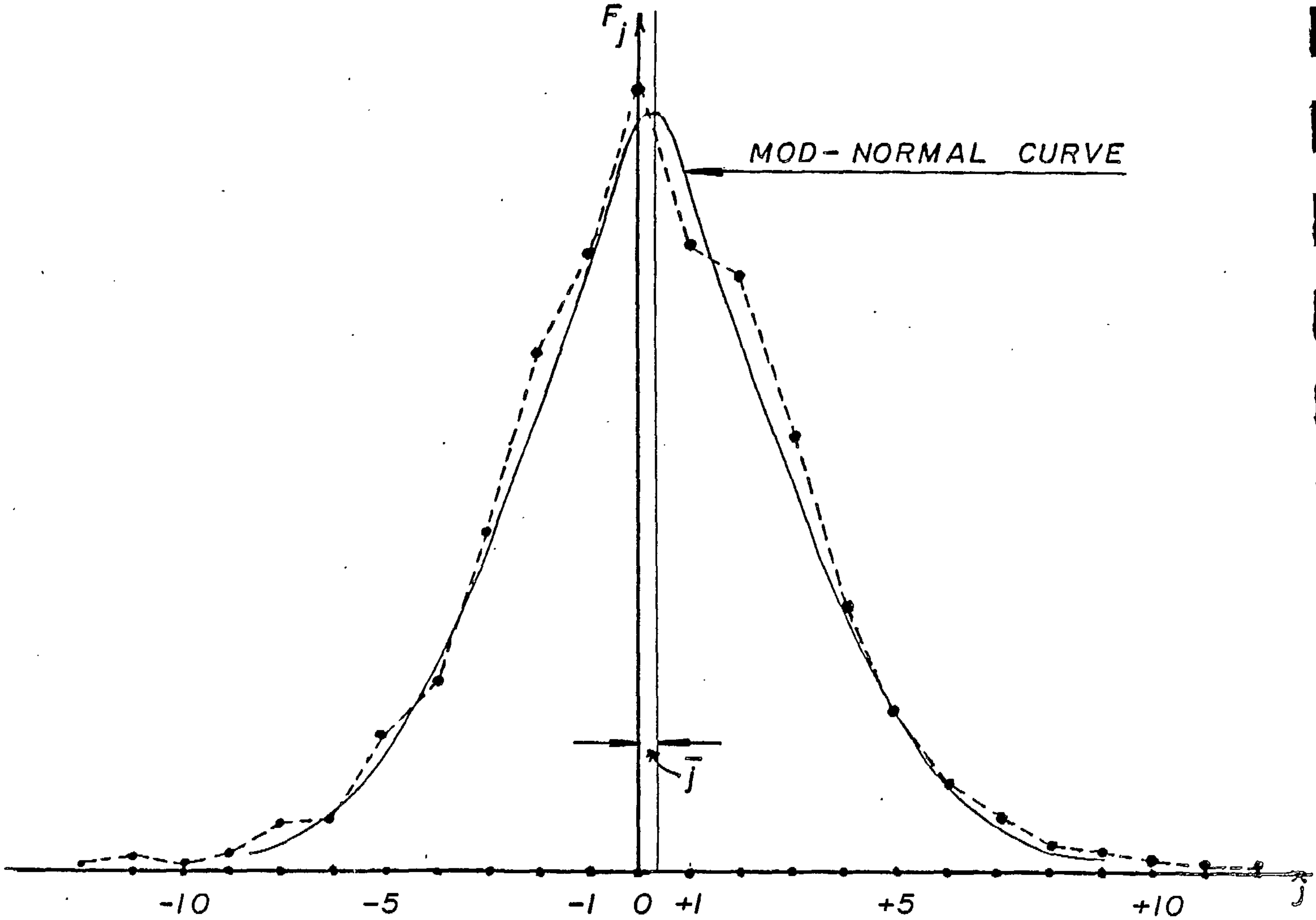


Fig. App. VII Sample of Gravimetric Residuals

Another outstanding case of leptokurtosis is described at the end of this Appendix. As the sample analyzed in Exercise 11, it comes from the domain of gravimetry.

Statistical Analysis of the Residuals in a Gravimetric Survey

The observed data concerning this survey have been communicated to the author by the staff of the Gravity Data Centre . They constitute the outcome of an operation which is not only very vast but also outstanding by the quality of the observations and the sophistication of the instruments used.

The most fundamental originality of the operation is that it has been performed on the sea. When the depth of the sea is more than 500 m, the gravimeter cannot be lowered to the sea bottom, but must be placed on a ship and mounted on a gyro-stabilized platform which keeps the instrument level and, as much as possible, isolated from the motion of the ship. It is obvious that the causes which affect the readings of a gravimeter are much more numerous and more difficult to control when the instrument is on a ship than when it is on solid ground. Thus the readings are performed every few seconds and averaged at intervals of a few minutes to smooth out "noise" due, for instance, to the vibrations produced by the ship engines and a variety of other multiple causes.

A typical marine survey consists of a series of parallel tracks with a number of cross tracks spaced at wider intervals, producing an approximately rectangular grid pattern. The gravity difference is measured between successive crossovers along the ship's track. Each pair of successive measurements leads to an equation (equation of condition) which contains the unknowns for the gravity values at two contiguous crossovers.

Some of the crossovers are located at port stations which belong to the national gravimetric network. Regular passages of the ship through these ports, introduces into the system of equations values of g that are known "without error", thus transforming these equations into equations of definition.

A vast calculation by the method of least squares of all observed data, attributes to the gravity at each crossover an adjusted value g(adj) from which also all adjusted values of the differences

$$\Delta g\ (\mathrm{adj}) = g_i\ (\mathrm{adj}) - g_j\ (\mathrm{adj})$$

can be calculated. The final step is the calculation of the *residuals* of the survey. A residual is defined as being equal to

$$\epsilon = \Delta g \text{ (obs)} - \Delta g \text{ (adj)},$$

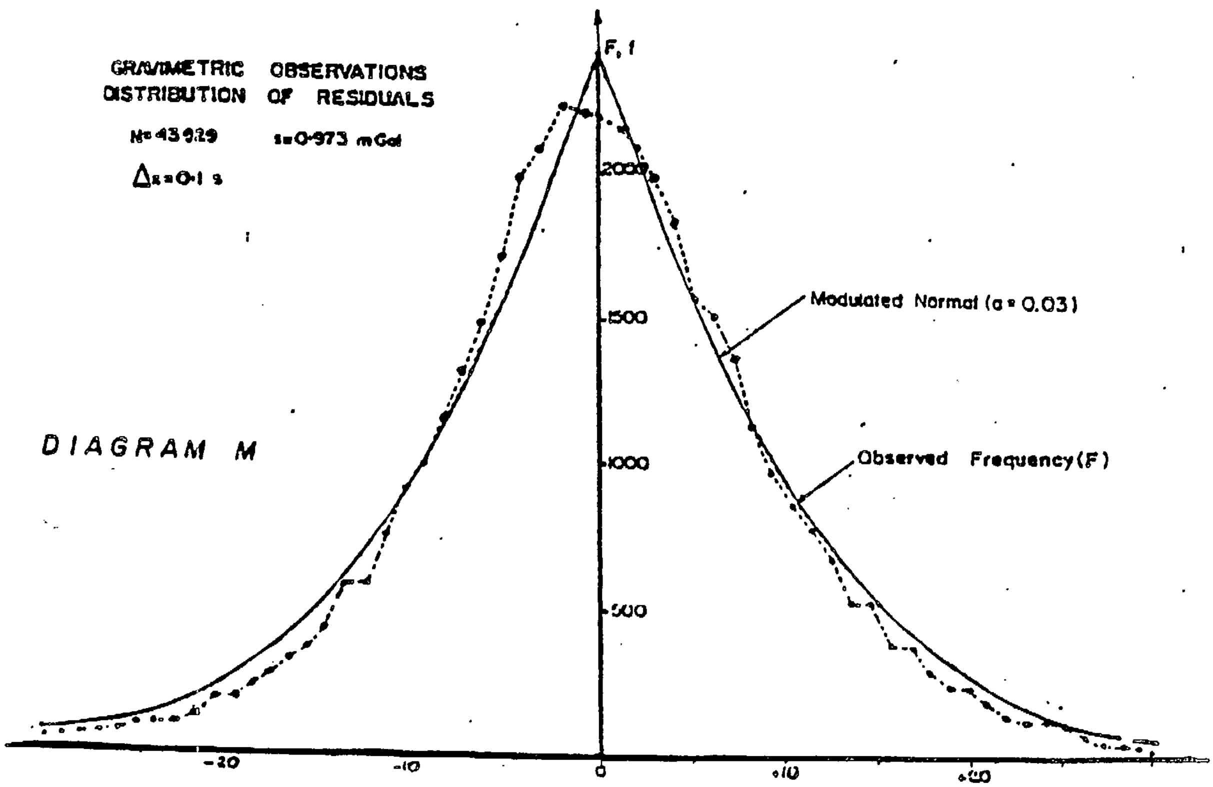
Δg(obs) being directly calculated from the raw data *i.e.* measurements recorded by the ship on any pair of two adjacent crossovers.

The sample here analyzed contains 43 929 residuals. It is, so far, the largest sample to which the modulation theory has been applied. The residuals are classified by means of an interval equal to one-tenth of the standard deviation s. The value of s has been calculated directly from the sample elements; it is equal to $s = 0.973$ milligal so that

$$\Delta x = 0.0973 \text{ mGal}$$

The Diagram M represents the modulated normal curve with the modulator a very close to zero (actually $a = 0.03$). In spite of the fact that the curve has a somewhat too pointed top, it fits into the observed line much better than the normal curve.

The Diagram N represents the normal curve that fits into the observed line (black dots on the horizontal axis are the centres of the classification intervals). Clearly, the sample is strongly leptokurtic.



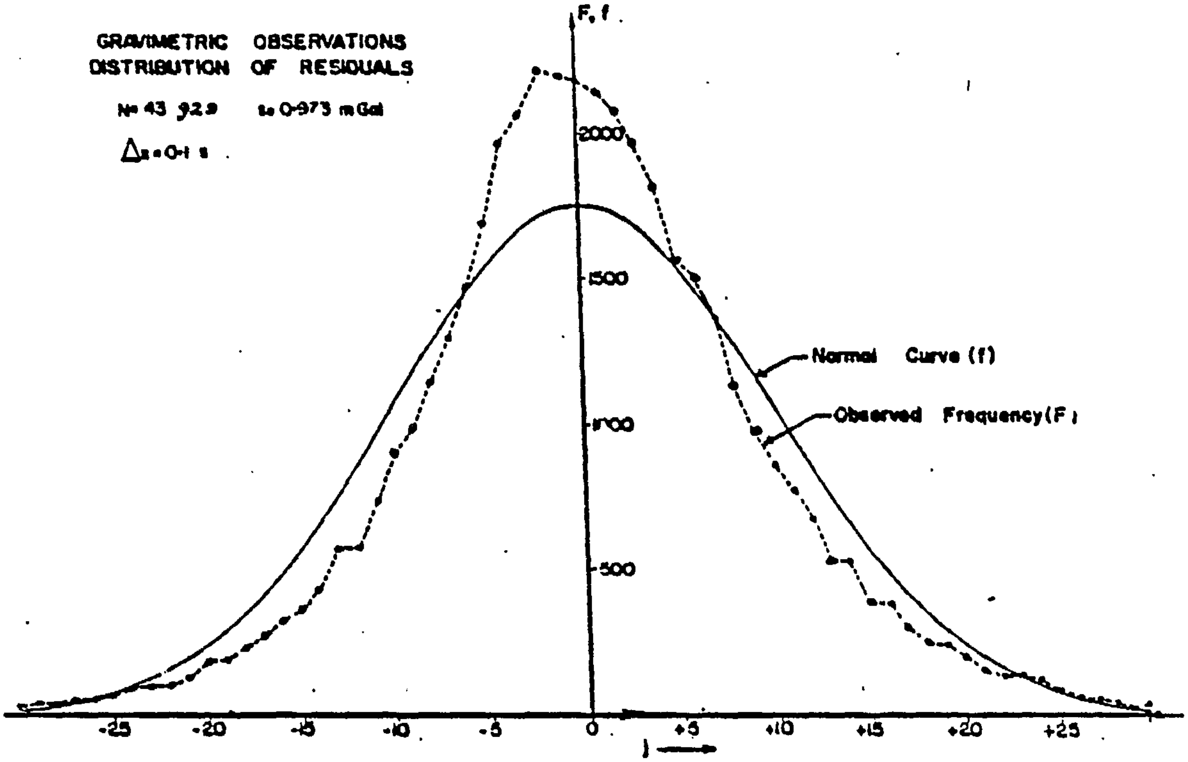

GRAVIMETRIC OBSERVATIONS
DISTRIBUTION OF RESIDUALS
N= 43 929   s= 0·973 mGal
Δx = 0·1 s
F, f
2000
1500
1000
500
Normal Curve (f)
Observed Frequency(F)
-25
-20
-15
-10
-5
0
+5
+10
+15
+20
+25

DIAGRAM N

Exercises

Note

It is strongly recommended that the first part of each exercise (separated from the rest by a line of asterisks) be examined first in itself. The reader should then try to solve the presented problem until he reaches the limits of his own capacity to progress.

In this manner he will, step by step, reach the end of the exercise and should then tackle the final questions and suggestions.

Some exercises contain significant extensions to the theory which is therefore illustrated, completed and clarified.

Nov. 12. 1733.

APPROXIMATIO AD

SUMMAM TERMINORUM BINOMII

$\overline{a+b}^n$ in Seriem expanſi,

Autore *A. D. M.* R. S. S.

QUanquam ſolutio Problematum ad ſortem ſpectantium non raro exigit ut plures Termini Binomii $\overline{a+b}^n$ in ſummam colligantur; attamen in poteſtatibus excelſis res adeo laborioſa videtur, ut perpauci hoc opus aggredi curaverint; *Jacobus* & *Nicolaus Bernoulli* viri Doctiſſimi primi quod ſciam tentarunt quid ſua induſtria in hoc genere præſtare poſſet, in quo etiamſi uterque propoſitum ſumma cum laude ſit aſſecutus, aliquid tamen ultra poteſt requiri, hoc eſt approximatio ad ſummam; non enim tam de approximatione videntur fuiſſe ſolliciti quam de aſſignandis certis limitibus quos Summa Terminorum neceſſario tranſcenderet. Quam vero viam illi tenuerint, breviter in. Miſcellaneis meis expoſui * quæ conſulat Lector ſi vacat, quod ipſi tamen ſcripſerint melius erit fortaſſe conſulere: Ego quoque in hanc diſquiſitionem incubui; quod autem eo me primum impulit non profectum fuit ab opinione me cæteros anteiturum, ſed ab obſequio in Digniſſimum virum qui mihi autor fuerat ut hæc ſuſciperem; Quicquid eſt, novas cogitationes prioribus ſubnecto, ſed eo ut connexio poſtremorum cum primis melius appareat, mihi neceſſe eſt ut pauca jampridem a me tradita denuo proferam.

I. Duodecim jam ſunt anni & amplius cum illud inveneram; ſi Binomium $1+1$ ad poteſtatem n permagnam attollatur, ratio quam Terminus Medius habet ad ſummam Terminorum omnium, hoc eſt ad 2^n, ad hunc modum poterit exprimi $\frac{2A \times \overline{n-1}^n}{n^n \sqrt{n-1}}$, ubi A eum numerum exponit cujus Logarithmus

A

* Vide Miſcellanea Analytica pag. 96. 97. 98. 99.

Abraham de Moivre's work on Bernoulli Formula

(Ars Conjectandi, 1713)

Exercise 1 - Alarm devices

A fire alarm device has a 90 percent chance of responding to an emergency. A house contains three such devices: A, B, C; a positive response is denoted by A+ and a failure by A−. Examine various possible cases, their respective probabilities and the total degree of protection the devices can offer. Consider also other numbers of devices.

First case:

All three (separate) devices operate: A+, B+, C+.

Probability: $\frac{9}{10}\times\frac{9}{10}\times\frac{9}{10} = 0.729$

Second case:

Two devices operate: A+ B+ C-
A+ B- C+
A- B+ C+

Probability: $3\times(\frac{9}{10}\times\frac{9}{10}\times\frac{1}{10}) = 0.243$

Third case:

One device operates.

Probability: $3\times(\frac{9}{10}\times\frac{1}{10}\times\frac{1}{10}) = 0.027$

The total probability that at least one device will operate is therefore $0.729 + 0.243 + 0.027 = 0.999$.

Fourth case:

Total failure.

Probability: $\frac{1}{10}\times\frac{1}{10}\times\frac{1}{10} = 0.001$

Similarly two devices would lead to $0.81 + 0.18 = 0.99$ and one single device to 0.9.

Exercise 2 - Lottery Tickets

A pack of 100 lottery tickets contains three winners. How many tickets should one buy in order to have the greatest probability of getting one (and only one) winning ticket.

Let us first establish the expression for the probability that in a set of, say, 20 tickets one (and only one) be a winner. As a first step let us assume that it is the first ticket which will be a winner. The compound probability of one winner and 19 losers takes the form of a product of fractions:

$$p = \frac{3}{100}\times\frac{97}{99}\times\frac{96}{98}\times\frac{95}{97}\times\frac{94}{96}\times\cdots\times\frac{83}{85}\times\frac{82}{84}\times\frac{81}{83}\times\frac{80}{82}\times\frac{79}{81}$$

One readily sees that the symbols "80" and "79" in the numerator are actually designating the differences:

$$80 = 100-20 ,$$

$$79 = 100-(20+1) .$$

Simplifying and designating by x the number of bought tickets (here $x=20$) we obtain

$$\frac{3}{100}\times\frac{80}{99}\times\frac{79}{98} = \frac{3}{100\times 99\times 98}(100-x)(99-x) .$$

If in the product, the winning ticket takes the second position, this product takes the form:

$$\frac{97}{100}\times\frac{3}{99}\times\frac{96}{98}\times\frac{95}{97}\times\cdots\times(100-x)(99-x) ,$$

which is identical to the product p above as the only difference is that the symbols "100" and "97" have interchanged their positions.

There are $x=20$ such products, so the total probability will be equal to:

$$\frac{3}{100\times 99\times 98}\times(100-x)(99-x)x .$$

For $x=20$, it is equal to

$$\frac{3\times 80\times 79}{100\times 99\times 98}\times 20 = 0.391 \text{ or } 39.1 \text{ percent} .$$

To find the value of x having the maximum probability we have to study the function

$$y = (100-x)(99-x)x ,$$

$$y = x^3 - 199x^2 + 9900x .$$

The derivatives of this cubic are:

$$\frac{dy}{dx} = 3x^2 - 398x + 9900 ,$$

$$\frac{d^2y}{dx^2} = 6x - 398 .$$

The equation $\frac{dy}{dx} = 0$ indicates that the nearest *integer* solution is $x=33$ and the sign of the second derivative indicates that it actually leads to a maximum of y. One can reach the same results by calculating the numerical values for y when x is made equal to 32, 33 and 34.

The values of the probabilities for these three values of x are:

$$P_{32} = 0.4508 ,$$

$$P_{33} = 0.4512 \ (maximum) ,$$

$$P_{34} = 0.4510 .$$

It is interesting to notice that the curve y has in the vicinity of $x=33$ a rather flat top. It is of interest to investigate the cases where the number of winning tickets is different from 3. Thus we can notice that the order of the polynomial y is equal to the number of winning tickets:

$$\text{with } x = 2, \ y = (100-x)x, \ x(max) = 50 ;$$

$$\text{with } x = 4, \ y = (100-x)(99-x)(98-x)x, \ x(max) = 25 .$$

Exercise 3 - Numerical Example of Bernoulli Trials

Part A

Using the formula (2), trace the diagram of P_X as a function of X, for $k=12$, $p=\frac{1}{2}$ and $k=12$, $p=\frac{1}{3}$.

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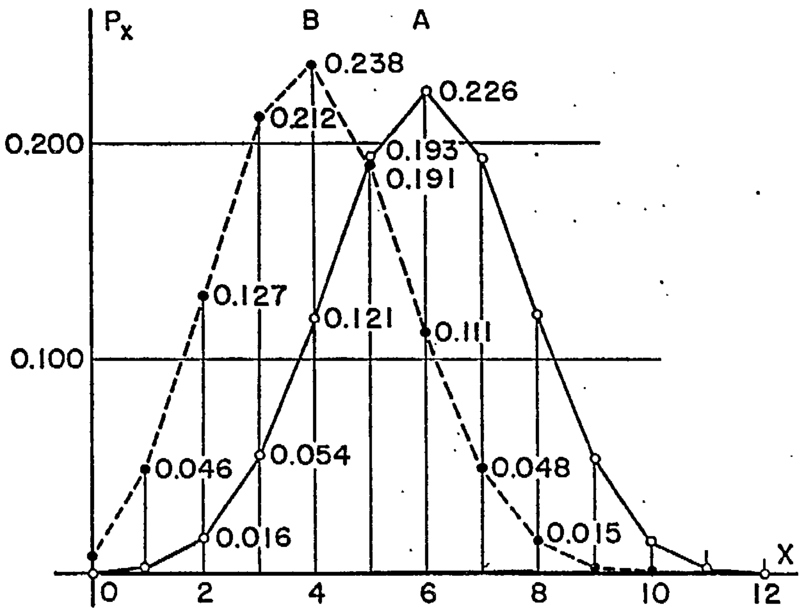

Polygon A. $P_X = \frac{12!}{X!(12-X)!} \cdot \left(\frac{1}{2}\right)^X \left(\frac{1}{2}\right)^{12-X}$

Polygon B. $P_X = \frac{12!}{X!(12-X)!} \cdot \left(\frac{1}{3}\right)^X \cdot \left(\frac{2}{3}\right)^{12-X}$

Fig. Ex. 3. Bernoulli Trials.

Part B

In Bernoulli Trials with $k=11$ and $p=\frac{1}{3}$ determine the value of X for which P_X is a maximum. Prove by a direct substitution in the expression of P_X.

* * * * * * * * * *

The relation

$$kp + p > X > kp - q$$

gives

$$\frac{11+1}{3} \geq X \geq \frac{11-2}{3}$$

$$X = 4 \text{ and } 3.$$

Check: 1) $X = 4$

$$\frac{11!}{4!\,7!}\cdot\left(\frac{1}{3}\right)^4\cdot\left(\frac{2}{3}\right)^7 = \frac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot4\times1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}\cdot\frac{2^7}{3^{11}}$$

$$= \frac{11\cdot10\cdot9\cdot8}{1\cdot2\cdot3\cdot4}\cdot\frac{2^7}{3^{11}} = \frac{11\cdot10\cdot9}{1\cdot2\cdot3}\cdot\frac{2^8}{3^{11}}$$

2) $X = 3$

$$\frac{11!}{3!\,8!}\cdot\left(\frac{1}{3}\right)\cdot\left(\frac{2}{3}\right)^8 = \frac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\times1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}\cdot\frac{2^8}{3^{11}}$$

$$= \frac{11\cdot10\cdot9}{1\cdot2\cdot3}\cdot\frac{2^8}{3^{11}}$$

$$P_{X=4} = P_{X=3} = 0.238 \cdots \quad (23.8 \text{ percent}).$$

Exercise 4 - Screws Fabricating Machine

The production of a screws fabricating machine is distributed into 100 boxes, the nominal weight of a box being 7 kg. In reality, this weight is a variable the extreme values of which are $x = 6.20$ kg and $x = 7.80$ kg. The weights of the boxes are distributed into 17 classes by means of an interval equal to 0.10 kg. The classes are numbered, the interval (7.00 to 7.10) being considered as the "central" interval. Its rank j is equal to zero (0) and its centre is at $x = 7.05$; the centres of extreme classes are: 6.25 (rank $j = -8$) and 7.85 (rank $j = +8$).

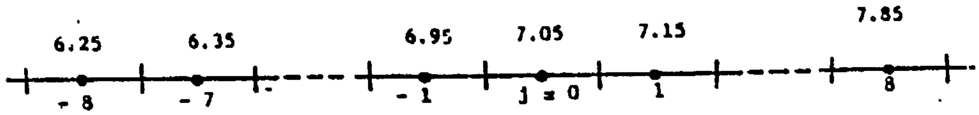

Fig. Ex. 4 - A. j-axis for distribution of screws.

In all calculations that follow the rank j (from -8 to $+8$) is considered as the independent variable and it is only at the end of the calculations that the results (expressed in terms of j) may be converted into kilograms.

Draw an exact diagram using the class frequencies F_j presented in the Table that follows. Then compute the values of $\omega_1 = \overline{j}$, ω_2, μ_2 (moment about $\overline{j}$). Check the relation $\mu_2 = \omega_2 - \omega_1^2$.

* * * * * * * * * * *

Table of Calculations

j	F_j	jF	$j-\bar{j}$	$(j-\bar{j})^2$	$(j-\bar{j})^2 F_j$
−8	0	0	−8.4	70.56	0.00
−7	0	0	−7.4	54.76	0.00
−6	1	−6	−6.4	40.96	40.96
−5	3	−15	−5.4	29.16	87.48
−4	4	−16	−4.4	19.36	77.44
−3	8	−24	−3.4	11.56	92.48
−2	9	−18	−2.4	5.76	51.84
−1	12	−12	−1.4	1.96	23.52
0	10	0	-0.4	0.16	1.60
+1	18	+18	+0.6	0.36	6.48
+2	14	+28	+1.6	2.56	35.84
+3	9	+27	+2.6	6.76	60.84
+4	6	+24	+3.6	12.96	77.76
+5	4	+20	+4.6	21.16	84.64
+6	1	+6	+5.6	31.36	31.36
+7	0	0	+6.6	43.56	0.00
+8	1	+8	+7.6	57.76	57.76
	100	+40			730.00

Formulae

$$\bar{j} = \frac{1}{N}\sum_j jF_j = \omega_1 \qquad \bar{j} = +\frac{40}{100} = +0.40$$

$$\omega_2 = \frac{1}{N}\sum_j j^2 F_j$$

$$\mu_2 = \frac{1}{N-1}\sum_j (j-\bar{j})^2 F_j \qquad \mu_2 = \omega_2 - \bar{j}^2$$

$$\mu_2 = \frac{730}{99} = 7.374 \qquad \sqrt{\mu_2} = 2.7155$$

Calculations of ω_2: $\sum_j j^2 F_j = 746$

$$\omega_2 = 7.46$$

$$\mu_2 = \omega_2 - \omega_1^2 = 7.46 - (0.4)^2$$

$$= 7.46 - 0.16 = 7.30$$

If the intervals are numbered from -6 to $+10$ the mean is equal to $\bar{j} = +\frac{240}{100} = 2.4$. Thus the value of the extreme negative abscissa is equal to $-6 - 2.4 = -8.4$ *i.e.* is the same as above. This will lead to the same value of μ_2, *i.e.* 7.374.

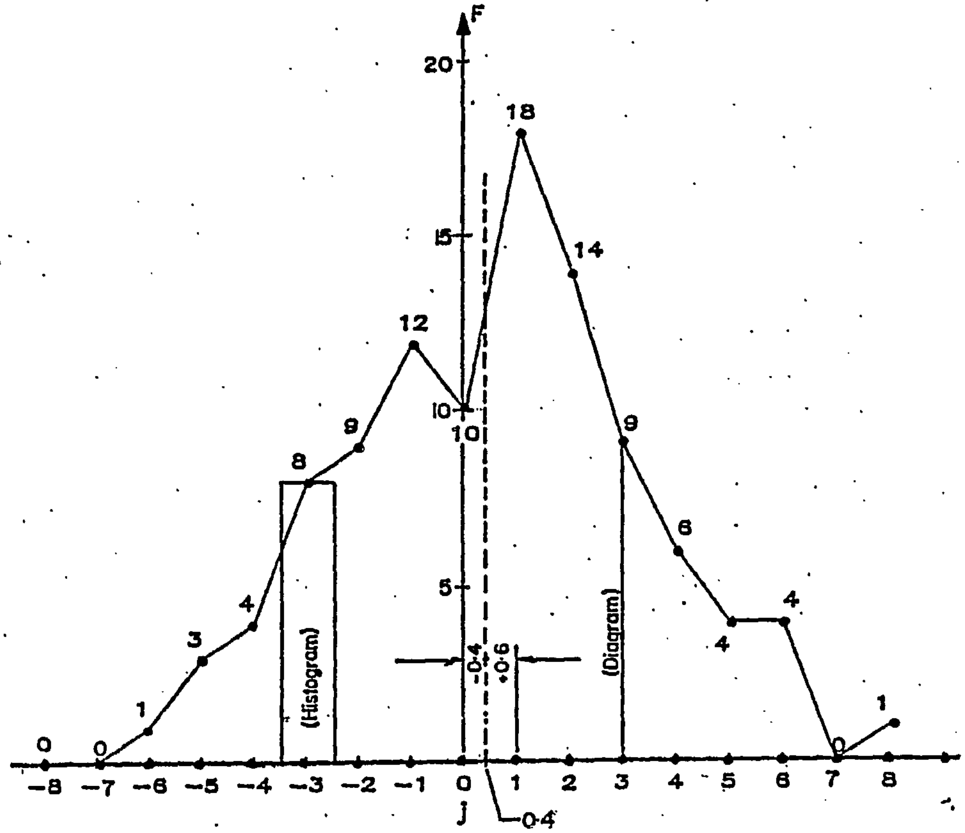

Fig. Ex. 4 - B. Output of screws machine.

For the fitting of a theoretical normal curve into this diagram, see Exercise 10.

Exercise 5 - Calculation of Moments of X when $k=7$

When $k = 7$, the expression (7b) is

$$(q+p)^7 = q^7 + \frac{7}{1}pq^6 + \frac{7\cdot 6}{1\cdot 2}p^2q^5 + \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}p^3q^4 + $$

$$+ \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}p^4q^3 + \frac{7\cdot 6}{1\cdot 2}p^5q^2 + \frac{7}{1}p^6q + p^7$$

If the successive terms are multiplied by the numbers 0, 1, 2, ... 7, respectively, to form the expression for ω_1, (top p.17A) we obtain

$$\omega_1 = 0\cdot q^7 + 1\cdot\frac{7}{1}pq^6 + 2\cdot\frac{7\cdot 6}{1\cdot 2}p^2q^5 + 3\cdot\frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}p^3q^4 + $$

$$+ 4\cdot\frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}p^4q^3 + 5\cdot\frac{7\cdot 6}{1\cdot 2}p^5q^2 + 6\cdot\frac{7}{1}p^6q + 7\cdot p^7$$

After the obvious simplifications, it becomes possible to put $7p$ out of the brackets:

$$\omega_1 = 7p\left(q^6 + \frac{6}{1}pq^5 + \frac{6\cdot 5}{1\cdot 2}p^2q^6 + \cdots + \frac{6}{1}p^5q + p^6\right)$$

Thus we obtain $\omega_1 = 7p(p+q)^6 = 7p$.

The expression for ω_2 is readily deducible from that of ω_1 by replacing the values of the variable (*i.e.* 0, 1, ... 7) by their squares 0^2, 1^2, $2^2 \cdots 7^2$. Thus

$$\omega_2 = 1\times 1\cdot\frac{7}{1}pq^6 + 2\times 2\frac{7\cdot 6}{1\cdot 2}p^2q^5 + 3\times 3\cdot\frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}p^3q^4 + $$

$$+ 4\times 4\cdot\frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}p^4q^3 + 5\times 5\cdot\frac{7\cdot 6}{1\cdot 2}p^5q^2 + 6\times 6\cdot\frac{7}{1}p^6q + 7\times 7\cdot p^7$$

The terms are now simplified and arranged as follows:

$$\omega_2 = 1\cdot\frac{7}{1}pq^6 + 2\cdot\frac{7\cdot 6}{1}p^2q^5 + 3\cdot\frac{7\cdot 6\cdot 5}{1\cdot 2}p^3q^4 + 4\cdot\frac{7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3}p^4q^3 + $$

$$+ 5\cdot\frac{7\cdot 6\cdot 5}{1\cdot 2}p^5q^2 + 6\cdot\frac{7\cdot 6}{1}p^6q + 7\cdot 7p^7$$

Now, as in the previous calculation, the factor $7p$ can be put out of the brackets:

$$\omega_2 = 7p\left(1\cdot q^6 + 2\cdot\frac{6}{1}pq^5 + 3\cdot\frac{6\cdot 5}{1\cdot 2}p^2q^4 + 4\cdot\frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}p^3q^3 + \right.$$

$$\left. + 5\cdot\frac{6\cdot 5}{1\cdot 2}p^4q^2 + 6\cdot\frac{6}{1}p^5q + 7p^6\right)$$

At this point of the analysis it becomes less clearly visible what further transformation may be appropriate. Probably a certain number of attempts had been made before it was discovered that all factors 1, 2, 3, 4, 5, 6, 7 should be replaced by (0+1), (1+1), (2+1), (3+1) ... (6+1), respectively. This gives

$$\omega_2 = 7p\left[\left(0\cdot q^6 + 1\cdot\frac{6}{1}pq^5 + 2\cdot\frac{6\cdot 5}{1\cdot 2}p^2q^4 + 3\cdot\frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}p^3q^3 + 4\cdot\frac{6\cdot 5}{1\cdot 2}p^4q^2 + 5\cdot\frac{6}{1}p^5q + 6\cdot p^6\right) + \right. \quad (A)$$

$$\left. + \left(q^6 + \frac{6}{1}pq^5 + \frac{6\cdot 5}{1\cdot 2}p^2q^4 + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}p^3q^3 + \frac{6\cdot 5}{1\cdot 2}p^4q^2 + \frac{6}{1}p^5q + p^6\right)\right] \quad (B)$$

By examining closely the contents of the brackets (A) and (B) we will notice that (A) represents the first moment ω_1 with $k=6$. So that (A) $= 6p$. On the other hand, we have simply (B) $= (p+q)^6 = 1$.

Hence,

$$\omega_2 = 7p\left[6p + (p+q)^6\right] = 7p(6p+1)\ .$$

This conforms to the formula (11): $\omega_2 = kp[(k-1)p+1]$.

Exercise 6 - Triangular Distribution

A distribution is termed triangular if its representative polygon has the form of an isoceles triangle AA'T. The extreme class frequencies (here, $x = \pm 5$) are equal to 0 and the top T is on the y-axis. The class frequencies $F(x)$ are given in Table A.

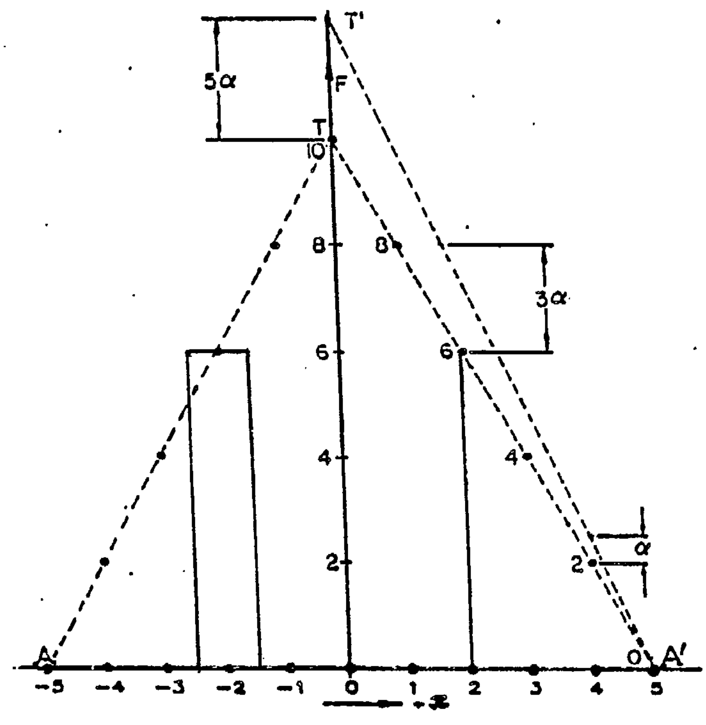

Fig. Ex. 6. Triangular distribution.

Calculate the second moment μ_0 of x and show that it remains constant when the top T changes its position on the F-axis.

* * * * * * * * * * *

A		B		C	D
x	F	x^2	Fx^2	F'	$F'x^2$
-5	0	25	0	$0 + 0\alpha$	0
-4	2	16	32	$2 + 1\alpha$	$32 + 16\alpha$
-3	4	9	36	$4 + 2\alpha$	$36 + 18\alpha$
-2	6	4	24	$6 + 3\alpha$	$24 + 12\alpha$
-1	8	1	8	$8 + 4\alpha$	$8 + 4\alpha$
0	10	0	0	$10 + 5\alpha$	0
+1	8	1	8	$8 + 4\alpha$	$8 + 4\alpha$
+2	6	4	24	$6 + 3\alpha$	$24 + 12\alpha$
+3	4	9	36	$4 + 2\alpha$	$36 + 18\alpha$
+4	2	16	32	$2 + 1\alpha$	$32 + 16\alpha$
+5	0	25	0	$0 + 0\alpha$	0
$N=50$			200	$50+25\alpha$	$200+100\alpha$

Calculation of the second moments:

$$\mu_2 = \frac{1}{N} \sum Fx^2 = \frac{200}{50} = 4$$

$$\mu'_2 = \frac{1}{N'} \sum F'x^2 = \frac{200+100\alpha}{50+25\alpha} = 4$$

As α can take any value, μ_2 is independent of α and thus is the same in all positions of the top T.

Exercise 7 - Distribution of the Variate V

As a complete numerical example of the distribution of V, consider the Bernoulli trials with $k=576$ and $p = q = \frac{1}{2}$ (cards). In the diagram the classification interval will be taken equal to $\Delta V = 4$.

* * * * * * * * * * *

For the moments ω_1, ω_2, μ_2 we have

$$\omega_1 = \bar{x} = kp = 288 \;;$$

$$\mu_2 = kpq = 144 \;; \qquad \sqrt{\mu_2} = 12 \,.$$

The probabilities in successive intervals are computed for the following values of the variables:

$$V = 0\,,\ \pm\Delta V\,,\ \pm 2\Delta V\,,\ \cdots \qquad v = 0\,,\ \pm\frac{\Delta V}{\sqrt{\mu_2}}\,,\ \pm\frac{2\Delta V}{\sqrt{\mu_2}}\,,\ \cdots$$

$$V = 0\,,\quad \pm 4\,,\quad \pm 8\ \cdots \qquad v = 0\,,\ \pm\frac{1}{3}\,,\quad \pm\frac{2}{3}\,,\ \cdots$$

They are given in the table of the normal function

$$\phi(v) = \frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}\,, \qquad \Delta v = \frac{4}{12} = \frac{1}{3}\,.$$

For instance, $P_0 = 0.3989\times\frac{1}{3} = 0.133$, *i.e.* 13.3 percent. All other values of P are given in the diagram.

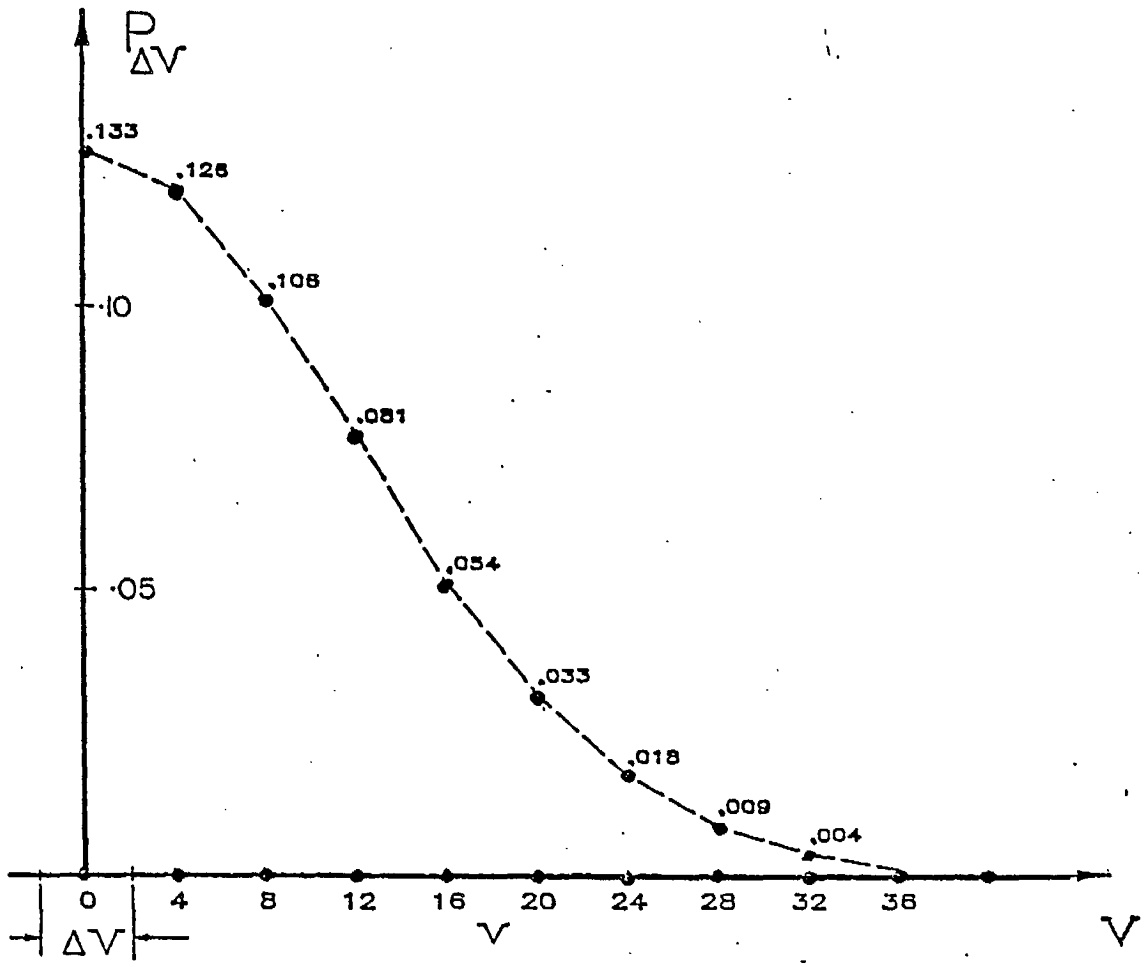

Fig. Ex. 7. Variate V.

Note: in diagrams in which the values of the variable V are integers the curve joining the points representing the probability P has no real meaning. It may be a broken line helping the eye to see the evolution of the ordinate.

Exercise 8 - A Model in Hagen's Theory

Assume that in a certain type of high-precision length measurements the total number k of elementary errors is equal to 40,000 and that the value of ϵ is equal to 0.10 μm. Calculate the values of various parameters.

* * * * * * * * * *

According to Hagen's theory the variance of H is

$$S^2 = \epsilon^2 kpq = 0.01 \times \frac{40\,000}{2\times 2} = 100$$

so that the standard deviation is

$$S = 10\mu\text{m} .$$

If we take $\Delta H = 2\mu$m, then $\frac{\Delta H}{S} = \Delta h = 0.2$.

If the deviations are classified by means of an interval $\Delta H = 2\mu$m then the reduced value of an interval can be designated by Δh and the reduced abscissae of intervals centres are $h = 0, 0.2, 0.4, 0.6, \cdots$ Their corresponding probability densities may be calculated by means of the tables of the normal function $\psi(h)$ and the probabilities by the relation

$$\Delta P_h = \phi(h)\Delta h .$$

Exercise 9 - Exercise on Moments

A variate x can take two values x_1 and x_2 the probabilities being p and q, respectively $(p+q = 1)$. Establish the expression for ω_1, ω_2, ω_3 and solve them for x_1, x_2, p considered as unknowns. Apply the results to the case where $\omega_1 = 1.75$, $\omega_2 = 3.25$, $\omega_3 = 6.25$.

* * * * * * * * * *

The system of equations is

$$\left.\begin{aligned} \omega_1 &= px_1 + qx_2 \\ \omega_2 &= px_1^2 + qx_2^2 \\ \omega_3 &= px_1^3 + qx_2^3 \end{aligned}\right\} \quad p + q = 1$$

To solve this system let us multiply both the first and the second equations by (x_1+x_2):

$$\begin{cases} \omega_1(x_1+x_2) = (px_1+qx_2)(x_1+x_2) = px_1^2+qx_2^2+x_1x_2 \\ \omega_2(x_1+x_2) = px_1^3+qx_2^3+x_1x_2(px_1+qx_2) \end{cases}$$

$$\begin{cases} \omega_1(x_1+x_2) = \omega_2+x_1x_2 \\ \omega_2(x_1+x_2) = \omega_3+\omega_1x_1x_2 \end{cases}$$

This is a system with two unknowns

$$x_1 + x_2 = y \ , \qquad x_1x_2 = z \ ,$$

$$\begin{cases} \omega_1 y = \omega_2+z \\ \omega_2 y = \omega_3+\omega_1 z \ . \end{cases}$$

The solutions are

$$y = \frac{\omega_1\omega_2 - \omega_3}{\omega_1^2 - \omega_2}, \qquad z = \frac{\omega_2^2 - \omega_1\omega_3}{\omega_1^2 - \omega_2},$$

and the unknowns x_1 and x_2 are respectively equal to the roots X' and X'' of the quadratic equation

$$X^2 - yX + z = 0 . \qquad (\textit{Solutions:}\, X' \text{ and } X'') .$$

It is easy to check that the above given values for ω_1, ω_2, ω_3 and p lead to the solutions

$$X' = x_1 = 1 , \quad X'' = x_2 = 2 , \quad p = \frac{1}{4} , \quad q = \frac{3}{4} , \quad y = 3 , \quad z = 2 .$$

Exercise 10 - Theoretical Curve for "Screws Machine"

This is an extension of Ex. 4: Fitting of a Normal Curve

$$\mu_2 = 7.374\,, \qquad \sigma = \sqrt{\mu_2} = 2.7155\,, \qquad \bar{j} = +0.40\,.$$

* * * * * * * * * *

j	$j-\bar{j}$	$\lambda_j = \frac{j-\bar{j}}{\sigma}$	$\phi(\lambda_j)$	f_j
-8	−8.4	−3.0934	0.003 335	0.12
-7	−7.4	−2.7251	0.009 736	0.36
-6	−6.4	−2.3569	0.024 813	0.91
-5	−5.4	−1.9886	0.055 234	2.03
-4	−4.4	−1.6203	0.107 354	3.95
-3	−3.4	−1.2521	0.182 171	6.71
-2	−2.4	−0.8838	0.269 957	9.94
-1	−1.4	−0.5156	0.349 285	12.86
0	−0.4	−0.1473	0.394 634	14.53
+1	+0.6	+0.2210	0.389 316	14.34
+2	+1.6	+0.5892	0.335 370	12.35
+3	+2.6	+0.9575	0.252 248	9.29
+4	+3.6	+1.3257	0.165 685	6.10
+5	+4.6	+1.6940	0.095 014	3.50
+6	+5.6	+2.0623	0.047 575	1.75
+7	+6.6	+2.4305	0.020 804	0.77
+8	+7.6	+2.7988	0.007 942	0.29

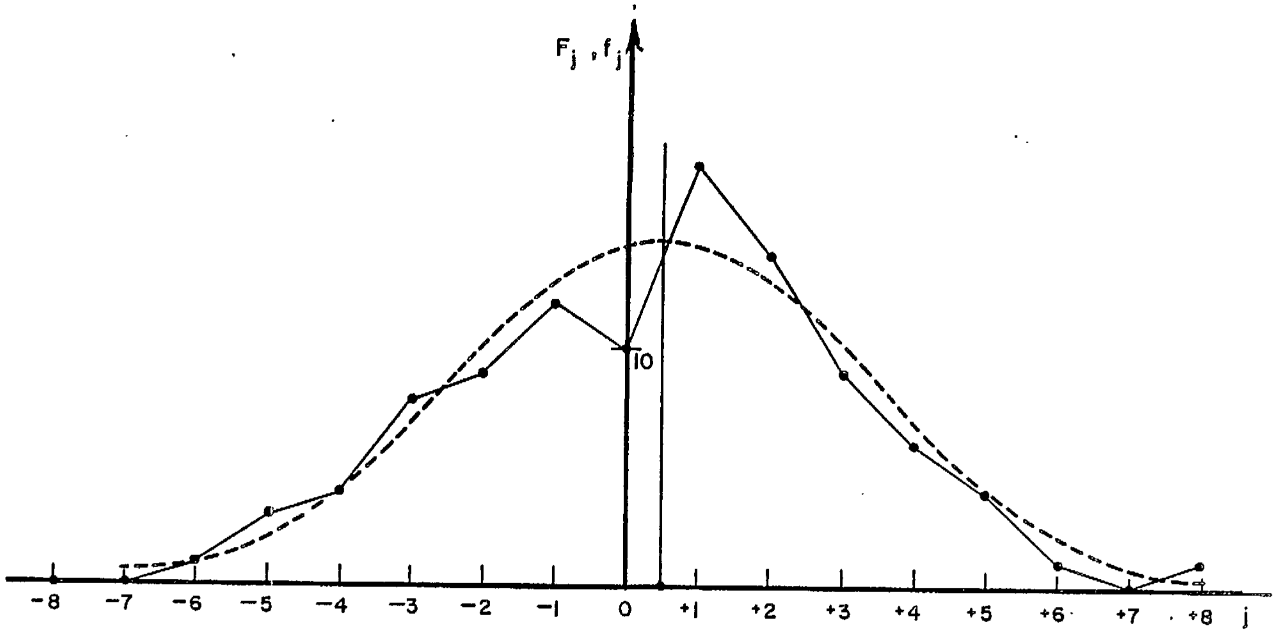

FIG. 2

Fig. Ex. 10. Fitting a normal curve into Ex. 4 - B.

The test of the hypothesis that the sample is drawn from a normal population is performed in Ex. 23.I.

Exercise 11 - Sample of Residuals in a Gravimetric Operation

This sample consists of 1992 residuals obtained in the adjustment of the results of measurements of "g" (Gravitational Constant) made along the line joining Ottawa and Washington. There are twenty stations on this line. The values of g in the extreme stations (O. and W.) are known from absolute determinations. As a final outcome of the operation, each station is attributed an adjusted value of g. The difference between the adjusted values in two consecutive stations can now be compared with the difference directly indicated by the gravimeters. The discrepancy between these two differences is termed "residual".

The residuals, denoted by the symbol x are expressed in a unit of acceleration equal to $10^{-5}\ \frac{\text{m}}{\text{s}^2}$. This unit is equivalent to the old "milligal" $(10^{-3}\ \frac{\text{cm}}{\text{s}^2})$ which officially is no longer part of the SI system.

The population is distributed into 25 classes, from $j=-12$ to $j=+12$ $(\Delta j=1)$, each class interval being equivalent to Δx = 0.015 mGal. Assuming that the observed F_j indicate that the distribution is close to normality, calculate the parameters of the normal curve that fit into the diagram and the theoretical class frequencies f_j.

* * * * * * * * * *

The calculation of the parameters leads to the following results:

$$\bar{j} = 0.2595 \Delta j$$

$$\mu_2 = \sigma^2 = 9.1677 \cdot (\Delta j)^2$$

$$\sqrt{\mu_2} = \sigma = 3.0278\ \Delta j$$

Normal Curve		
j	F_j	f_j
−12	2	0.07
−11	3	0.26
−10	4	0.84
−9	2	2.45
−8	6	6.36
−7	18	14.82
−6	21	30.97
−5	54	58.06
−4	76	97.57
−3	132	147.03
−2	204	198.67
−1	246	240.71
0	315	261.50
+1	251	254.73
+2	238	222.49
+3	171	174.25
+4	104	122.37
+5	65	77.05
+6	35	43.50
+7	21	22.02
+8	10	10.00
+9	7	4.07
+10	4	1.49
+11	1	0.49
+12	2	0.14
	1992	1991.91

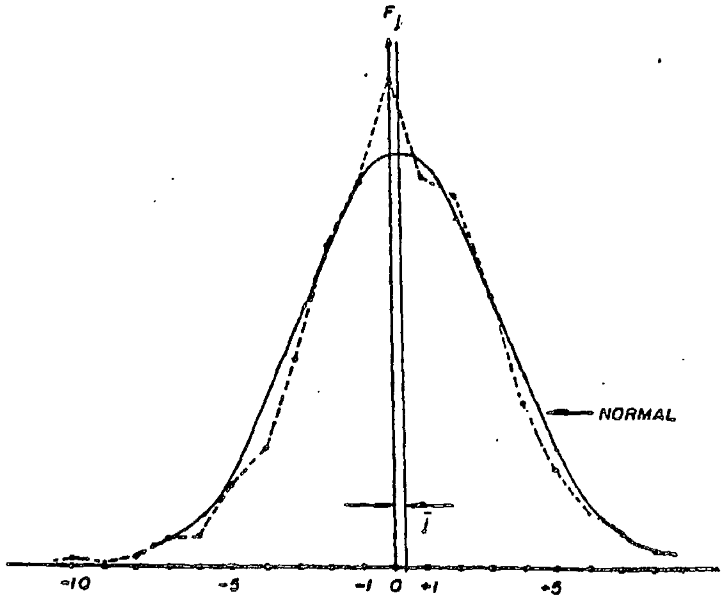

Fig. Ex. 11. Sample of Gravimetric Residuals.

Exercise 12 - Recurrence Formulae for Integrals

Part I

Establish the recurrence formula for the indefinite integral $I_n = \int z^n e^{-z^2} dz$ and then transform it into the formula for the definite integral $J_n = \int_0^\infty z^n e^{-z^2} dz$.

* * * * * * * * * *

The starting function for this operation is

$$\Phi(z) = z^{n-1} e^{-z^2}$$

and its differential

$$\Phi'(z)dz = -2z^n e^{-z^2} dz + (n-1)z^{n-2} e^{-z^2} dz .$$

Integrating both sides we obtain

$$z^{n-1} e^{-z^2} = -2\int z^n e^{-z^2} dz + (n-1)\int z^{n-2} e^{-z^2} dz$$

and, solving for I_n,

$$I_n = \int z^n e^{-z^2} dz = \frac{n-1}{2}\int z^{n-2} e^{-z^2} dz - \frac{1}{2} z^{n-1} e^{-z^2} .$$

Hence the recurrence formula

$$I_n = \frac{n-1}{2} I_{n-2} - \frac{1}{2} z^{n-1} e^{-z^2} .$$

The only integral that can be obtained from this relation is I_1; for $n = 1$ we have

$$I_1 = -\frac{1}{2} e^{-z^2}$$

From I_1, all integrals of odd orders (I_3, I_5 ...) can be calculated step by step.

For instance,

$$I_3 = I_1 - \frac{1}{2} z^2 e^{-z^2}$$

$$I_3 = -\frac{1}{2} e^{-z^2}(1+z^2)$$

The expressions for integrals of even orders cannot be established.

From the recurrence formula established above, the formula for definite integrals is readily deduced by noticing that the term

$$-\frac{1}{2}\left| z^{n-1} e^{-z^2} \right|_0^\infty$$

is also equal to zero. It is obviously equal to zero for $z=0$ and, for $z=\infty$, it can be put under the form

$$\frac{z^{n-1}}{e^{z^2}}$$

Now it is known that an exponential (such as e^{z^2}) tends towards infinity faster than any finite power of z: the fraction, when $n \to \infty$, tends very fast toward zero.

The recurrence formula is thus reduced to

$$J_n = \frac{n-1}{2} J_{n-2} .$$

An interesting feature of the definite integral J_n is that there is a simple expression for J_0:

$$J_0 = \frac{\sqrt{\pi}}{2}$$

This leads to the expressions of all integrals J_n with even values of n; for instance $J_2 = \frac{\sqrt{\pi}}{4}$, $J_4 = \frac{3\sqrt{\pi}}{8}$ etc.

Part II

Establish the recurrence formula for the indefinite integral $I_n = \int x^n e^x dx$ and the definite integral $J_n = \int_0^1 x^n e^x dx$.

The starting function is

$$\Phi(x) = x^n e^x$$

the differential of which is

$$\Phi'(x)dx = x^n e^x dx + nx^{n-1} e^x dx .$$

Integrating both sides we obtain

$$x^n e^x = \int x^n e^x dx + n\int x^{n-1} e^x dx$$

and the recurrence formula

$$I_n = x^n e^x - nI_{n-1} .$$

Hence

$$I_0 = e^x , \quad I_1 = e^x(x-1) , \quad I_2 = e^x(x^2-2x+2) \quad etc.$$

For the definite integral

$$J_n = \int_0^1 x^n e^x dx$$

the recurrence formula is

$$J_n = \left| x^n e^x \right|_0^1 - nJ_{n-1}$$

and, as

$$\left| x^n e^x \right|_0^1 = e ,$$

it reduces to $J_n = e - nJ_{n-1}$. $J_0 = e$, $J_1 = 0$, $J_2 = e$, $J_3 = -2e$, *etc.*

Exercise 13 - Mixture and Dichotomy of Non-coaxial Samples

A theoretical sample of $N = 2000$ elements is considered as being a mixture of two normal sub-samples. It is distributed in 41 classes, from $j = -20$ to $j = +20$. As it is postulated that nothing is known about the parameters, Pearson's system will contain six equations in p_1, p_2, m_1, m_2, σ_1, σ_2. It will be of the form (53b).

j	F_j	j	F_j
−20	0.12	+1	141.93
−19	0.25	+2	145.96
−18	0.48	+3	145.06
−17	0.89	+4	138.57
−16	1.58	+5	126.61
−15	2.72	+6	110.20
−14	4.48	+7	91.08
−13	7.11	+8	71.29
−12	10.83	+9	52.76
−11	15.88	+10	36.86
−10	22.40	+11	24.29
−9	30.45	+12	15.08
−8	39.95	+13	8.82
−7	50.67	+14	4.86
−6	62.32	+15	2.52
−5	74.58	+16	1.23
−4	87.15	+17	0.57
−3	99.78	+18	0.24
−2	112.17	+19	0.10
−1	123.87	+20	0.04
0	134.15		

This sample has been used to test the correctness of all operations related to Pearson's equations. From the values of F_j it was easy to calculate the mean $\bar{j}$, which is equal to $\bar{j} = +1$, and then all moments about $\bar{j}$: μ_0, μ_1, μ_2, μ_3, μ_4, μ_5. The solution of the resulting system (53.) was then performed by Pearson's method based on the solution of the nonic.

The results showed a satisfactory agreement between the calculated values of the parameters p_1, p_2, m_1, m_2, σ_1, σ_2 and the values which have been used to calculate the class frequencies. Pearson's method is not described in the present work; the reader can nevertheless, as an excellent exercise, calculate the values of the μ-moments on the one hand from the class frequencies F_j and, on the other hand, from the parameters that have been used to form the system.

These parameters are:

$$p_1 = 0.5\,, \quad m_1 = -3\,, \quad \sigma_1 = 5$$

$$p_2 = 0.5\,, \quad m_2 = +3\,, \quad \sigma_2 = 4$$

They lead to the following values of the moments:

$\mu_0 =$	1.0	$\mu_3 =$	-40.5
$\mu_1 =$	0.0	$\mu_4 =$	+2752.5
$\mu_2 =$	+29.5	$\mu_5 =$	-9517.5

Note: The solution of a complete system of Pearson's equations (through the nonic) is now available in the library of routine operations at the N.R.C. Computation Centre. It is the outcome of the work performed by Dr. S. Baxter, mathematician at the Centre.

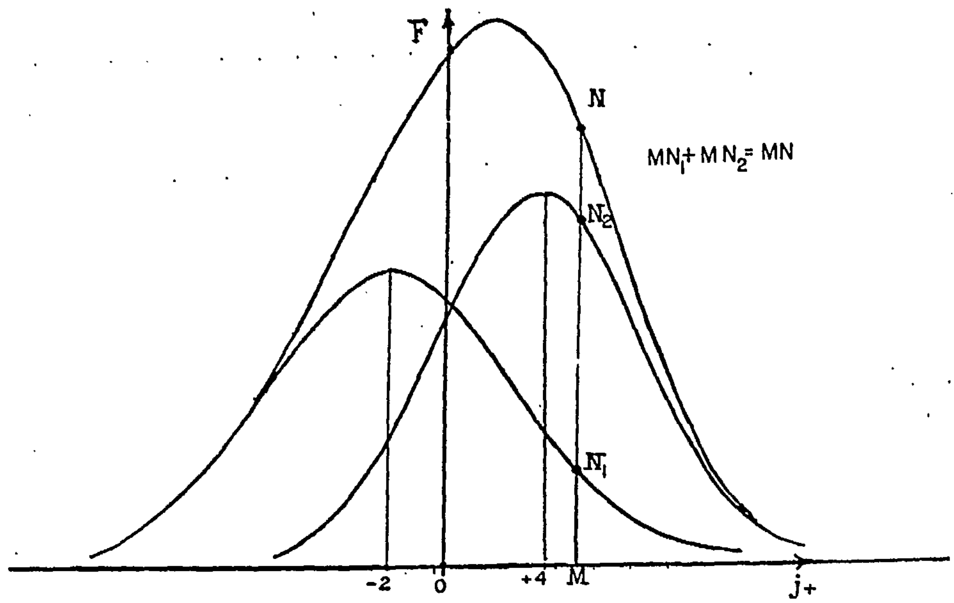

Fig. Ex. 13. Mixture of Non-coaxial Samples.

Exercise 14 - Mixture of Two Lots of Pills

A pharmacist has produced two lots of pills of 100 units each. The mass of a pill may be assumed to be normally distributed about the mean of the lot to which it belongs. The variance is the same in all lots. After the two lots have been mixed, the suspicion arose that one of them may be slightly heavier than the other. As the lots cannot be physically separated, show that it is possible, by statistical methods, to evaluate the difference between their weights.

It may be assumed that the nominal mass of a pill is of the order of 1 g and that the largest deviations from the average are of the order of 15 mg.

* * * * * * * * * *

The first operation consists in weighing all of the pills and recording the obtained masses m_i ($i = 1, 2, \cdots 200$). In order to simplify the operations, the mean may be computed before the classification

$$\overline{m} = \frac{1}{200} \sum_i m_i \, .$$

The deviations are

$$x_i = m_i - \overline{m}$$

and are expressed in milligrams (mg) with 2 decimals. Now x_i may be classified by means of an interval $\Delta x = 1$ mg, the central interval extending from -0.5 to $+0.5$ mg. It receives the rank $j = 0$ and thus the whole sample is distributed into 31 intervals: from $j = -15$ to $j = +15$.

The second step is the calculation of the second moment μ_2 and the fourth moment μ_4 (with respect to $\overline{j} = 0$).

As the sizes (N_1 and N_2) of both lots are identical, ($N_1 = N_2 = 100$) the centres of the components are located symmetrically with respect to the centre of the mixture. Let the abscissae of these centres be $+a$ and $-a$ ($\alpha > 0$) and let the variance of both representative curves be σ^2. According to Pearson's equations we have the following system of equations:

$$p_1 = p_2 = \frac{1}{2}$$

$$\frac{1}{2}(\sigma^2 + a^2) + \frac{1}{2}(\sigma^2 + a^2) = \mu_2 ,$$

$$\frac{1}{2}(3\sigma^4 + 6a^2\sigma^2 + a^4) + \frac{1}{2}(3\sigma^4 + 6a^2\sigma^2 + a^4) = \mu_4 .$$

The simplified form is:

$$\sigma^2 + a^2 = \mu_2 ,$$

$$3\sigma^4 + 6a^2\sigma^2 + a^4 = \mu_4 .$$

The solution presents no difficulty: the first equation is squared, multiplied by 3 and the result is subtracted from the second equation. This gives

$$2a^4 = 3\mu_2^2 - \mu_4 \rightarrow \quad a = \left(\frac{3\mu_2^2 - \mu_4}{2}\right)^{\frac{1}{4}}$$

and leads to the expression for σ^2:

$$\sigma^2 = \mu_2 - a^2 = \mu_2 - \left(\frac{3\mu_2^2 - \mu_4}{2}\right)^{\frac{1}{2}} .$$

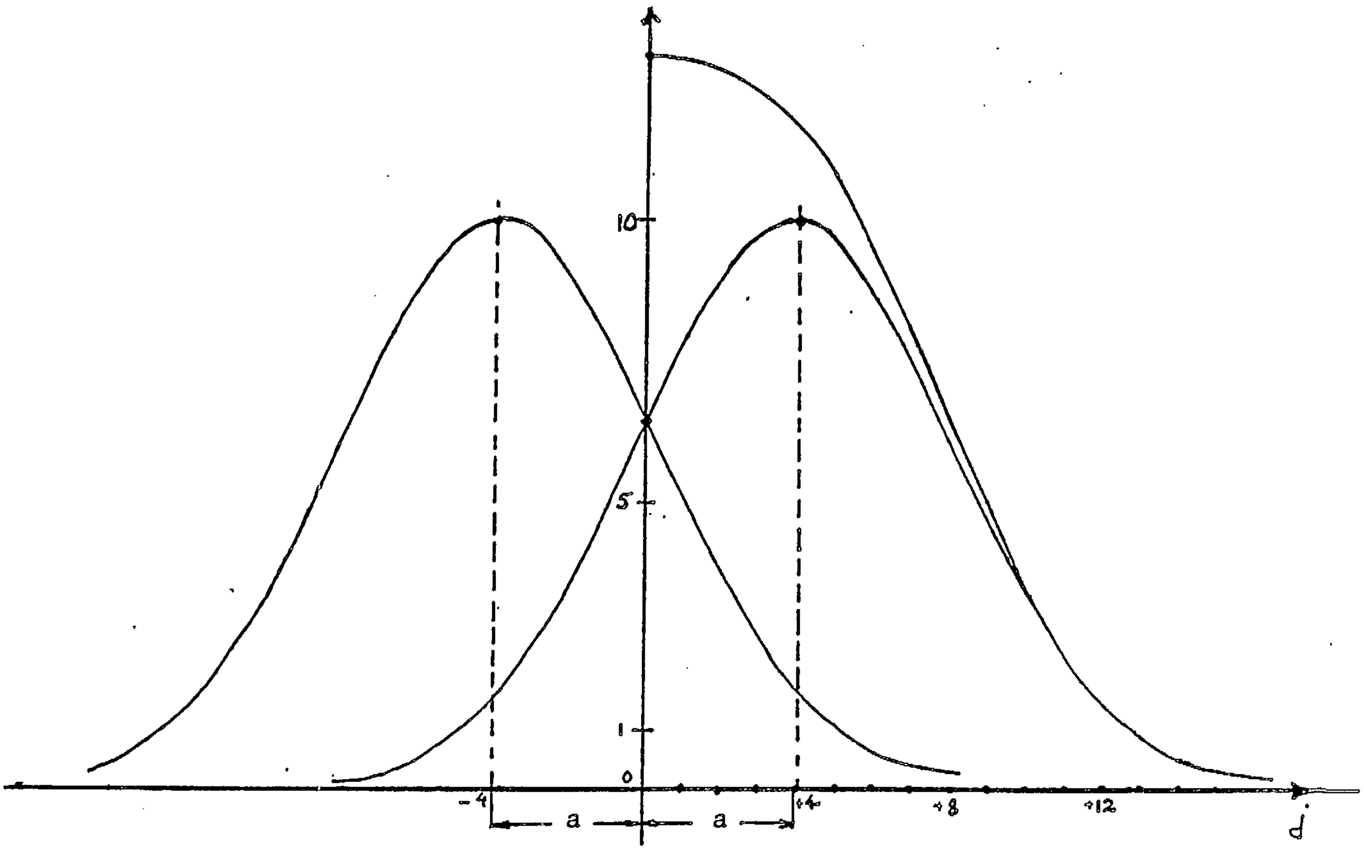

Fig. Ex. 14. Mixture of two lots of pills.

Exercise 15 - Calibration of Masses of Mercury

A mass of mercury (1 kg nominal) has been divided into four nominally equal parts. The results ofthe weighings made on a medium precision balance are

$$
\begin{array}{lllllllll}
X_1 & & & & & = m_1 & = 250.032 \text{ g} & \\
 & & X_2 & & & = m_2 & = 249.979 \text{ g} & \\
 & & & & X_3 & = m_3 & = 250.063 \text{ g} & \quad \ldots(A) \\
 & & & & X_4 & = m_4 & = 249.948 \text{ g} & \\
X_1 & + & X_2 & + & X_3 + X_4 & = m_5 & = 1{,}000.0165 \text{ g} &
\end{array}
$$

All results are considered as of the same degree of precision.

* * * * * * * * * *

The system (A) formed of five equations of condition contains four unknowns. It is treated as follows. The first equation is multiplied by 1, the following 3 equations are multiplied by 0 and the last is also multiplied by 1. The summation produces the first normal equation (in X_1):

$$2X_1 + X_2 + X_3 + X_4 = 1.250.0485 \ .$$

In the same manner are formed the other 3 normal equations so the total system takes the form

$$
\begin{aligned}
2X_1 + X_2 + X_3 + X_4 &= m_1 + m_5 \\
X_1 + 2X_2 + X_3 + X_4 &= m_2 + m_5 \qquad \ldots(B) \\
X_1 + X_2 + 2X_3 + X_4 &= m_3 + m_5 \\
X_1 + X_2 + X_3 + 2X_4 &= m_4 + m_5
\end{aligned}
$$

The determinant of the system is not equal to 0:

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} \neq 0$$

To solve the system add all equations in (B) and divide by 5. This gives

$$X_1 + X_2 + X_3 + X_4 = \frac{1}{5}\left(m_1 + m_2 + m_3 + m_4 + 4m_5\right)$$

Subtract this from each of the normal equations to obtain, for X_1 and similarly for other unknowns

$$\begin{aligned} X_1 &= \frac{1}{5}\left(4m_1 - m_2 - m_3 - m_4 + m_5\right) = 250.0309 \\ X_2 &= \frac{1}{5}\left(4m_2 - m_1 - m_3 - m_4 + m_5\right) = 249.9779 \\ X_3 &= \frac{1}{5}\left(4m_3 - m_1 - m_2 - m_4 + m_5\right) = 250.0619 \\ X_4 &= \frac{1}{5}\left(4m_4 - m_1 - m_2 - m_3 + m_5\right) = 249.9469 \end{aligned} \qquad \text{...(C)}$$

In order to apply the method of undetermined coefficients (λ, μ, ν, θ) let us write the equations of condition (A) under the form

$$\begin{array}{ccccl} (a) & (b) & (c) & (d) & \\ 1X_1 & + 0X_2 & + 0X_3 & + 0X_4 & = m_1 \\ 0X_1 & + 1X_2 & + 0X_3 & + 0X_4 & = m_2 \\ 0X_1 & + 0X_2 & + 1X_3 & + 0X_4 & = m_3 \\ 0X_1 & + 0X_2 & + 0X_3 & + 1X_4 & = m_4 \\ 0X_1 & + 1X_2 & + 1X_3 & + 1X_4 & = m_5 \end{array} \qquad \text{...(D)}$$

The double products are therefore:

$$\left\{ \begin{array}{l} (aa) = (bb) = (cc) = (dd) = 2 \\ \text{All other products, } (ab), (ca), \cdots \text{ are equal to } 1 \end{array} \right.$$

and

$$(am) = m_1 + m_5\,,$$
$$(bm) = m_2 + m_5\,,$$
$$(cm) = m_3 + m_5\,, \qquad \text{...(E)}$$
$$(dm) = m_4 + m_5\,.$$

To solve the system (B) for X_1 requires the solution of the auxiliary system

$$2\lambda + \mu + \nu + \theta = 1$$
$$\lambda + 2\mu + \nu + \theta = 0$$
$$\lambda + \mu + 2\nu + \theta = 0$$
$$\lambda + \mu + \nu + 2\theta = 0$$

The solutions are $\lambda = \frac{4}{5}$, $\mu = \nu = \theta = -\frac{1}{5}$.

In conformity with (73):

$$\overline{X}_1 = \lambda(am) + \mu(bm) + \nu(cm) + \theta(dm)\,,$$
$$= \frac{4}{5}(m_1+m_5) - \frac{1}{5}(m_2+m_5) - \frac{1}{5}(m_3+m_5) - \frac{1}{5}(m_4+m_5)\,,$$
$$\overline{X}_1 = \frac{1}{5}\left(4m_1 - m_2 - m_3 - m_4 + m_5\right)\,.$$

The expressions for other unknowns are established in the same manner. The results confirm the set (C).

To find the residuals the solutions are substituted into the equations of condition: *e.g.*

$$v_1 = X_1 - \overline{X}_1 = 250.0320 - 25.0309 = +0.0011\,.$$

All residuals are identical in the first four equations:

$$v_1 = v_2 = v_3 = v_4 = +1.1\times 10^{-3}$$

and

$$v_5 = -1.1\times 10^{-3} .$$

Hence

$$(vv) = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 = 6.05\times 10^{-6} .$$

the variance is equal to

$$\sigma_m^2 = \frac{(vv)}{6-5} = 6.05\times 10^{-6} .$$

$$\sigma_m = 2.46\times 10^{-3} .$$

The variances on all individual unknowns are identical to each other. The theorem of propagation of variance yields, *e.g.* for $\sigma_{X_1}^2$:

$$\sigma_{X_1}^2 = \left(\frac{1}{5}\right)^2 \left(4^2+1+1+1+1\right)\sigma_m^2 = \frac{4}{5}\sigma_m^2 ,$$

$$\sigma_{X_4}^2 = 4.84\times 10^{-6} ,$$

$$\sigma_{X_1} = 2.20\times 10^{-3} .$$

Exercise 16 - Calibration of Mass Standards

The most convenient system of values for establishing the submultiples and the multiples of the kilogram is

$$(5),\ (2),\ (2'),\ (1),\ (1')\ .$$

The system of equations of condition is the same for the constitution of submultiples and of multiples. Let us first establish this system and then the system of normal equations.

* * * * * * * * * *

Diagram N.

Equations of Condition

$$\begin{array}{rrrrrl}
+(5) & -(2) & -(2') & -(1) & & = m_1 \\
+(5) & -(2) & -(2') & & -(1') & = m_2 \\
 & +(2) & -(2') & +(1) & -(1') & = m_3 \\
 & +(2) & -(2') & -(1) & +(1') & = m_4 \\
 & +(2) & -(2') & & & = m_5 \\
 & +(2) & & -(1) & -(1') & = m_6 \\
 & & +(2') & -(1) & -(1') & = m_7 \\
 & & & +(1) & -(1') & = m_8\ .
\end{array} \qquad \text{...(A)}$$

Normal Equations

$$
\begin{array}{rrrrrcl}
2(5) & -2(2) & -2(2') & -(1) & -(1') & = & N_1 \\
-2(5) & +6(2) & -(2') & & & = & N_2 \\
-2(5) & -(2) & +6(2') & & & = & N_3 \\
-(5) & & & +6(1) & -(1') & = & N_4 \\
-(5) & & & -(1) & +6(1') & = & N_5 \, .
\end{array}
\qquad \ldots(B)
$$

The symbols $N_1 \cdots N_5$ are equal to the linear combinations of the observed values m_i as they are given in the following table

$$
\begin{array}{rrrrrr}
 & +m_1 & -m_1 & -m_1 & -m_1 & \\
 & +m_2 & -m_2 & -m_2 & & -m_2 \\
 & & +m_3 & -m_3 & +m_3 & -m_3 \\
 & & +m_4 & -m_4 & -m_4 & +m_4 \\
 & & +m_5 & -m_5 & & \\
 & & +m_6 & & -m_6 & -m_6 \\
 & & & +m_7 & -m_7 & -m_7 \\
 & & & & +m_8 & -m_8 \\
\hline
\text{Sums:} & N_1 & N_2 & N_3 & N_4 & N_5
\end{array}
\qquad \ldots(C)
$$

At this point, the analysis is divided into two branches: a) that of sub-multiple and b) that of multiples.

a) Calibration of submultiples

To the system (B) is joined the equation of definition

$$(5) + (2) + (2') + (1) = M \quad (\approx 10 \text{ hectograms})$$

so that (B) becomes

$$\begin{array}{rrrrrl} 2(5) & -2(2) & -2(2') & -(1) & -(1') & = N_1 \\ -2(5) & +6(2) & -(2') & & & = N_2 \\ -2(5) & -(2) & +6(2') & & & = N_3 \\ -(5) & & & +6(1) & -(1') & = N_4 \\ -(5) & & & -(1) & +6(1') & = N_5 \\ (5) & +(2) & +(2') & +(1) & & = M \end{array} \qquad \ldots(D)$$

There is an important difference between this system of normal equations and the system (64): while in the latter the determinant Δ is not equal to zero, the determinant D *is* equal to zero. This property (easy to check numerically) indicates that there is a linear relation between the values N_1, N_2, N_3, N_4 and N_5. In other words, one of the normal equations is redundant and that, therefore, one of the unknowns can be be given any arbitrary value. The relation between N's is

$$5N_1 + 2N_2 + 2N_3 + N_4 + N_5 = 0 .$$

The fact that N's are interconnected linearly does not mean that one of them must be ignored. Bu it means that the unknowns will acquire finite and well-defined values only if they are completed by one equation of definition. The algorithm (F) uses all five forms of the normal equations.

$$\begin{array}{ccccc} N_1 & N_2 & N_3 & N_4 & N_5 \\ \hline +7N_1 & +N_1 & +N_1 & +7N_1 & +7N_1 \\ & +5N_2 & & & \\ & & +5N_3 & & \\ -N_4 & -N_4 & -N_4 & +23N_4 & +3N_4 \\ +N_5 & & & +5N_5 & +25N_5 \\ \hline S_1 & S_2 & S_3 & S_4 & S_5 \\ \hline \frac{S_1}{28} & \frac{S_2}{35} & \frac{S_3}{35} & \frac{S_4}{140} & \frac{S_5}{140} \\ \frac{M}{2} & \frac{M}{5} & \frac{M}{5} & \frac{M}{10} & \frac{M}{10} \\ \hline (5) & (2) & (2') & (1) & (1') \end{array} \qquad \ldots(E)$$

Numerical Example

	N_1	N_2	N_3	N_4	N_5	
$m_1 = -1.4$ mg	-1.4	$+1.4$	$+1.4$	$+1.4$		
$m_2 = -0.6$ mg	-0.6	$+0.6$	$+0.6$		$+0.6$	
$m_3 = +4.4$ mg		$+4.4$	-4.4	$+4.4$	-4.4	
$m_4 = +2.2$ mg		$+2.2$	-2.2	-2.2	$+2.2$	...(F)
$m_5 = +3.4$ mg		$+3.4$	-3.4			
$m_6 = +3.2$ mg		$+3.2$		-3.2	-3.2	
$m_7 = 0.0$ mg			0.0	0.0	0.0	
$m_8 = +1.4$ mg				$+1.4$	-1.4	
	-2.0	$+15.2$	-8.0	$+1.8$	-6.2	

N_1	N_2	N_3	N_4	N_5
-2.0	$+15.2$	-8.0	$+1.8$	-6.2
-14.0	-2.0	-2.0	-14.0	-14.0
	$+76.0$			
		-40.0		
-1.8	-1.8	-1.8	$+41.4$	$+5.4$
-6.2			-31.0	-155.0
$S_1=-22.0$	$S_2=+72.2$	$S_3=-43.8$	$S_4=-3.6$	$S_5=-163.6$
-0.786	$+2.063$	-1.251	-0.026	-1.169
$[\mu]-3.165$	-1.266	-1.266	-0.633	-0.633
-3.951	$+0.797$	-2.517	-0.659	-1.802

The equation of definition in this table is $M = (1-6.33\times10^{-6})$ kg. The line $[\mu]$ consists of the parts of the quantity $\mu = -6.33$ mg. The final results are presented under the following form

$$(5) = (0.5 - 3.951 \times 10^{-6})\text{kg}$$
$$(2) = (0.2 + 0.797 \times 10^{-6})\text{kg}$$
$$(2') = (0.2 - 2.517 \times 10^{-6})\text{kg} \qquad ...(G)$$
$$(1) = (0.1 - 0.659 \times 10^{-6})\text{kg}$$
$$(1') = (0.1 - 1.802 \times 10^{-6})\text{kg} .$$

If these values are introduced into the equations of condition, we obtain the adjusted values m'_i, the deviations $v_i = m'_i - m_i$ and the squares v_i^2 :

$$\begin{array}{lll} m'_1 = -1.6 & v_1 = +0.2 & v_1^2 = 0.04 \\ m'_2 = -0.4 & v_2 = -0.2 & v_2^2 = 0.04 \\ m'_3 = +4.5 & v_3 = -0.1 & v_3^2 = 0.01 \\ m'_4 = +2.2 & v_4 = 0.0 & v_4^2 = 0.00 \\ m'_5 = +3.3 & v_5 = +0.1 & v_5^2 = 0.01 \\ m'_6 = +3.3 & v_6 = -0.1 & v_6^2 = 0.01 \\ m'_7 = -0.1 & v_7 = +0.1 & v_7^2 = 0.01 \\ m'_8 = +1.1 & v_8 = +0.3 & v_8^2 = 0.09 . \end{array} \qquad ...(H)$$

The value of the group variance s_m^2 is therefore

$$s_m^2 = \frac{1}{\nu}\sum_i v_i^2 = \frac{1}{4}\times 0.21 = 0.0525 . \qquad ...(I)$$

Note that here the number of masses is equal to 5 but only 4 are independent because of the equation of definition, therefore $\nu = 8-4 = 4$.

The calculation of the variances on individual weights is performed by establishing, for each weight, its algebraic expression in terms of m_i and M. For instance, from (C):

$$N_2 = - m_1 - m_2 + m_3 + m_4 + m_5 + m_6 ,$$

and, from (E):

$$S_2 = N_1 + 5N_2 - N_4 .$$

The expressions for S_1 , $\cdots$ S_5 are

...(J)

$$\begin{array}{lllllllll}
S_1 = & +8m_1 & +6m_2 & -2m_3 & +2m_4 & & & & -2m_8, \\
S_2 = & -3m_1 & -4m_2 & +4m_3 & +6m_4 & +5m_5 & +6m_6 & +m_7 & -m_8 \\
S_3 = & -3m_1 & -4m_2 & -6m_3 & -4m_4 & -5m_5 & +m_6 & +6m_7 & -m_8 \\
S_4 = & -16m_1 & +2m_2 & +18m_3 & -18m_4 & & -28m_6 & -28m_7 & +18m_8 \\
S_5 = & +4m_1 & -18m_2 & -22m_3 & +22m_4 & & -28m_6 & -28m_7 & -22m_8 .
\end{array}$$

so that, by the theorem of propagation of variance, we have, for instance for s_1^2 (variance of S_1):

$$s_1^2 = \left(8^2 + 6^2 + 2^2 + 2^2 + 2^2\right) s_m^2 = 112 s_m^2 .$$

The final table (below) contains also the variance on M. This variance, denoted by s_M^2 has been obtained by a high precision calibration performed by a national standardizing laboratory. It is $s_M^2 = 0.008(\mathrm{mg})^2$. The value for s_m^2 is given in (I): $s_m^2 = 0.052$

Table of Variances

$$s_{(5)}^2 = \left(\frac{1}{2}\right)^2 s_M^2 + \left(\frac{1}{28}\right)^2 112 s_m^2 = 0.25 s_M^2 + 0.143 s_m^2 = 0.0094 \ (\mathrm{mg})^2$$

$$s_{(2)}^2 = \left(\frac{1}{5}\right)^2 s_M^2 + \left(\frac{1}{35}\right)^2 140 s_m^2 = 0.04 s_M^2 + 0.114 s_m^2 = 0.006 \quad ''$$

$$s_{(2')}^2 = \left(\frac{1}{5}\right)^2 s_M^2 + \left(\frac{1}{35}\right)^2 140 s_m^2 = 0.04 s_M^2 + 0.114 s_m^2 = 0.006 \quad '' \qquad \ldots(K)$$

$$s_{(1)}^2 = \left(\frac{1}{10}\right)^2 s_M^2 + \left(\frac{1}{10}\right)^2 2836 s_m^2 = 0.01 s_M^2 + 0.14 s_m^2 = 0.007 \quad ''$$

$$s_{(1')}^2 = \left(\frac{1}{10}\right)^2 s_M^2 + \left(\frac{1}{10}\right)^2 3360 s_m^2 = 0.01 s_M^2 + 0.17 s_m^2 = 0.009 \quad '' .$$

b) Calibration of Multiples

The equation of definition is here

$$(1') = M \quad (= 1\text{kg} + 0.05 \text{ mg}) ,$$

$$\mu = 0.05 \text{ mg} ,$$

so that (B) becomes

$$\begin{array}{cccccll}
2(5) & -2(2) & -2(2') & -(1) & -(1') & = N_1 & \\
-2(5) & +6(2) & -(2') & & & = N_2 & \\
-2(5) & -(2) & +6(2') & & & = N_3 & \ldots(L) \\
-(5) & & & +6(1) & -(1') & = N_4 & \\
-(5) & & & -(1) & +6(1') & = N_5 & \\
 & & & & (1') & = M & .
\end{array}$$

The algorithm for the solution:

$$\begin{array}{ccccc l}
N_1 & N_2 & N_3 & N_4 & N_5 & \\
\hline
 & 12N_1 & 12N_1 & & & \\
 & 6N_2 & 5N_2 & & & \\
 & 5N_3 & 6N_3 & & & \\
-N_4 & 2N_4 & 2N_4 & +N_4 & & \\
-6N_5 & & & -N_5 & & \ldots(M) \\
\hline
S_1 & S_2 & S_3 & S_4 & & \\
\frac{S_1}{7} & \frac{S_2}{7} & \frac{S_3}{7} & \frac{S_4}{7} & & \\
5M & 2M & 2M & M & M & \\
\hline
(5) & (2) & (2') & (1) & (1') &
\end{array}$$

The numerical example treated below follows the same pattern as that for submultiples.

Solution (Unit = 1 mg)

$m_1 = -25.0$ mg	−25.0	+25.0	+25.0	+25.0		
$m_2 = -35.0$ mg	−35.0	+35.0	+35.0		+35.0	
$m_3 = -5.8$ mg		−5.8	+5.8	−5.8	+5.8	
$m_4 = -6.1$ mg		+6.1	−6.1	−6.1	+6.1	
$m_5 = -0.3$ mg		−0.3	+0.3			
$m_6 = -0.9$ mg		−0.9		+0.9	+0.9	
$m_7 = -0.8$ mg			−0.8	+0.8	+0.8	
$m_8 = -5.6$ mg				−5.6	+5.6	
	−60.0	+59.1	+59.2	+9.2	+54.2	
		−720.0	−720.0			...(N)
		+364.6	+305.0			
		+296.0	+355.2	+9.2		
	−9.2	+18.4	+18.4	−54.2		
	−325.2					
	−334.4	−51.0	−50.9	−45.0		
	−47.71	−7.29	−7.27	−5.43		
	* +0.25	+0.10	+0.10	+0.05	+0.05	
	−47.52	−7.19	−7.17	−6.38	+0.05	

* The values on this line are expressed in terms of μ with μ = 0.05 mg.

$$(5) = 5 \text{ kg.} - 47.52 \text{ mg}$$

$$(2) = 2 \text{ ''} - 17.19 \text{ mg}$$

$$(2') = 2 \text{ ''} - 7.17 \text{ mg} \qquad \text{...(O)}$$

$$(1) = 1 \text{ ''} - 6.38 \text{ mg}$$

$$(1') = 1 \text{ ''} - 0.05 \text{ mg} \quad \text{(by definition)} .$$

The substitution of these values in the equations of condition leads to the following table of residuals v_i, their squares, and the value of the group variance s_m^2:

$$\begin{aligned}
v_1 &= -25.0 + 26.8 = +1.8 & v_1^2 &= 3.24 \\
v_2 &= -35.0 + 33.2 = -1.8 & v_2^2 &= 3.24 \\
v_3 &= -5.8 + 6.4 = +0.6 & v_3^2 &= 0.36 \\
v_4 &= +6.1 - 6.4 = -0.3 & v_4^2 &= 0.09 \\
v_5 &= -0.3 + 0.0 = -0.3 & v_5^2 &= 0.09 \\
v_6 &= -0.9 + 0.9 = 0.0 & v_6^2 &= 0.00 \\
v_7 &= -0.8 + 0.8 = 0.0 & v_7^2 &= 0.00 \\
v_8 &= -5.6 + 6.4 = +0.8 & v_8^2 &= 0.64 .
\end{aligned} \qquad \text{...(P)}$$

$$(vv) = 7.66,$$

$$s_m^2 = \frac{(vv)}{8-4} = \frac{7.66}{4} = 1.92 \text{ (mg)}^2.$$

The variance on the calibration of $M = (1')$ has been found equal to

$$s_M^2 = 0.008 \text{ (mg)}^2 . \qquad \text{...(Q)}$$

Exercise 17 - Measurement of "g"

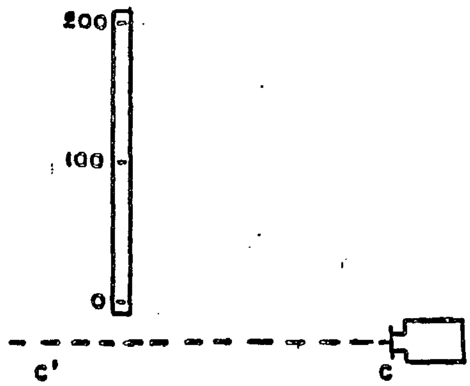

Fig. Ex. 17

A 2 meter graduated rule is held vertically above the horizontal axis CC′ of a camera C. When the rule is released, it falls vertically and as soon as the origin of the graduation O passes CC′ the rule is illuminated every tenth of a second by an electric spark. Each spark produces a photograph of a small portion of the rule's graduation on which the projection of the camera's fiducial line determines the point termed "station" which at the time of the flash was on the axis CC′. There are seven stations $s_0, s_1, \cdots s_6$ hence six time intervals $t_1, t_2, t_3, t_4, t_5, t_6$ the nominal values of which are 0.1, 0.2, ... 0.6 respectively.

The distance of points $s_1, s_2, \ldots s_6$ from the initial point s_0 are designated by the symbols $m_1, m_2, \cdots m_6$. The numerical data are presented in the following table

		(observed)
t_1 = 0.09999914 s	m_1 =	8.838 407 cm
t_2 = 0.19999922	m_2 =	27.482 803
t_3 = 0.29999895	m_3 =	55.933 421
t_4 = 0.39999971	m_4 =	94.190 082
t_5 = 0.50000007	m_5 =	142.253 322
t_6 = 0.59999922	m_6 =	200.121 939

It is important to underline that, by a fundamental assumption, all errors in the measurements of lengths will be of Hagen's type and that the intervals of time will be considered as "without error".

* * * * * * * * * * *

According to the laws of the free fall (constant acceleration) the quantities t and m are connected by the equation

$$t_i X + t_i^2 Y = m_i$$

in which X is the velocity at the time S_0 crosses CC′ and Y is equal to $\frac{g}{2}$, g being the acceleration due to gravity. Here g is expressed in $\frac{\text{cm}}{\text{s}^2}$ (gal). Treating the six equations of the system as equations of conditions we obtain the following system of normal equations in which the unknowns are X and Y:

$$X\sum_i t_i^2 + Y\sum_i t_i^3 = \sum_i m_i t_i$$
$$X\sum_i t_i^3 + Y\sum_i t_i^4 = \sum_i m_i t_i^2$$

The numerical coefficients of X and Y are:

t	t^2	t^3	t^4
0.099 999 14	0.009 999 828	0.000 999 974	0.000 099 996
0.199 999 22	0.039 999 688	0.007 999 906	0.001 599 975
0.299 998 95	0.089 999 370	0.026 999 716	0.008 099 887
0.399 999 71	0.159 999 768	0.063 999 861	0.025 599 926
0.500 000 07	0.250 000 070	0.125 000 052	0.062 500 036
0.599 999 22	0.359 999 064	0.215 999 158	0.129 599 326
Sums:	0.909 997 788	0.440 998 667	0.227 499 146

m	mt	mt^2
8.838 407	0.883 833 1	0.088 382 55
27.482 803	5.496 539 2	1.099 303 54
55.933 421	16.779 967 6	5.033 972 65
94.190 082	37.676 005 5	15.070 391 27
142.253 322	71.126 671 0	35.563 340 46
200.121 939	120.073 007 3	72.043 710 73
Sums:	252.036 023 7	128.899 101 19

The Normal equations are therefore:

$$0.909\,997\,788\, X + 0.440\,998\,667\, Y = 252.036\,023\,7\,,$$

$$0.440\,998\,667\, X + 0.227\,499\,146\, Y = 128.899\,101\,2\,.$$

The elimination of X leads to the value of Y and the substitution into both equations yields X:

$$X = 39.353\,67,$$

$$Y = 490.305\,95.$$

The value of g, *i.e.* $2Y$ is

$$g = 2Y = 980.612\ gal,\ \left(\frac{\text{cm}}{\text{s}^2}\right).$$

and the value of X indicates the velocity in $\frac{\text{cm}}{\text{s}}$ with which S_0 crosses the axis CC'.

If we substitute the values $\bar{X}$ and $\bar{Y}$ into the equations of condition we find the compensated values of the various observed lengths m_i:

39.353 671 ×	0.099 999 14	+ 490.305 95 ×	0.009 999 828 =	8.838 308 cm
"	0.199 999 22	"	0.039 999 688 =	27.482 788
"	0.299 998 95	"	0.089 999 370 =	55.933 287
"	0.399 999 71	"	0.159 999 768 =	94.190 295
"	0.500 000 07	"	0.250 000 070 =	142.253 360
"	0.599 999 22	"	0.359 999 064 =	200.121 855

The residual errors are computed by forming the differences $v_i = m_i \,(\textit{observed}) - m_i \,(\textit{computed})$, for instance

$$v_1 = 8.838\,407 - 8.838\,308 = 0.000\,099\,,$$

$$v_1 = 0.99\mu\text{m}\,.$$

To calculate the variance σ_m^2 we form the following table of residual errors and their squares:

		v_i^2
$v_1 = +0.000\,099$ cm	$v_1 = +0.99\mu$m	0.980 1 $(\mu\text{m})^2$
$v_2 = +0.000\,015$	$v_2 = +0.15$	0.022 5
$v_3 = +0.000\,134$	$v_3 = +1.34$	1.795 6
$v_4 = -0.000\,213$	$v_4 = -2.13$	4.536 9
$v_5 = -0.000\,038$	$v_5 = -0.38$	0.144 4
$v_6 = +0.000\,084$	$v_6 = +0.84$	0.705 6
		$(vv) = 8.185\,1$

The variance σ_m^2 is therefore

$$\sigma_m^2 = \frac{8.1851}{(6-2)} = 2.0463\,; \quad \sigma_m = 1.43\mu\text{m}\,.$$

As the main objective of the experiment is to determine "g" *i.e.* $2Y$, the final step is to solve the system

$$0.909\ 997\ 788\ \lambda' + 0.440\ 998\ 667\ \mu' = 0$$

$$0.440\ 998\ 667\ \lambda' + 0.227\ 499\ 146\ \mu' = 1\ .$$

We obtain first

$$\mu' = \frac{1}{0.013\ 784\ 534} = 72.545\ 071\ 1\ ,$$

and then, by the first of the above equations, the value of λ', which is found to be equal to

$$\lambda' = -\frac{72.545\ 071\ 1 \times 0.440\ 998\ 667}{0.909\ 997\ 788} = -35.156\ 436\ 7\ .$$

Hence $\sigma_y = \sigma_m \sqrt{\mu'} = 1.43\sqrt{72.545} = 12.179$ milligals.

Exercise 18. Sum of Squares of Deviations from the Mean

A set of fixed points is distributed randomly on the x-axis. Let ζ designate the abscissa of a mobile point P which may be located anywhere on this axis. Denote by y the sum of squares:

$$y = \sum_{i=1}^{n} (x_i - \zeta)^2 .$$

y can be represented by an ordinate through ζ. It is a well known fact that when

$$\zeta = \overline{x} = \frac{1}{n}\sum_i x_i$$

then the sum $y = \sum_i (x - \overline{x})^2$ is a minimum (y_m). Study the evolution of y considered as a function of the variable ζ

* * * * * * * * * * *

As a first step let us consider the following set of x_i:

$$-12 \quad -10 \quad -9 \quad -6 \quad -4 \quad -3 \quad +4 \quad +7 \quad +8 \quad +10 \quad +15, \qquad \sum x_i = 0 .$$

The fact that $\sum x_i = 0$ shows that $\overline{x} = 0$. We also have for the sum of squares:

$$y(0) = \sum_i x_i^2 = 480 .$$

Let us now make $\zeta = +1$:

$$y(+1) = \sum_i [x_i - 1]^2 = \sum x_i^2 - 2\sum x_i + 11 \times (-1)^2 .$$

But $\sum_i x_i = 0$, so that

$$y(+1) = \sum_i x_i^2 + 11 \; .$$

Similarly, for $\zeta = -1$ we have

$$y(-1) = \sum[x+1]^2 = \sum_i x_i^2 + 11\times(-1)^2 = \sum_i x_i^2 + 11 \; .$$

Hence

$$y(+1) = y(-1) \; .$$

Obviously the numeral 11 indicates the total number n of points and we can write

$$y(\zeta) = \sum_i x_i^2 + n\zeta^2 \; .$$

For instance, for $\zeta = \pm 2$

$$y(\pm 2) = \sum_i x_i^2 + 11\times 2^{\cdot} = \sum_i x_i^2 + 11\times 4 = \sum x^2 + 44 \; .$$

The curve $y(\zeta)$ is a parabola, the axis of which is vertical and passes through $\overline{x}$.

Exercise 19. Small Samples

Samples of three elements m_1, m_2, m_3 are drawn from the population of a normal variate M, (variance σ^2, mean a). Give the characteristics of the variate S the element s of which is equal to

$$s = m_1 + m_2 + m_3 .$$

Then, calculate

1) the probability that s will be inside the limits $a-\sigma$ and $a+\sigma$.

2) the probability that the mean of the sample will be inside the same limits.

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According to Chapter V, Section 1 (87), the variable s is normally distributed about the mean $A = 3a$ with a variance σ_S^2 equal to

$$\sigma_S^2 = 3\sigma^2 \qquad i.e. \quad \sigma = \frac{\sigma_S}{\sqrt{3}} .$$

Hence,

$$dP_S = \frac{1}{\sigma_S\sqrt{2\pi}} e^{-\frac{(s-A)^2}{2\sigma_S^2}} ds .$$

The calculations will be simplified by the change of variable

$$x = s - A$$

which leads to

$$dP_x = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}} dx$$

with $\sigma_x^2 = \sigma_S^2$. As now the centre is at $x = 0$, the above mentioned condition (*i.e.* that

x be between $-\sigma$ and $+\sigma$) requires the evaluation of the integral

$$P_{-\sigma}^{+\sigma} = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{+\sigma} e^{-\frac{x^2}{2\sigma_x^2}} \frac{dx}{\sigma_x} .$$

The limits of integration, by the change of variable $\frac{x}{\sigma_x} = \lambda$, become

$$x_1 = -\sigma , \quad \lambda_1 = -\frac{\sigma}{\sigma_x} = -\frac{\sigma}{\sqrt{3}\sigma} = -\frac{1}{\sqrt{3}} ,$$

$$x_2 = +\sigma , \quad \lambda_2 = +\frac{\sigma}{\sigma_x} = +\frac{\sigma}{\sqrt{3}\sigma} = +\frac{1}{\sqrt{3}} ,$$

Hence,

$$P_{\lambda_1}^{\lambda_2} = \frac{2}{\sqrt{2\pi}} \int_{0}^{+\frac{1}{\sqrt{3}}} e^{-\frac{\lambda^2}{2}} d\lambda .$$

As $\frac{1}{\sqrt{3}} = 0.580$, the area under the normal curve is equal to

$$P_{\lambda_1}^{\lambda_2} = \frac{1}{\sqrt{2\pi}} \int_{0}^{0.580} e^{-\frac{\lambda^2}{2}} d\lambda = 0.219$$

Therefore the probability that s will be between $-\sigma$ and $+\sigma$ is equal to $2\times0.219 = 0.438$, *i.e.* 43.8 percent.

Now, instead of considering the variate sum, $s = m_1 + m_2 + m_3$, we shall consider the variate "mean" s' *i.e.*

$$s' = \frac{m_1 + m_2 + m_3}{3} = \frac{s}{3} .$$

This is equivalent to forming the linear equation

$$s' = (\frac{1}{3})m_1 + (\frac{1}{3})m_2 + (\frac{1}{3})m_3 .$$

According to the formulae given at the end of Chapter V, Section 1 we have now:

$$A' = \frac{A}{3} = (\frac{1}{3})a + (\frac{1}{3})a + (\frac{1}{3})a = a \; ,$$

$$\sigma_{s'}^2 = (\frac{1}{3})^2\sigma^2 + (\frac{1}{3})^2\sigma^2 + (\frac{1}{3})^2\sigma^2 = \frac{3\sigma^2}{9} = \frac{\sigma^2}{3} \; ,$$

$$\sigma_{s'} = \frac{\sigma}{\sqrt{3}} \; .$$

The expression for $dP_{s'}$ takes the form

$$dP_{s'} = \frac{1}{\sigma_{s'}\sqrt{2\pi}} \, e^{-\frac{(s'-a)^2}{2\sigma_{s'}^2}} \, ds'$$

which, by the substitution $s' - a = x$, becomes

$$dP_x = \frac{1}{\sigma_x\sqrt{2\pi}} \, e^{-\frac{x^2}{2\sigma_x^2}} \, dx \; .$$

The usual change of variables $\frac{x}{\sigma_x} = \lambda$ leads to the form dP_x which is tabulated. The limits of the integration of dP_x *i.e.* $-\sigma$ to $+\sigma$, are calculated as follows.

Lower Limit: $x_1 = -\sigma$, hence $\lambda_1 = \frac{x_1}{\sigma_{s'}} = \frac{-\sigma}{\sigma_{s'}}$. But $\sigma_{s'} = \frac{\sigma}{\sqrt{3}}$, so that

$$\lambda_1 = \frac{-\sigma}{(\frac{\sigma}{\sqrt{3}})} = -\sqrt{3} \; .$$

Upper Limit: $x_1 = +\sigma$, $\lambda_2 = +\sqrt{3}$. The expression for the total probability $P_{-\sigma}^{+\sigma}$ is:

$$P_{-\sigma}^{+\sigma} = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{3}}^{+\sqrt{3}} e^{-\frac{\lambda^2}{2}} \, d\lambda = \frac{2}{\sqrt{2\pi}} \int_0^{1.732} e^{-\frac{\lambda^2}{2}} \, d\lambda \; .$$

In the table of areas we find that

$$\frac{1}{\sqrt{2\pi}} \int_0^{1.732} e^{-\frac{\lambda^2}{2}} \, d\lambda = 0.4584 \longrightarrow P_{-\sigma}^{+\sigma} = 0.9168 \; .$$

Exercise 20. Variate X ($x_1 = 1$, $x_2 = 0$).

A variate X can take two values: $x_1 = 1$, with constant probability p, and $x_2 = 0$, probability $q = 1-p$. Calculate

1) the expectation of x and

2) the expectation of x^2.

Consider then two such variables X and Y with p_x and p_y, respectively. Give the expressions for the combinations: $X + Y$, XY, $X^2 + Y^2$, $(X + Y)^2$

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1a) $E(x) = p_x x_1 + q_x x_2 = p_x \times 1 + q_x \times 0 = p_x$,

1b) $E(x^2) = p_x x_1^2 + q_x x_2^2 = p_x \times 1^2 + q_x \times 0^2 = p_x$.

Combinations

2a) $E(X+Y) = E(x) + E(y) = p_x + p_y$

2b) $E(XY) = E(x)E(y) = p_x p_y$

2c) $E(X^2+Y^2) = E(x^2) + E(y^2) = p_x + p_y$

2d) $E\left[(X+Y)^2\right] = E(x^2) + E(y^2) + 2E(XY)$

$$= p_x + p_y + 2p_x p_y .$$

Exercise 21. Multiplication of a Variate by a Constant

A normal variate X is centered on a. Establish the expression dP_z for the variate Z that is defined by the product $Z = \alpha X$, α being a constant.

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The probability dP_z that z will fall into dz is deduced from dP_x through the relations $z = \alpha x$, $dz = \alpha dx$, $\sigma^2 = \alpha^2 \sigma^2$ which are first written as follows:

$$x = \frac{z}{\alpha}, \quad dx = \frac{dz}{\alpha}, \quad \sigma_x^2 = \frac{\sigma_z^2}{\alpha^2}, \quad \sigma_x = \frac{\sigma_z}{\alpha}.$$

When these expressions are introduced into dP_x we obtain

$$dP_z = \frac{1}{\left(\frac{\sigma_z}{\alpha}\right)\sqrt{2\pi}} e^{-\frac{\left(\frac{z}{\alpha} - a\right)^2}{2\left(\frac{\sigma_z}{\alpha}\right)^2}} \cdot \frac{dz}{\alpha} .$$

After all simplifications have been performed, the expression for P_z becomes

$$dP_z = \frac{1}{\sigma_z\sqrt{2\pi}} e^{-\frac{(z-\alpha a)^2}{2\sigma_z^2}} \cdot dz .$$

This shows that z is centered on $c = \alpha a$.

Exercise 22. Numerical Examples and Basic Values of the Gamma Function

A. Establish the expression for the probability density dP_u of the Gamma variate which is derived from a normal variate $\Phi(x)$ with a variance $\sigma^2 = 25$. Calculate the numerical value of dP_u for $x = 5$.

B. Using the expressions for J_0, J_1, J_2, J_3 (as in Chapter II, Table II), transform these integrals into Gamma integrals and deduce their numerical values.

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A. The formula for u and dP_u are

$$u = \frac{x^2}{2\sigma^2} \quad \text{and} \quad dP_u = \frac{e^{-u}u^{\frac{1}{2}-1}}{\sqrt{\pi}}du .$$

Hence for $\sigma^2 = 25$ and $x = 5$,

$$u = 0.5 \quad \text{and} \quad dP_u = \frac{e^{-\frac{1}{2}}(0.5)^{-\frac{1}{2}}}{\sqrt{\pi}}du ,$$

and

$$dP_u = \frac{du}{\sqrt{2.71828}\sqrt{0.5}\sqrt{3.14159}} = 0.484 .$$

B. The recurrence relation (42) can be put under the form

$$J_{n+2} = \frac{n+1}{2}J_n ,$$

so that, by (146):

$$\begin{cases} J_n = \dfrac{1}{2}\Gamma\left(\dfrac{n+1}{2}\right). \\ J_{n+2} = \dfrac{1}{2}\Gamma\left(\dfrac{n+2+1}{2}\right) = \dfrac{1}{2}\Gamma\left(\dfrac{n+1}{2}+1\right). \end{cases}$$

Hence

$$\frac{1}{2}\Gamma\left(\frac{n+1}{2}+1\right) = \frac{n+1}{2}\cdot\frac{1}{2}\Gamma\left(\frac{n+1}{2}\right).$$

Let us now introduce the symbol α for $\frac{n+1}{2}$:

$$\frac{n+1}{2} = \alpha .$$

The relation above takes the form

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) ,$$

which is identical to (99).

Exercise 23. Examples of Chi-square Tests

Apply the χ^2- test to the hypothesis that the samples of errors described in Exercises 10 and 11 can be considered as drawn from normally distributed populations.

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I. Exercise 16: Screws Machine Output

Four extreme classes on the negative wing ($F = 0+0+1+3 = 4$), and three extreme classes on the positive wing ($F = 1+0+1 = 2$) are grouped in one single class. The calculation of the Chi square is performed as follows:

j	F	f	$\lvert F-f \rvert$	$\Delta\chi^2$
−5	4	3.42	0.58	0.10
...	...			
−4	4	3.95	0.05	0.00
−3	8	6.71	1.29	0.25
−2	9	9.94	0.94	0.09
−1	12	12.86	0.86	0.06
0	10	14.53	4.53	1.41
+1	18	14.34	3.66	0.93
+2	14	12.35	1.65	0.22
+3	9	9.29	0.29	0.00
+4	6	6.10	0.10	0.00
+5	4	3.50	0.50	0.07
...	...			
+6	2	2.83	0.83	0.24
				$\chi^2 = 3.37$

For $\nu = 11$, this is an extremely small χ^2; the hypothesis is very likely to be correct. The probability P of exceeding 3.37 in repeated sampling is very large: $P > 95$

percent.

II. Exercise 11: Gravimetric Observations

The calculation of χ^2 from the data (F_j and f_j) displayed in Exercise 11 follows the same pattern as that of Exercise 10. However it is necessary, in order to avoid the presence on both wings of thinly populated classes, to perform the following groupings:

j	F_j	f_j
−12	2	0.07
−11	3	0.26
−10	4	0.84
−9	2	2.45
	11	3.62

j	F_j	f_j
+9	7	4.07
+10	4	1.49
+11	1	0.49
+12	2	0.14
	14	6.19

Thus the number of classes is reduced from 25 to 19 ($\nu = 18$). The result is

$$\chi^2 = 54.$$

This is a very high value for χ^2 and P is very small. This totally rejects the hypothesis that the sample is drawn from a normal population: the probability P of exceeding 54 in repeated sampling is much smaller than 1 percent.

Exercise 24. Another derivation of the Recurrence Relation for ν_n Moments.

The recurrence relation for ν_n moments has been established using the expressions in which ν_n is formulated in terms of the integrals J_n (Chapter II, Section 3). Establish now this relation using the expressions based on Gamma functions

* * * * * * * * * *

In the formula

$$\nu_n = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \sigma^n \Gamma\left(\frac{n}{2} + \frac{1}{2}\right)$$

we replace n by $(n+2)$

$$\nu_{n+2} = 2^{\frac{n+2}{2}} \sigma^{n+2} \Gamma\left(\frac{n+2}{2} + \frac{1}{2}\right)$$

and form the ratio

$$\frac{\nu_{n+2}}{\nu_n} = 2\sigma^n \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2} + 1\right)}{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)} = 2\sigma^2 \left(\frac{n+1}{2}\right) = (n+1)\sigma^2.$$

The recurrence formula is :

$$\nu_{n+2} = \nu_n (n+1)\sigma^2 .$$

It can, of course, take the form (45):

$$\nu_n = \nu_{n-2}(n-1)\sigma^2 ,$$

Bibliography

The number of books on statistics published in the last two or three decades is surprisingly large. This can be partially explained by the fact that this branch of mathematics has become an integral part of many scientific domains (e.g. biology, psychology, chemistry, etc.) and is also extensively used in various activities of technical and economic nature. One consequence of this is the proliferation of courses in statistics (on various levels) in the schools and universities.

It is also surprising that among all these books only a very few devote a significant part of the text to the problems found in metrology and this in spite of the fact that the importance of the latter is constantly growing.

Some of the most classical of the metrologically oriented books are given below and can be used by all those who have a reasonable training in the calculus. It is obvious that the study of any classical book on higher statistics will be extremely beneficial to a metrologist. However, those quoted here, contain examples which are more directly helpful to those who start worrying about the "errors" in the "observations" they perform.

List of Books

[1] M.W. Smart, *Combination of observations,* Cambridge Univ. Press.

[2] E. Borel, R. Deltheil, R. Huron, *Probabilité Erreurs,* Armand Colin.

[3] M. Romanowski, *Theory of random errors and the influence of modulation on their distribution,* Konrad Wittwer Verlag.

[4] C.F. Dietrich, *Uncertainty, calibration and probability: the statistics of scientific and industrial measurement,* Wiley, New York.

[5] C.E. Weatherburn, *A first course in mathematical statistics,* Cambridge University Press.

Tables of modulated functions

I Tables for ordinates

These tables are analogous to the table of $\Phi(v)$, page 16. The expressions for $\phi(a, \tau, \lambda)$ are obtained by attributing to the modulator the numerical values $a = 0, 0.25, 0.50, 1.00$.

The tables have been calculated, first in 1965, by F. Farrell (Computation Centre, National Research Council) using numerical integration methods. The calculation has been repeated and completed later by J. Halpenny (Earth Physics Branch) using the expressions quoted in [3].

II Tables for areas

The function $\Phi(\lambda)$ in these tables is defined by the integral

$$\Phi(\lambda) = \int_0^\lambda \phi(\lambda) d\lambda ,$$

$\phi(\lambda)$ being the expression (VII.9), p. 121. It represents therefore the so-called "area under the curve" $\phi(a,\lambda)$ contained between the ordinates at $\lambda = 0$ and $\lambda = \lambda$.

These tables are particularly necessary in those cases where the interval of classification $\Delta\lambda$ is not constant.

Articles on Orthogonal Systems

M. GRABE, "Note on the Application of the Method of Least Squares", Metrologia 14; p. 143-6. (1978)

M. ZUKER, and al. "Systematic Search for Orthogonal Systems in the Calibration of Submultiples and Multiples of the Unit of Mass", Metrologia 16, p. 51-54. (1980)

M. ROMANOWSKI and G. MIHAILOV, "New Developments in the Metrology of Mass Standards", Legal Metrology Branch, Holland Avenue, Ottawa, Ontario, Canada K1A 0C9

G.D. CHAPMAN, "Calibration of Kilogram Submultiples", NRCC 25819 (1987) National Research Council, Division of Physics, Ottawa, Ontario, Canada K1A 0R6

ORDINATE FOR a=0.00

λ	0.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5642	.5592	.5542	.5493	.5444	.5395	.5347	.5299	.5251	.5203
.1	.5156	.5109	.5062	.5016	.4970	.4924	.4878	.4833	.4788	.4743
.2	.4698	.4654	.4610	.4566	.4523	.4480	.4437	.4394	.4352	.4310
.3	.4268	.4227	.4186	.4145	.4104	.4064	.4024	.3984	.3944	.3905
.4	.3866	.3827	.3789	.3751	.3713	.3675	.3638	.3601	.3564	.3527
.5	.3491	.3455	.3419	.3384	.3348	.3313	.3279	.3244	.3210	.3176
.6	.3142	.3109	.3076	.3043	.3010	.2978	.2945	.2913	.2882	.2850
.7	.2819	.2788	.2758	.2727	.2697	.2667	.2638	.2608	.2579	.2550
.8	.2521	.2493	.2465	.2437	.2409	.2381	.2354	.2327	.2300	.2274
.9	.2247	.2221	.2195	.2170	.2144	.2119	.2094	.2069	.2045	.2021
1.0	.1996	.1973	.1949	.1925	.1902	.1879	.1856	.1834	.1812	.1789
1.1	.1767	.1746	.1724	.1703	.1682	.1661	.1640	.1620	.1599	.1579
1.2	.1559	.1540	.1520	.1501	.1482	.1463	.1444	.1425	.1407	.1389
1.3	.1371	.1353	.1336	.1318	.1301	.1284	.1267	.1250	.1234	.1217
1.4	.1201	.1185	.1169	.1153	.1138	.1123	.1107	.1092	.1078	.1063
1.5	.1048	.1034	.1020	.1006	.0992	.0978	.0965	.0951	.0938	.0925
1.6	.0912	.0899	.0886	.0874	.0861	.0849	.0837	.0825	.0813	.0802
1.7	.0790	.0779	.0767	.0756	.0745	.0734	.0724	.0713	.0703	.0692
1.8	.0682	.0672	.0662	.0652	.0642	.0633	.0623	.0614	.0605	.0596
1.9	.0587	.0578	.0569	.0560	.0552	.0543	.0535	.0527	.0518	.0510
2.0	.0503	.0495	.0487	.0479	.0472	.0464	.0457	.0450	.0443	.0436
2.1	.0429	.0422	.0415	.0409	.0402	.0396	.0389	.0383	.0377	.0371
2.2	.0365	.0359	.0353	.0347	.0341	.0336	.0330	.0325	.0319	.0314
2.3	.0309	.0304	.0299	.0294	.0289	.0284	.0279	.0274	.0270	.0265
2.4	.0260	.0256	.0252	.0247	.0243	.0239	.0235	.0231	.0227	.0223
2.5	.0219	.0215	.0211	.0208	.0204	.0200	.0197	.0193	.0190	.0186
2.6	.0183	.0180	.0177	.0173	.0170	.0167	.0164	.0161	.0158	.0155
2.7	.0153	.0150	.0147	.0144	.0142	.0139	.0137	.0134	.0132	.0129
2.8	.0127	.0124	.0122	.0120	.0117	.0115	.0113	.0111	.0109	.0107
2.9	.0105	.0103	.0101	.0099	.0097	.0095	.0093	.0091	.0090	.0088
3.0	.0086	.0085	.0083	.0081	.0080	.0078	.0077	.0075	.0074	.0072
3.1	.0071	.0069	.0068	.0067	.0065	.0064	.0063	.0061	.0060	.0059
3.2	.0058	.0057	.0055	.0054	.0053	.0052	.0051	.0050	.0049	.0048
3.3	.0047	.0046	.0045	.0044	.0043	.0042	.0041	.0040	.0040	.0039
3.4	.0038	.0037	.0036	.0036	.0035	.0034	.0033	.0033	.0032	.0031
3.5	.0031	.0030	.0029	.0029	.0028	.0027	.0027	.0026	.0026	.0025
3.6	.0025	.0024	.0024	.0023	.0022	.0022	.0022	.0021	.0021	.0020
3.7	.0020	.0019	.0019	.0018	.0018	.0018	.0017	.0017	.0016	.0016
3.8	.0016	.0015	.0015	.0015	.0014	.0014	.0014	.0013	.0013	.0013
3.9	.0012	.0012	.0012	.0012	.0011	.0011	.0011	.0011	.0010	.0010
4.0	.0010	.0010	.0009	.0009	.0009	.0009	.0008	.0008	.0008	.0008
4.1	.0008	.0007	.0007	.0007	.0007	.0007	.0007	.0006	.0006	.0006
4.2	.0006	.0006	.0006	.0006	.0005	.0005	.0005	.0005	.0005	.0005
4.3	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004
4.4	.0004	.0004	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003

ORDINATE FOR a=0.25

λ	0.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.4956	.4949	.4938	.4924	.4908	.4889	.4870	.4849	.4827	.4804
.1	.4780	.4755	.4730	.4703	.4677	.4649	.4622	.4593	.4565	.4535
.2	.4506	.4476	.4446	.4416	.4385	.4354	.4323	.4292	.4260	.4229
.3	.4197	.4165	.4133	.4101	.4069	.4037	.4004	.3972	.3939	.3907
.4	.3875	.3842	.3810	.3777	.3745	.3712	.3680	.3647	.3615	.3583
.5	.3551	.3518	.3486	.3454	.3422	.3391	.3359	.3327	.3296	.3264
.6	.3233	.3202	.3170	.3139	.3109	.3078	.3047	.3017	.2986	.2956
.7	.2926	.2896	.2867	.2837	.2808	.2778	.2749	.2720	.2691	.2663
.8	.2634	.2606	.2578	.2550	.2522	.2494	.2467	.2440	.2412	.2386
.9	.2359	.2332	.2306	.2280	.2254	.2228	.2202	.2177	.2152	.2127
1.0	.2102	.2077	.2052	.2028	.2004	.1980	.1956	.1933	.1909	.1886
1.1	.1863	.1841	.1818	.1796	.1773	.1751	.1729	.1708	.1686	.1665
1.2	.1644	.1623	.1602	.1582	.1562	.1542	.1522	.1502	.1482	.1463
1.3	.1444	.1425	.1406	.1387	.1369	.1350	.1332	.1314	.1297	.1279
1.4	.1262	.1245	.1228	.1211	.1194	.1178	.1161	.1145	.1129	.1113
1.5	.1098	.1082	.1067	.1052	.1037	.1022	.1007	.0993	.0979	.0965
1.6	.0951	.0937	.0923	.0910	.0896	.0883	.0870	.0857	.0844	.0832
1.7	.0819	.0807	.0795	.0783	.0771	.0759	.0748	.0736	.0725	.0714
1.8	.0703	.0692	.0681	.0671	.0660	.0650	.0640	.0630	.0620	.0610
1.9	.0600	.0591	.0581	.0572	.0563	.0554	.0545	.0536	.0527	.0519
2.0	.0510	.0502	.0494	.0485	.0477	.0470	.0462	.0454	.0446	.0439
2.1	.0432	.0424	.0417	.0410	.0403	.0396	.0390	.0383	.0376	.0370
2.2	.0364	.0357	.0351	.0345	.0339	.0333	.0327	.0321	.0316	.0310
2.3	.0305	.0299	.0294	.0289	.0284	.0278	.0273	.0269	.0264	.0259
2.4	.0254	.0250	.0245	.0241	.0236	.0232	.0228	.0223	.0219	.0215
2.5	.0211	.0207	.0203	.0199	.0196	.0192	.0188	.0185	.0181	.0178
2.6	.0174	.0171	.0168	.0165	.0161	.0158	.0155	.0152	.0149	.0146
2.7	.0144	.0141	.0138	.0135	.0133	.0130	.0127	.0125	.0122	.0120
2.8	.0118	.0115	.0113	.0111	.0108	.0106	.0104	.0102	.0100	.0098
2.9	.0096	.0094	.0092	.0090	.0088	.0086	.0084	.0083	.0081	.0079
3.0	.0078	.0076	.0074	.0073	.0071	.0070	.0068	.0067	.0065	.0064
3.1	.0063	.0061	.0060	.0059	.0057	.0056	.0055	.0054	.0053	.0051
3.2	.0050	.0049	.0048	.0047	.0046	.0045	.0044	.0043	.0042	.0041
3.3	.0040	.0039	.0038	.0038	.0037	.0036	.0035	.0034	.0034	.0033
3.4	.0032	.0031	.0031	.0030	.0029	.0028	.0028	.0027	.0027	.0026
3.5	.0025	.0025	.0024	.0024	.0023	.0023	.0022	.0021	.0021	.0020
3.6	.0020	.0019	.0019	.0019	.0018	.0018	.0017	.0017	.0016	.0016
3.7	.0016	.0015	.0015	.0015	.0014	.0014	.0013	.0013	.0013	.0013
3.8	.0012	.0012	.0012	.0011	.0011	.0011	.0010	.0010	.0010	.0010
3.9	.0009	.0009	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0008
4.0	.0007	.0007	.0007	.0007	.0007	.0006	.0006	.0006	.0006	.0006
4.1	.0006	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0004
4.2	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0003	.0003
4.3	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003
4.4	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002

ORDINATE FOR α=0.50

λ	0.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.4635	.4634	.4630	.4624	.4617	.4609	.4599	.4588	.4576	.4563
.1	.4549	.4534	.4518	.4501	.4484	.4466	.4447	.4427	.4407	.4387
.2	.4365	.4344	.4322	.4299	.4276	.4252	.4228	.4204	.4179	.4154
.3	.4129	.4103	.4077	.4051	.4024	.3997	.3970	.3943	.3916	.3888
.4	.3860	.3832	.3804	.3776	.3748	.3719	.3690	.3662	.3633	.3604
.5	.3575	.3546	.3517	.3488	.3458	.3429	.3400	.3371	.3341	.3312
.6	.3283	.3254	.3224	.3195	.3166	.3137	.3108	.3079	.3050	.3021
.7	.2992	.2963	.2934	.2906	.2877	.2849	.2820	.2792	.2764	.2736
.8	.2708	.2680	.2652	.2624	.2597	.2569	.2542	.2515	.2488	.2461
.9	.2434	.2407	.2381	.2355	.2328	.2302	.2277	.2251	.2225	.2200
1.0	.2174	.2149	.2124	.2100	.2075	.2051	.2026	.2002	.1978	.1954
1.1	.1931	.1907	.1884	.1861	.1838	.1815	.1793	.1771	.1748	.1726
1.2	.1704	.1683	.1661	.1640	.1619	.1598	.1577	.1557	.1536	.1516
1.3	.1496	.1476	.1457	.1437	.1418	.1399	.1380	.1361	.1342	.1324
1.4	.1306	.1288	.1270	.1252	.1235	.1217	.1200	.1183	.1166	.1150
1.5	.1133	.1117	.1101	.1085	.1069	.1054	.1038	.1023	.1008	.0993
1.6	.0978	.0964	.0949	.0935	.0921	.0907	.0894	.0880	.0866	.0853
1.7	.0840	.0827	.0814	.0802	.0789	.0777	.0765	.0753	.0741	.0729
1.8	.0717	.0706	.0695	.0684	.0673	.0662	.0651	.0640	.0630	.0620
1.9	.0609	.0599	.0590	.0580	.0570	.0561	.0551	.0542	.0533	.0524
2.0	.0515	.0506	.0498	.0489	.0481	.0472	.0464	.0456	.0448	.0441
2.1	.0433	.0425	.0418	.0410	.0403	.0396	.0389	.0382	.0375	.0369
2.2	.0362	.0355	.0349	.0343	.0336	.0330	.0324	.0318	.0312	.0307
2.3	.0301	.0295	.0290	.0285	.0279	.0274	.0269	.0264	.0259	.0254
2.4	.0249	.0244	.0240	.0235	.0230	.0226	.0222	.0217	.0213	.0209
2.5	.0205	.0201	.0197	.0193	.0189	.0186	.0182	.0178	.0175	.0171
2.6	.0168	.0164	.0161	.0158	.0155	.0151	.0148	.0145	.0142	.0139
2.7	.0137	.0134	.0131	.0128	.0126	.0123	.0120	.0118	.0115	.0113
2.8	.0111	.0108	.0106	.0104	.0102	.0099	.0097	.0095	.0093	.0091
2.9	.0089	.0087	.0085	.0083	.0082	.0080	.0078	.0076	.0075	.0073
3.0	.0071	.0070	.0068	.0067	.0065	.0064	.0062	.0061	.0060	.0058
3.1	.0057	.0056	.0054	.0053	.0052	.0051	.0050	.0048	.0047	.0046
3.2	.0045	.0044	.0043	.0042	.0041	.0040	.0039	.0038	.0037	.0036
3.3	.0036	.0035	.0034	.0033	.0032	.0032	.0031	.0030	.0029	.0029
3.4	.0028	.0027	.0027	.0026	.0025	.0025	.0024	.0024	.0023	.0022
3.5	.0022	.0021	.0021	.0020	.0020	.0019	.0019	.0018	.0018	.0017
3.6	.0017	.0016	.0016	.0016	.0015	.0015	.0015	.0014	.0014	.0013
3.7	.0013	.0013	.0012	.0012	.0012	.0011	.0011	.0011	.0011	.0010
3.8	.0010	.0010	.0010	.0009	.0009	.0009	.0009	.0008	.0008	.0008
3.9	.0008	.0007	.0007	.0007	.0007	.0007	.0007	.0006	.0006	.0006
4.0	.0006	.0006	.0006	.0005	.0005	.0005	.0005	.0005	.0005	.0005
4.1	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0004	.0003	.0003
4.2	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003
4.3	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
4.4	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0001	.0001	.0001

ORDINATE FOR a= 1.00

λ	0.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.4343	.4343	.4341	.4339	.4336	.4333	.4328	.4323	.4317	.4310
.1	.4303	.4294	.4286	.4276	.4266	.4255	.4244	.4232	.4219	.4206
.2	.4192	.4178	.4163	.4147	.4132	.4115	.4098	.4081	.4063	.4045
.3	.4026	.4007	.3988	.3968	.3948	.3927	.3906	.3885	.3864	.3842
.4	.3819	.3797	.3774	.3751	.3728	.3704	.3680	.3656	.3632	.3608
.5	.3583	.3558	.3533	.3508	.3482	.3457	.3431	.3405	.3379	.3353
.6	.3327	.3301	.3274	.3248	.3221	.3195	.3168	.3141	.3114	.3087
.7	.3061	.3034	.3007	.2980	.2953	.2926	.2899	.2872	.2845	.2818
.8	.2791	.2764	.2737	.2710	.2683	.2657	.2630	.2603	.2577	.2550
.9	.2524	.2498	.2471	.2445	.2419	.2393	.2367	.2341	.2316	.2290
1.0	.2265	.2239	.2214	.2189	.2164	.2139	.2114	.2090	.2065	.2041
1.1	.2017	.1993	.1969	.1945	.1921	.1898	.1874	.1851	.1828	.1805
1.2	.1783	.1760	.1738	.1716	.1694	.1672	.1650	.1628	.1607	.1586
1.3	.1565	.1544	.1523	.1503	.1482	.1462	.1442	.1422	.1403	.1383
1.4	.1364	.1345	.1326	.1307	.1289	.1270	.1252	.1234	.1216	.1198
1.5	.1181	.1164	.1146	.1129	.1113	.1096	.1080	.1063	.1047	.1031
1.6	.1016	.1000	.0985	.0970	.0954	.0940	.0925	.0910	.0896	.0882
1.7	.0868	.0854	.0840	.0827	.0813	.0800	.0787	.0774	.0762	.0749
1.8	.0737	.0725	.0712	.0701	.0689	.0677	.0666	.0654	.0643	.0632
1.9	.0621	.0611	.0600	.0590	.0580	.0569	.0559	.0550	.0540	.0530
2.0	.0521	.0512	.0502	.0493	.0484	.0476	.0467	.0459	.0450	.0442
2.1	.0434	.0426	.0418	.0410	.0403	.0395	.0388	.0380	.0373	.0366
2.2	.0359	.0352	.0346	.0339	.0332	.0326	.0320	.0313	.0307	.0301
2.3	.0295	.0290	.0284	.0278	.0273	.0267	.0262	.0257	.0252	.0246
2.4	.0241	.0237	.0232	.0227	.0222	.0218	.0213	.0209	.0205	.0200
2.5	.0196	.0192	.0188	.0184	.0180	.0176	.0173	.0169	.0165	.0162
2.6	.0158	.0155	.0152	.0148	.0145	.0142	.0139	.0136	.0133	.0130
2.7	.0127	.0124	.0122	.0119	.0116	.0114	.0111	.0109	.0106	.0104
2.8	.0101	.0099	.0097	.0095	.0092	.0090	.0088	.0086	.0084	.0082
2.9	.0080	.0078	.0077	.0075	.0073	.0071	.0070	.0068	.0066	.0065
3.0	.0063	.0062	.0060	.0059	.0057	.0056	.0055	.0053	.0052	.0051
3.1	.0050	.0048	.0047	.0046	.0045	.0044	.0043	.0042	.0041	.0040
3.2	.0039	.0038	.0037	.0036	.0035	.0034	.0033	.0032	.0031	.0031
3.3	.0030	.0029	.0028	.0028	.0027	.0026	.0026	.0025	.0024	.0024
3.4	.0023	.0022	.0022	.0021	.0021	.0020	.0020	.0019	.0019	.0018
3.5	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014	.0014
3.6	.0013	.0013	.0013	.0012	.0012	.0012	.0011	.0011	.0011	.0010
3.7	.0010	.0010	.0009	.0009	.0009	.0009	.0008	.0008	.0008	.0008
3.8	.0008	.0007	.0007	.0007	.0007	.0007	.0006	.0006	.0006	.0006
3.9	.0006	.0005	.0005	.0005	.0005	.0005	.0005	.0005	.0004	.0004
4.0	.0004	.0004	.0004	.0004	.0004	.0004	.0003	.0003	.0003	.0003
4.1	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002	.0002	.0002
4.2	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
4.3	.0002	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001
4.4	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001

AREA FOR $\alpha = 0.00$

λ	0.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0056	.0112	.0167	.0222	.0276	.0330	.0383	.0436	.0488	.0540
.1	.0591	.0642	.0692	.0742	.0792	.0841	.0889	.0937	.0985	.1032
.2	.1079	.1125	.1171	.1217	.1262	.1306	.1350	.1394	.1437	.1480
.3	.1523	.1565	.1606	.1648	.1689	.1729	.1769	.1809	.1848	.1887
.4	.1925	.1963	.2001	.2038	.2075	.2112	.2148	.2184	.2219	.2254
.5	.2289	.2323	.2357	.2391	.2424	.2457	.2490	.2522	.2554	.2586
.6	.2617	.2648	.2679	.2709	.2739	.2768	.2798	.2827	.2855	.2884
.7	.2912	.2939	.2967	.2994	.3021	.3047	.3074	.3099	.3125	.3150
.8	.3176	.3200	.3225	.3249	.3273	.3297	.3320	.3343	.3366	.3389
.9	.3411	.3433	.3455	.3477	.3498	.3519	.3540	.3560	.3581	.3601
1.0	.3621	.3640	.3660	.3679	.3698	.3716	.3735	.3753	.3771	.3789
1.1	.3806	.3824	.3841	.3858	.3874	.3891	.3907	.3923	.3939	.3955
1.2	.3970	.3986	.4001	.4016	.4030	.4045	.4059	.4073	.4087	.4101
1.3	.4115	.4128	.4142	.4155	.4168	.4180	.4193	.4205	.4218	.4230
1.4	.4242	.4253	.4265	.4276	.4288	.4299	.4310	.4321	.4331	.4342
1.5	.4352	.4363	.4373	.4383	.4393	.4402	.4412	.4421	.4431	.4440
1.6	.4449	.4458	.4467	.4475	.4484	.4492	.4501	.4509	.4517	.4525
1.7	.4533	.4540	.4548	.4556	.4563	.4570	.4577	.4585	.4591	.4598
1.8	.4605	.4612	.4618	.4625	.4631	.4638	.4644	.4650	.4656	.4662
1.9	.4668	.4673	.4679	.4684	.4690	.4695	.4701	.4706	.4711	.4716
2.0	.4721	.4726	.4731	.4736	.4740	.4745	.4749	.4754	.4758	.4763
2.1	.4767	.4771	.4775	.4779	.4783	.4787	.4791	.4795	.4798	.4802
2.2	.4806	.4809	.4813	.4816	.4820	.4823	.4826	.4829	.4833	.4836
2.3	.4839	.4842	.4845	.4848	.4851	.4853	.4856	.4859	.4862	.4864
2.4	.4867	.4869	.4872	.4874	.4877	.4879	.4881	.4884	.4886	.4888
2.5	.4890	.4892	.4894	.4897	.4899	.4901	.4902	.4904	.4906	.4908
2.6	.4910	.4912	.4913	.4915	.4917	.4919	.4920	.4922	.4923	.4925
2.7	.4926	.4928	.4929	.4931	.4932	.4934	.4935	.4936	.4938	.4939
2.8	.4940	.4941	.4943	.4944	.4945	.4946	.4947	.4948	.4949	.4950
2.9	.4951	.4952	.4953	.4954	.4955	.4956	.4957	.4958	.4959	.4960
3.0	.4961	.4962	.4962	.4963	.4964	.4965	.4966	.4966	.4967	.4968
3.1	.4968	.4969	.4970	.4970	.4971	.4972	.4972	.4973	.4974	.4974
3.2	.4975	.4975	.4976	.4976	.4977	.4977	.4978	.4978	.4979	.4979
3.3	.4980	.4980	.4981	.4981	.4982	.4982	.4982	.4983	.4983	.4984
3.4	.4984	.4984	.4985	.4985	.4985	.4986	.4986	.4986	.4987	.4987
3.5	.4987	.4988	.4988	.4988	.4988	.4989	.4989	.4989	.4989	.4990
3.6	.4990	.4990	.4990	.4991	.4991	.4991	.4991	.4992	.4992	.4992
3.7	.4992	.4992	.4992	.4993	.4993	.4993	.4993	.4993	.4994	.4994
3.8	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995	.4995	.4995
3.9	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4996
4.0	.4996	.4996	.4996	.4997	.4997	.4997	.4997	.4997	.4997	.4997
4.1	.4997	.4997	.4997	.4997	.4997	.4997	.4998	.4998	.4998	.4998
4.2	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
4.3	.4998	.4998	.4998	.4998	.4998	.4999	.4999	.4999	.4999	.4999
4.4	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999

AREA FOR $a = 0.25$

λ	0.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0050	.0099	.0148	.0197	.0246	.0295	.0344	.0392	.0440	.0488
.1	.0536	.0583	.0631	.0677	.0724	.0770	.0816	.0862	.0908	.0953
.2	.0998	.1043	.1087	.1131	.1175	.1218	.1261	.1304	.1346	.1388
.3	.1430	.1472	.1513	.1554	.1594	.1634	.1674	.1714	.1753	.1792
.4	.1831	.1869	.1907	.1944	.1982	.2019	.2055	.2092	.2128	.2163
.5	.2199	.2234	.2268	.2303	.2337	.2370	.2404	.2437	.2470	.2502
.6	.2534	.2566	.2598	.2629	.2660	.2691	.2721	.2751	.2781	.2810
.7	.2839	.2868	.2897	.2925	.2953	.2980	.3008	.3035	.3062	.3088
.8	.3114	.3140	.3166	.3191	.3216	.3241	.3266	.3290	.3314	.3338
.9	.3361	.3384	.3407	.3430	.3452	.3474	.3496	.3518	.3539	.3560
1.0	.3581	.3602	.3622	.3642	.3662	.3682	.3702	.3721	.3740	.3758
1.1	.3777	.3795	.3813	.3831	.3849	.3866	.3883	.3900	.3917	.3934
1.2	.3950	.3966	.3982	.3998	.4013	.4029	.4044	.4059	.4073	.4088
1.3	.4102	.4116	.4130	.4144	.4158	.4171	.4184	.4197	.4210	.4223
1.4	.4236	.4248	.4260	.4272	.4284	.4296	.4307	.4319	.4330	.4341
1.5	.4352	.4363	.4373	.4384	.4394	.4404	.4414	.4424	.4434	.4443
1.6	.4453	.4462	.4471	.4480	.4489	.4498	.4506	.4515	.4523	.4532
1.7	.4540	.4548	.4556	.4563	.4571	.4578	.4586	.4593	.4600	.4607
1.8	.4614	.4621	.4628	.4635	.4641	.4648	.4654	.4660	.4666	.4673
1.9	.4678	.4684	.4690	.4696	.4701	.4707	.4712	.4718	.4723	.4728
2.0	.4733	.4738	.4743	.4748	.4752	.4757	.4762	.4766	.4771	.4775
2.1	.4779	.4783	.4788	.4792	.4796	.4800	.4803	.4807	.4811	.4815
2.2	.4818	.4822	.4825	.4829	.4832	.4835	.4839	.4842	.4845	.4848
2.3	.4851	.4854	.4857	.4860	.4863	.4865	.4868	.4871	.4873	.4876
2.4	.4878	.4881	.4883	.4886	.4888	.4890	.4893	.4895	.4897	.4899
2.5	.4901	.4903	.4905	.4907	.4909	.4911	.4913	.4915	.4917	.4918
2.6	.4920	.4922	.4923	.4925	.4927	.4928	.4930	.4931	.4933	.4934
2.7	.4936	.4937	.4938	.4940	.4941	.4942	.4944	.4945	.4946	.4947
2.8	.4948	.4949	.4951	.4952	.4953	.4954	.4955	.4956	.4957	.4958
2.9	.4959	.4960	.4961	.4961	.4962	.4963	.4964	.4965	.4966	.4966
3.0	.4967	.4968	.4969	.4969	.4970	.4971	.4971	.4972	.4973	.4973
3.1	.4974	.4975	.4975	.4976	.4976	.4977	.4977	.4978	.4979	.4979
3.2	.4980	.4980	.4981	.4981	.4981	.4982	.4982	.4983	.4983	.4984
3.3	.4984	.4984	.4985	.4985	.4985	.4986	.4986	.4987	.4987	.4987
3.4	.4987	.4988	.4988	.4988	.4989	.4989	.4989	.4990	.4990	.4990
3.5	.4990	.4991	.4991	.4991	.4991	.4991	.4992	.4992	.4992	.4992
3.6	.4992	.4993	.4993	.4993	.4993	.4993	.4994	.4994	.4994	.4994
3.7	.4994	.4994	.4995	.4995	.4995	.4995	.4995	.4995	.4995	.4995
3.8	.4996	.4996	.4996	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.9	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997
4.0	.4997	.4997	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
4.1	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
4.2	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
4.3	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
4.4	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999

AREA FOR $a = 0.50$

λ	0.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0046	.0093	.0139	.0185	.0231	.0277	.0323	.0369	.0415	.0460
.1	.0506	.0551	.0596	.0641	.0686	.0730	.0775	.0819	.0863	.0907
.2	.0950	.0993	.1037	.1079	.1122	.1164	.1207	.1249	.1290	.1332
.3	.1373	.1414	.1454	.1495	.1535	.1575	.1614	.1654	.1693	.1731
.4	.1770	.1808	.1846	.1883	.1921	.1958	.1995	.2031	.2067	.2103
.5	.2139	.2174	.2209	.2244	.2278	.2312	.2346	.2380	.2413	.2446
.6	.2479	.2511	.2543	.2575	.2607	.2638	.2669	.2699	.2730	.2760
.7	.2790	.2819	.2848	.2877	.2906	.2934	.2962	.2990	.3017	.3045
.8	.3072	.3098	.3125	.3151	.3177	.3202	.3227	.3252	.3277	.3302
.9	.3326	.3350	.3373	.3397	.3420	.3443	.3466	.3488	.3510	.3532
1.0	.3554	.3575	.3596	.3617	.3638	.3658	.3678	.3698	.3718	.3737
1.1	.3756	.3775	.3794	.3812	.3831	.3849	.3867	.3884	.3902	.3919
1.2	.3936	.3952	.3969	.3985	.4001	.4017	.4033	.4048	.4064	.4079
1.3	.4093	.4108	.4123	.4137	.4151	.4165	.4179	.4192	.4205	.4219
1.4	.4231	.4244	.4257	.4269	.4282	.4294	.4306	.4317	.4329	.4340
1.5	.4352	.4363	.4374	.4384	.4395	.4405	.4416	.4426	.4436	.4446
1.6	.4455	.4465	.4474	.4484	.4493	.4502	.4511	.4519	.4528	.4537
1.7	.4545	.4553	.4561	.4569	.4577	.4585	.4592	.4600	.4607	.4614
1.8	.4621	.4628	.4635	.4642	.4649	.4655	.4662	.4668	.4674	.4681
1.9	.4687	.4693	.4698	.4704	.4710	.4715	.4721	.4726	.4731	.4737
2.0	.4742	.4747	.4752	.4757	.4761	.4766	.4771	.4775	.4780	.4784
2.1	.4788	.4792	.4797	.4801	.4805	.4809	.4812	.4816	.4820	.4824
2.2	.4827	.4831	.4834	.4838	.4841	.4844	.4847	.4851	.4854	.4857
2.3	.4860	.4863	.4865	.4868	.4871	.4874	.4876	.4879	.4882	.4884
2.4	.4887	.4889	.4891	.4894	.4896	.4898	.4900	.4903	.4905	.4907
2.5	.4909	.4911	.4913	.4915	.4916	.4918	.4920	.4922	.4924	.4925
2.6	.4927	.4929	.4930	.4932	.4933	.4935	.4936	.4938	.4939	.4940
2.7	.4942	.4943	.4944	.4946	.4947	.4948	.4949	.4951	.4952	.4953
2.8	.4954	.4955	.4956	.4957	.4958	.4959	.4960	.4961	.4962	.4963
2.9	.4964	.4965	.4965	.4966	.4967	.4968	.4969	.4969	.4970	.4971
3.0	.4971	.4972	.4973	.4974	.4974	.4975	.4975	.4976	.4977	.4977
3.1	.4978	.4978	.4979	.4979	.4980	.4980	.4981	.4981	.4982	.4982
3.2	.4983	.4983	.4984	.4984	.4984	.4985	.4985	.4986	.4986	.4986
3.3	.4987	.4987	.4987	.4988	.4988	.4988	.4989	.4989	.4989	.4989
3.4	.4990	.4990	.4990	.4991	.4991	.4991	.4991	.4991	.4992	.4992
3.5	.4992	.4992	.4993	.4993	.4993	.4993	.4993	.4993	.4994	.4994
3.6	.4994	.4994	.4994	.4994	.4995	.4995	.4995	.4995	.4995	.4995
3.7	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4996	.4996	.4996
3.8	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997
3.9	.4997	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
4.0	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4999
4.1	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
4.2	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
4.3	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
4.4	.4999	.4999	.4999	.4999	.5000	.5000	.5000	.5000	.5000	.5000

AREA FOR α = 1.00

λ	0.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0043	.0087	.0130	.0174	.0217	.0260	.0304	.0347	.0390	.0433
.1	.0476	.0519	.0562	.0604	.0647	.0689	.0732	.0774	.0816	.0858
.2	.0900	.0942	.0983	.1025	.1066	.1107	.1148	.1189	.1229	.1270
.3	.1310	.1350	.1389	.1429	.1468	.1508	.1547	.1585	.1624	.1662
.4	.1700	.1738	.1776	.1813	.1850	.1887	.1924	.1960	.1996	.2032
.5	.2068	.2104	.2139	.2174	.2208	.2243	.2277	.2311	.2345	.2378
.6	.2411	.2444	.2477	.2509	.2541	.2573	.2604	.2636	.2667	.2697
.7	.2728	.2758	.2788	.2818	.2847	.2876	.2905	.2934	.2962	.2990
.8	.3018	.3045	.3073	.3100	.3126	.3153	.3179	.3205	.3230	.3256
.9	.3281	.3306	.3330	.3355	.3379	.3402	.3426	.3449	.3472	.3495
1.0	.3518	.3540	.3562	.3584	.3605	.3626	.3647	.3668	.3689	.3709
1.1	.3729	.3749	.3768	.3788	.3807	.3826	.3844	.3863	.3881	.3899
1.2	.3917	.3934	.3951	.3968	.3985	.4002	.4018	.4034	.4050	.4066
1.3	.4082	.4097	.4112	.4127	.4142	.4156	.4171	.4185	.4199	.4212
1.4	.4226	.4239	.4252	.4265	.4278	.4291	.4303	.4316	.4328	.4339
1.5	.4351	.4363	.4374	.4385	.4396	.4407	.4418	.4429	.4439	.4449
1.6	.4459	.4469	.4479	.4489	.4498	.4507	.4517	.4526	.4534	.4543
1.7	.4552	.4560	.4569	.4577	.4585	.4593	.4601	.4608	.4616	.4623
1.8	.4631	.4638	.4645	.4652	.4659	.4665	.4672	.4678	.4685	.4691
1.9	.4697	.4703	.4709	.4715	.4721	.4726	.4732	.4737	.4743	.4748
2.0	.4753	.4758	.4763	.4768	.4773	.4778	.4782	.4787	.4791	.4796
2.1	.4800	.4804	.4808	.4812	.4816	.4820	.4824	.4828	.4832	.4835
2.2	.4839	.4842	.4846	.4849	.4852	.4856	.4859	.4862	.4865	.4868
2.3	.4871	.4874	.4876	.4879	.4882	.4885	.4887	.4890	.4892	.4895
2.4	.4897	.4899	.4902	.4904	.4906	.4908	.4910	.4912	.4914	.4916
2.5	.4918	.4920	.4922	.4924	.4926	.4928	.4929	.4931	.4933	.4934
2.6	.4936	.4937	.4939	.4940	.4942	.4943	.4944	.4946	.4947	.4948
2.7	.4950	.4951	.4952	.4953	.4954	.4955	.4957	.4958	.4959	.4960
2.8	.4961	.4962	.4963	.4964	.4965	.4965	.4966	.4967	.4968	.4969
2.9	.4970	.4970	.4971	.4972	.4973	.4973	.4974	.4975	.4975	.4976
3.0	.4977	.4977	.4978	.4978	.4979	.4979	.4980	.4981	.4981	.4982
3.1	.4982	.4983	.4983	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.2	.4986	.4987	.4987	.4987	.4988	.4988	.4988	.4989	.4989	.4989
3.3	.4990	.4990	.4990	.4990	.4991	.4991	.4991	.4992	.4992	.4992
3.4	.4992	.4992	.4993	.4993	.4993	.4993	.4993	.4994	.4994	.4994
3.5	.4994	.4994	.4994	.4995	.4995	.4995	.4995	.4995	.4995	.4996
3.6	.4996	.4996	.4996	.4996	.4996	.4996	.4996	.4996	.4997	.4997
3.7	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.8	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.9	.4998	.4998	.4998	.4998	.4998	.4999	.4999	.4999	.4999	.4999
4.0	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
4.1	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
4.2	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.5000
4.3	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
4.4	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000