INTERCOMPARISON AND SCALING OF MASSES

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Technical Memorandum Number 10

Legal Metrology and Laboratory Services Branch

Consumer and Corporate Affairs Canada

Consommation et Corporations Canada

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Foreword

The task of forming the multiples and the submultiples of a unit of mass is one of the fundamental problems of the scientific and the legal metrology. It is in the solution of the systems of equations related to the scaling of mass that the method of least squares has found one of its most striking and elegant applications.

Each metrological laboratory of the industrialized world has established its own patterns based on this method, for performing intercomparisons and solving the resulting linear equations. From this standpoint, those countries where the metric (i.e. decimal) system had been adopted from the start, had an easier and simpler task than those which like Canada, continued to use simultaneously various different systems.

Several men have played a prominent role in Canada's mass metrology. The first to mention is R.H. Field from the National Research Council who, in the years that precede and follow the second world war, contributed so strongly to the formation of the mass laboratory of the Standards Branch. A special recognition is due to the late Richard Reynolds, a metrologist totally dedicated to his activity, whose procedures, algorithms and forms constituted the foundation of the laboratory's work for several decades. W.J.S. Fraser, chief of the Standards Laboratory reworked and improved a large part of Reynolds' work and contributed seriously to establish the reliability of the operations.

At the time when this country started to move deliberately towards the total "metrication", a global and deep revision of the present status of the mass metrology in Canada seemed to be a perfectly justifiable undertaking. This memorandum treats only the SI unit of mass, i.e. the kilogram and its multiples and submultiples.

All calculations described in this work have been entirely redone starting from the basic principles of statistics and have been brought to their logical conclusion i.e. to the expressions of the standard deviations on all multiples and submultiples of the kilogram.

The authors hope that their work will be on the line of an already long "tradition" in the domain of mass calibration in Canada. They are conscious of what they owe to their predecessors and expect that this memorandum will play, in the years to come, the role of a centre about which will revolve all the activity of the Mass Laboratory. They also hope that it will contribute to the advancement of the Metric System in general.

Note

The present memorandum is the final form of the provisional version (1971) which had been reproduced only in a very restricted number of copies and in the foreword of which Mr. G. Jones, P. Eng (then Senior Engineer-Mass Calibration) has underlined the necessity to review from time to time the procedures of mass scaling in order to conform to the increased accuracies demanded in the field of modern metrology.

Since 1971, the provisional memorandum had been in use in the Mass Laboratory and has also been examined by other metrologists. The present version takes into account all comments that have been communicated to the authors and the conclusions the staff of the Mass Laboratory has reached by perusing the memorandum in the course of the last six years.

Intercomparison and Scaling of Masses

1. Introduction

In conformity with a resolution of the First General Conference of Weights and Measures held in Paris in 1889, the unit of mass in the metric system is defined by the mass of a certain platinum-iridium cylindrical weight named "International Kilogram" (IK). The mass of the IK is therefore, by definition, equal to 1 kg. This standard is in the custody of the International Bureau of Weights and Measures at Sèvres (France). The unit "kg" is now an integral part of the international system of units designated by the symbol SI.

It is obvious that the definition of an important unit by a unique material object presents some very weak points. For this reason, the International Kilogram is incorporated into a group of weights (of the same type and quality) and the above definition is considered as valid as long as the intercomparisons between the members of the group confirm the constancy of the mass of the International Kilogram.

The International Kilogram is used only in exceptionally important metrological operations; for instance, since 1889, it had been left untouched until after the end of the Second World War. Its constancy and, in general, the constancy of all platinum-iridium standards has always been found highly satisfactory.

Most of the older countries of the world possess a national platinum-iridium copy of IK. The equations of the copies are periodically verified at the International Bureau in terms of the International Kilogram and its witnesses. The Canadian national standard is in the custody of the National Research Council, Division of Physics. All other legal units in Canada are defined in terms of the International Kilogram.

There always was a strong and legitimate desire amongst the majority of physicists and metrologists to replace material prototypes by so-called "natural standards". While their efforts have been successful in the domain of the unit of length (metre), no method had yet been suggested for constituting a reliable "natural" unit of mass. The present International Kilogram is thus very likely to play its fundamental role for many years to come.

The object of this memorandum is the comparison of nominally equal masses; it serves as an introduction to methods for constituting standards of masses other than one kg. The described methods are generally performed with equal arm balances of the highest quality and sensitivity.

Although the idea that the comparison of masses (particularly with the purpose of "scaling") can be done only with the help of statistical methods is not absolutely justified, it must be acknowledged that the use of the method of least squares is so convenient that no mass metrology is presently conceivable without it. It provides valuable checks, increases the precision and permits to assess the overall accuracy of the obtained results. Its important feature is that it also eliminates all possibility of arbitrary operations so that it suffices to mention that an operation had been conducted in conformity with the method of least squares to be sure that any ambiguity (experimental and computational) is totally excluded.

So let us consider the fundamental problem of the comparison of masses from the standpoint of a metrologist who possesses in his laboratory a representative of the International Kilogram and who must establish the methods for constituting

a) other weights nominally equal to 1 kg,

b) weights of multiple values of 1 kg,

c) weights of submultiple values of 1 kg.

The three cases will be treated successively and it will be shown how the method of least squares helps the metrologist to reach the highest accuracy. In the following sections the standard that is the representative of the IK will be designated by the symbol RK and its mass by the symbol S.

2. Standardization of nominally equal masses

Although it is practically impossible to manufacture a standard which would be exactly equal to another standard, it is possible to make standards which are so close to each other that all intercomparisons could be carried out, even on the most sensitive balance, without the help of any additional mass. The only auxiliary mass that is required and without which no weighing is possible, is an appropriate "sensitivity mass" for instance of 1 or 1/2 milligram.

The simplest operation is to build and adjust one single weight and to compare it to RK. By repeating the comparison several times, we shall obtain some information concerning the precision of the operations, particularly if the weighings are made in somewhat different conditions of temperature, humidity etc. With three weights, (i.e. the RK and two copies), we can constitute one redundant relation, namely that obtained by intercomparing the two copies. This redundant equation can either be treated only as a check but not taken into account, or it can be incorporated into the calculation of the unknowns by the method of least squares.

The number of redundant equations, when nominally equal weights are intercompared (two by two), increases fast with n. If we have $(n - 1)$ weights to calibrate against the RK we need $n - 1$ equations; on the other hand, the number of possible intercomparisons is equal to $N = n(n - 1)/2$. For instance, with 5 weights we can make 10 intercomparisons, of which 6 are redundant; with 6 weights we would have $N = 15$, with 10 redundant.

The method of least squares will now be applied to the case where the number of weights is equal to five; the system of equations is then relatively small but the general features of its internal structure can be easily perceived. Let the symbols U, V, X, Y, Z represent the masses of the weights, Z being, by definition, the mass of the RK. The result of an intercomparison of, say, V and Y can be represented by a general linear relation of the form

$$V - Y = m_i \tag{1}$$

It corresponds to i = 6 in table (2) which contains all possible combinations of five weights taken two at a time. The subscript i takes on all the values ranging from 1 to 10.

<u>Equations of condition</u>

$$\begin{array}{ccccclcl}
U & - V & & & & = m_1 & & \\
U & & - X & & & = m_2 & & \\
U & & & - Y & & = m_3 & & \\
U & & & & - Z & = m_4 & & \\
 & V & - X & & & = m_5 & & (2) \\
 & V & & - Y & & = m_6 & & \\
 & V & & & - Z & = m_7 & & \\
 & & X & - Y & & = m_8 & & \\
 & & X & & - Z & = m_9 & & \\
 & & & Y & - Z & = m_{10} & &
\end{array}$$

This system has no solution in a strict mathematical sense because of the accidental errors that affect the observed values m_i. Any subsystem of four equations drawn from (2) has a well defined solution provided it is completed by an equation of definition (which plays the role of the fifth equation of the subsystem). This equation of definition is here

$$Z = S \tag{3}$$

The simplest subsystem is, of course:

$$\begin{aligned} U &= Z + m_4 &&= S + m_4, \\ V &= Z + m_7 &&= S + m_7, \\ X &= Z + m_9 &&= S + m_9, \\ Y &= Z + m_{10} &&= S + m_{10}. \end{aligned}$$

These values are not influenced by the results of all other intercomparisons (m_1, m_2, m_3, m_5, m_6, m_8).

To solve the system (2) by the method of least squares it is preferable to disregard the fact that Z has a special meaning; thus the form of the equation of definition is not specified beforehand and all quantities, U, V, X, Y, Z are first treated in the same way; the calculation is conducted in a "symmetrical" manner and the fact that Z = S is taken into account only at the end of the calculation.

The normal equations of the equations of condition (2) form the system (4).

<u>Normal equations (general form)</u>

$$\begin{aligned} (AA)U + (AB)V + (AC)X + (AD)Y + (AE)Z &= (Am) \\ (BA)U + (BB)V + (BC)X + (BD)Y + (BE)Z &= (Bm) \\ (CA)U + (CB)V + (CC)X + (CD)Y + (CE)Z &= (Cm) \\ (DA)U + (DB)V + (DC)X + (DD)Y + (DE)Z &= (Dm) \\ (EA)U + (EB)V + (EC)X + (ED)Y + (EE)Z &= (Em). \end{aligned} \tag{4}$$

where $(AA) = A_1^2 + A_2^2 + A_3^2 - \text{--------} + A_{10}^2,$

$(AB) = A_1B_1 + A_2B_2 + A_3B_3 - \text{--} + A_{10}B_{10}$, etc.

The first equation is the normal equation in U, the second equation is the normal equation in V etc. According to a classical rule, the normal equation in a certain variable, say U, is formed by summing all equations of condition after having multiplied each of them by its own coefficient of U. The system (2) will yield:

Normal equations

$$
\begin{aligned}
4U - V - X - Y - Z &= m_1 + m_2 + m_3 + m_4 = N_1 \\
-U + 4V - X - Y - Z &= -m_1 + m_5 + m_6 + m_7 = N_2 \\
-U - V + 4X - Y - Z &= -m_2 - m_5 + m_8 + m_9 = N_3 \\
-U - V - X + 4Y - Z &= -m_3 - m_6 - m_8 + m_{10} = N_4 \\
-U - V - X - Y + 4Z &= -m_4 - m_7 - m_9 - m_{10} = N_5.
\end{aligned}
\qquad (5)
$$

The reader can easily check that the determinant of (5) is equal to zero:

$$
\begin{vmatrix}
4 & -1 & -1 & -1 & -1 \\
-1 & 4 & -1 & -1 & -1 \\
-1 & -1 & 4 & -1 & -1 \\
-1 & -1 & -1 & 4 & -1 \\
-1 & -1 & -1 & -1 & 4
\end{vmatrix} = 0
$$

This confirms that the system (5) has no unique solution, unless an appropriate equation of definition is added to it, for instance a relation such as (3).

System (5) is easily solved and its solution may be put under a form which is very convenient for numerical calculations; this is obtained by the introduction of the auxiliary quantity

$$M = 1/5\ (U + V + X + Y + Z). \qquad (6)$$

The final results are

$$
\begin{aligned}
5U &= 5M + N_1, & U &= M + N_1/5, \\
5V &= 5M + N_2, & V &= M + N_2/5, \\
5X &= 5M + N_3, & X &= M + N_3/5, \\
5Y &= 5M + N_4, & Y &= M + N_4/5, \\
5Z &= 5M + N_5, & Z &= M + N_5/5.
\end{aligned}
\qquad (7)
$$

If the equation of definition (3) is taken into account, we have

$$M = S - N_5/5 \qquad (8)$$

and, therefore

$$\begin{aligned} U &= S + (N_1 - N_5)/5, \\ V &= S + (N_2 - N_5)/5, \\ X &= S + (N_3 - N_5)/5, \\ Y &= S + (N_4 - N_5)/5, \\ Z &= S + (N_5 - N_5)/5 = S. \end{aligned} \tag{9}$$

The computation of the right-hand terms of the normal equation (5) is generally made by means of the following table:

	U	V	X	Y	Z
- U	0	$-m_1$	$-m_2$	$-m_3$	$-m_4$
- V	m_1	0	$-m_5$	$-m_6$	$-m_7$
- X	m_2	m_5	0	$-m_8$	$-m_9$
- Y	m_3	m_6	m_8	0	$-m_{10}$
- Z	m_4	m_7	m_9	m_{10}	0
Sums	N_1	N_2	N_3	N_4	N_5
	$N_1/5$	$N_2/5$	$N_3/5$	$N_4/5$	$N_5/5$

(10)

The calculated values m'_i of the observed quantities m_i are computed from the bottom line of (10).

$$\begin{aligned} m'_1 &= (N_1-N_2)/5, & m'_6 &= (N_2-N_4)/5, \\ m'_2 &= (N_1-N_3)/5, & m'_7 &= (N_2-N_5)/5, \\ m'_3 &= (N_1-N_4)/5, & m'_8 &= (N_3-N_4)/5, \\ m'_4 &= (N_1-N_5)/5, & m'_9 &= (N_3-N_5)/5, \\ m'_5 &= (N_2-N_3)/5. & m'_{10} &= (N_4-N_5)/5. \end{aligned} \tag{11}$$

These values have to be introduced into the appropriate cases of table (10). The difference (observed - theoretical) in each case is termed residual deviation or, simply "residual". The sums of residuals in each column and each row of (10) should be equal to zero.

The computation of standard deviations is now done as follows. Calling v_i the residual

$$v_i = m_i - m_i' , \qquad i = 1, 2, --- 10. \qquad (12)$$

we obtain

$$(vv) = \sum v_i^2 , \qquad (13)$$

and

$$s_m^2 = \frac{(vv)}{10-4} = \frac{(vv)}{6} ,$$

$$s_m = \sqrt{\frac{(vv)}{6}} . \qquad (14)$$

10 = number of intercomparisons, 4 = number of unknown quantities: U, V, X, Y.

s_m is termed "group standard deviation". According to theory of probability, there is a 68% chance that the residual of a single intercomparison of any two masses taken from the set U, V, X, Y, Z, will be located inside the limits $\pm$ s_m.

The standard deviations on individual masses are computed from equations (9), for instance,

$$U = S + 1/5 \ (m_1 + m_2 + m_3 + m_4 + m_4 + m_7 + m_9 + m_{10}).$$

According to theory of progression of variance

$$s_U^2 = s_S^2 + (1/5)^2 8 s_m^2 = s_S^2 + 8/25 \cdot s_m^2. \qquad (15)$$

If S had been determined in terms of the IK in another laboratory, its certificate should mention the standard deviation s_S. If the latter is much smaller than s_m on the measured quantities m_i, it may be ignored. Then

$$s_U^2 = 8/25 \, . \, s_m^2 ,$$

$$s_U = 0.56 s_m . \qquad (16)$$

If the order of magnitude of s_S is similar to that of s_m, the complete formula (15) must be applied.

The formulas above apply to all intercompared masses

$$s_U = s_V = s_X = s_Y . \qquad (17)$$

These operations are illustrated in Appendix 1.

One of the most frequently found cases is that of three nominally equal masses, X, Y, Z.

We have now:

	X	Y	Z
- X	0	$-m_1$	$-m_2$
- Y	m_1	0	$-m_3$
- Z	m_2	m_3	0
	N_1	N_2	N_3
	$N_1/3$	$N_2/3$	$N_3/3$

<u>Equations of condition</u>

$$\begin{aligned} X - Y \qquad &= m_1 \\ X \qquad - Z &= m_2 \\ Y - Z &= m_3 \end{aligned}$$

<u>Normal equations</u>

$$\begin{aligned} 2X - Y - Z &= m_1 + m_2 = N_1 \\ -X + 2Y - Z &= m_1 + m_3 = N_2 \\ -X - Y + 2Z &= m_2 - m_3 = N_3 \end{aligned}$$

If

$$M = 1/3\cdot(X + Y + Z)$$

then,(as in (7)),

$$X = S + N_1/3 \ ,$$

$$Y = S + N_2/3 \ ,$$

$$Z = S + N_3/3 \ ,$$

and, if Z represents the reference standard (Z = S),

$$M = S - N_3/3 \ ,$$

and therefore

$$X = S + (N_1 - N_3)/3 \ ,$$

$$Y = S + (N_2 - N_3)/3 \ .$$

The computed values of the observed quantities are:

$$m'_1 = (N_1 - N_2)/3 \ ,$$
$$m'_2 = (N_1 - N_3)/3 \ ,$$
$$m'_3 = (N_2 - N_3)/3 \ .$$

and the standard deviation s_m is

$$s_m^2 = \frac{v_1^2 + v_2^2 + v_3^2}{3 - 2} = (m_1 - m'_1)^2 + (m_2 - m'_2)^2 + (m_3 - m'_3)^2 .$$

As

$$X = S + (m_1 + m_2 + m_2 + m_3)/3 \ ,$$

then

$$s_x^2 = s_S^2 + 4/9 . s_m^2 \ .$$

If s_S^2 is negligible, then, approximately $s_x = 2/3 . s_m$.

By constructing ten weights (one of which may be the RK) we can by the process described here constitute step by step the complete decimal scale of the multiples of the kilogram. By constructing ten nominally equal masses of one-tenth of a kilogram each and by comparing their sum to the RK, it is also possible to constitute the submultiples, i.e. 100, 200... 900 g. These procedures are not used in metrology because they require too many individual weights which cannot be easily placed on pans and safely handled during comparisons. The actually adopted methods are outlined in the following section.

3. Standardization of submultiples and multiples of the Unit of Mass

One of the series of weights that are the most commonly used for constituting a) multiples and b) submultiples of the kilogram, is based on the sequence 5, 2, 2, 1, 1. The set of intercomparisons to which this series actually leads is exactly the same in a) and in b) and is given in (22).

The true values of the weights will be designated by the numerals between brackets, e.g. (5) may designate the true value of a weight the nominal value of which is 5 kg. We shall thus have*

a) (5), (2), (2'), (1), (1'), ... in kilograms

b) (500), (200), (200'), (100), (100'), ... in grams

The influence of the equation of definition, which is not the same in a) and b), will be considered later. The five unknowns being

$$U = (5), \quad V = (2), \quad X = (2'), \quad Y = (1), \quad Z = (1'), \tag{18}$$

the general form of a linear equation of condition will be:

$$AU + BV + CX + DY + EZ = m_i; \tag{19}$$

for instance, if U = (5) is compared against the sum

$$V + X + Y = (2) + (2') + (1),$$

then

$$A_1 = 1, \; B_1 = -1, \; C_1 = -1, \; D_1 = -1, \; E_1 = 0. \tag{20}$$

i.e. we have

$$(5) - (2) - (2') - (1) = m \text{ (measured)}. \tag{21}$$

With the unknowns U, V, X, Y, Z, it is possible to perform the eight following comparisons:

(5)	against	(2) + (2') + (1)
(5)	"	(2) + (2') + (1')
(2) - (1)	"	(2') + (1')
(2) - (1')	"	(2') + (1)

(2)	against	(2')
(2)	"	(1) + (1')
(2')	"	(1) + (1')
(1)	"	(1')

---

* Other commonly used sequences are 5, 2, 1, 1, 1 and 5, 3, 2, 1, 1. They are treated in a separate memorandum (no. 11), in connection with other systems of weights.

They lead to the system:

Equations of condition

$$\begin{array}{rrrrrl}
(5) & -(2) & -(2') & -(1) & & = m_1 \\
(5) & -(2) & -(2') & & -(1') & = m_2 \\
& (2) & -(2') & +(1) & -(1') & = m_3 \\
& (2) & -(2') & -(1) & +(1') & = m_4 \\
& (2) & -(2') & & & = m_5 \\
& (2) & & -(1) & -(1') & = m_6 \\
& & (2') & -(1) & -(1') & = m_7 \\
& & & (1) & -(1') & = m_8,
\end{array} \qquad (22)$$

and to

Normal equations

$$\begin{array}{rrrrrl}
2(5) & -2(2) & -2(2') & -(1) & -(1') & = N_1 \\
-2(5) & +6(2) & -2(2') & & & = N_2 \\
-2(5) & -(2) & +6(2') & & & = N_3 \\
-(5) & & & +6(1) & -(1') & = N_4 \\
-(5) & & & -(1) & +6(1') & = N_5
\end{array} \qquad (23)$$

with $N_1 = m_1 + m_2$,

$$\begin{aligned}
N_2 &= -m_1 - m_2 + m_3 + m_4 + m_5 + m_6, \\
N_3 &= -m_1 - m_2 - m_3 - m_4 - m_5 + m_7, \\
N_4 &= -m_1 + m_3 - m_4 - m_6 - m_7 + m_8, \\
N_5 &= -m_2 - m_3 + m_4 - m_6 - m_7 - m_8.
\end{aligned} \qquad (24)$$

This system, as the system (5), has no single solution because one of the equations is redundant, i.e. is a linear combination of the other four equations. It is easy to check that the determinant of (23) is equal to zero and that there is the following linear relationship

$$5N_1 + 2N_2 + 2N_3 + N_4 + N_5 = 0. \qquad (25)$$

It is recommended, as an exercise, to perform all the calculations by substituting into (25) the left-hand and the right-hand terms of (23) and (24).

The algebraic treatment of the system (23) will strongly depend on which of the two following equations is adopted as the additional equation:

$$\text{A)} \quad (5) + (2) + (2') + (1) = M. \qquad (26)$$
$$\text{B)} \quad (1') = M.$$

M being nominally equal to one kilogram it is not necessary to perform two distinct algebraic operations: one system of solutions can be transformed into any other similar system. (See Note at the end of Appendix 3).

A. Standardization of submultiples

This operation is based on the equations (22), (23), (24), combined with the equation of definition A. It can be put under the form of a simple algorithm leading directly to the solutions that are given in (28). The algorithm and an example are presented in Appendix 2.

The system which has to be solved is therefore

$$\begin{aligned}
2(5) - 2(2) - 2(2') - (1) - (1') &= N_1 \\
-2(5) + 6(2) - (2') &= N_2 \\
-2(5) - (2) + 6(2') &= N_3 \\
-(5) + 6(1) - (1') &= N_4 \\
-(5) - (1) + 6(1') &= N_5 \\
(5) + (2) + (2') + (1) &= M
\end{aligned} \qquad (27)$$

in which the quantities N_1, N_2, N_3, N_4, N_5 are interconnected by the equation (25). The determinant of the system of five equations, the second terms of which are N_1, N_2, N_3, N_4, M (the fifth equation being omitted) is not equal to zero: this system has therefore a well defined solution.

The elimination method applied to (27) presents no difficulty provided the calculations are conducted in an orderly manner. They lead to the solutions:

$$
\begin{aligned}
(5) &= M/2 + 1/28\ (7N_1 - N_4 + N_5) = M/2 + 1/28.S_1,\\
(2) &= M/5 + 1/35\ (N_1 + 5N_2 - N_4) = M/5 + 1/35.S_2,\\
(2) &= M/5 + 1/35\ (N_1 + 5N_3 - N_4) = M/5 + 1/35.S_3, \qquad (28)\\
(1) &= M/10 + 1/140\ (7N_1 + 23N_4 + 5N_5) = M/10 + 1/140.S_4,\\
(1') &= M/10 + 1/140\ (7N_1 + 3N_4 + 25N_5) = M/10 + 1/140.S_5,
\end{aligned}
$$

in which S_1, S_2, S_3, S_4, S_5 are expressed directly in terms of the observed values m_i:

$$
\begin{aligned}
S_1 &= +8m_1 + 6m_2 - 2m_3 + 2m_4 - 2m_8,\\
S_2 &= -3m_1 - 4m_2 + 4m_3 + 6m_4 + 5m_5 + 6m_6 + m_7 - m_8,\\
S_3 &= -3m_1 - 4m_2 - 6m_3 - 4m_4 - 5m_5 + m_6 + 6m_7 - m_8, \qquad (29)\\
S_4 &= -16m_1 + 2m_2 + 18m_3 - 18m_4 - 28m_6 - 28m_7 + 18m_8,\\
S_5 &= +4m_1 - 18m_2 - 22m_3 + 22m_4 - 28m_6 - 28m_7 - 22m_8.
\end{aligned}
$$

For the determination of submultiples, the symbols (5), (2), (1) etc. designate the masses actually equal 0.5, 0.2, 0.1 kg or (500), (200), (100) in grams. The algorithm and an example are given in Appendix 2. The "additional" weight (100') generally plays an important role in the subsequent operations; for this purpose it is composed of the following weights: 50 g, 20 g, 20' g, 10 g, the sum of which is equal to 100 g.

Thence

$$\Sigma'(100) = (50) + (20) + (20') + (10),$$

and we can write down

$$\Sigma(100) = M' \qquad (30)$$

and consider this relation as the equation of definition that is necessary for the further downward calibration. The latter will, of course, require an additional weight (10') which, in its turn will be considered as yielding the third equation of definition of the type

$$\Sigma(10) = M''. \qquad (31)$$

To cover the whole range of submultiples, the sets of weights which are necessary are listed in the following table. From an experimental standpoint the whole operation requires four types of balances.

Table

I. Above 1 gram

$$\Sigma(1000) = M,$$

$$(500),\ (200),\ (200'),\ (100),\ (100')$$

where

$$(100') = \Sigma(100) = M'.$$

$$(50),\ (20),\ (20'),\ (10),\ (10')$$

where

$$(10') = \Sigma(10) = M''.$$

$$(5),\ (2),\ (2'),\ (1),\ (1')$$

where

$$(1') = \Sigma(1) = M'''.$$

II. Below 1 gram

$$\Sigma(1000) = M'''.$$

$$(500),\ (200),\ (200'),\ (100),\ (100')$$

where

$$(100') = \Sigma(100) = M^{IV}.$$

$$(50),\ (20),\ (20'),\ (10),\ (10')$$

where

$$(10') = \Sigma(10) = M^{V}.$$

$$(5),\ (2),\ (2'),\ (1),\ (1').$$

A weight of 1 mg cannot be subdivided into smaller invididual weights: (1') is a single piece weight.

The observer should always check that the (numerical) values (28) satisfy with high accuracy the normal equations (27). These values will not satisfy the equations of condition, at least not rigorously. This leads to the calculation of "residuals" in conformity with the definitions (36).

B. Standardization of multiples

We shall now pass to the calibration of multiples; the first step is to make weights with masses nominally equal to 2 kg and 5 kg and to form the following set:

(5), (2), (2'), (1), (1'),..............in kilograms.

The system of equations of condition is identical to (22) and the system of normal equations is identical to (23). The equation of definition however, is not (26)A but (26)B, i.e.

$$(1') = M \tag{32}$$

This M must be introduced into the system of normal equations (23) and, as the equations of the latter are interconnected by the relation (25), one of the equations of (23) may be replaced by (30). The system, the principal determinant of which is not equal to zero and which will lead to concrete values of individual unknowns in terms of M is, for instance:

$$\begin{aligned}
2(5) - 2(2) - 2(2') - (1) &= N_1 + M \\
-2(5) + 6(2) - (2') \qquad\qquad &= N_2 \\
-2(5) - (2) + 6(2') \qquad\qquad &= N_3 \\
-(5) \qquad\qquad\qquad + 6(1) &= N_4 + M.
\end{aligned} \tag{33}$$

The solutions are

$$\begin{aligned}
(5) &= 5M + S_1/7, \\
(2) &= 2M + S_2/7, \\
(2') &= 2M + S_3/7, \\
(1) &= M + S_4/7.
\end{aligned} \tag{34}$$

with

$$\begin{aligned}
S_1 &= + m_1 + 6m_2 + 5m_3 - 5m_4 \qquad + 7m_6 + 7m_7 + 5m_8, \\
S_2 &= - m_1 + m_2 + 3m_3 - m_4 + m_5 + 4m_6 + 3m_7 + 2m_8, \\
S_3 &= - m_1 + m_2 + m_3 - 3m_4 - m_5 + 3m_6 + 4m_7 + 2m_8, \\
S_4 &= - m_1 + m_2 + 2m_3 - 2m_4 \qquad\qquad\qquad + 2m_8,
\end{aligned} \tag{35}$$

The algorithm and a numerical example are given in Appendix 3.

Considering the mass of the sum (5) + (2) + (2') + (1) as the definition of the mass (10'), the reader should have no difficulty to accomplish the second step upwards, i.e. to calibrate the masses of the set

(50), (20), (20'), (10), (10'), in kilograms.

C. Calculation of Standard Deviations

The calculation of the so-called "group standard deviation" s_m (as defined in the preceding section) is the same in the case of submultiples as in the case of multiples. If, for instance, we consider the solutions (28), we know that they satisfy rigorously the normal equations (27) but do not satisfy rigorously the equations of condition (22). If for instance, the expression

(5) - (2) - (2') - (1)

is formed by means of the values given by (28) the result will not be exactly equal to m_1 (first equation of (22)) but to a slightly different value m_1'. The difference $m_1 - m_1'$, is called the residual v_1 of the first operation:

$$v_1 = m_1 - m_1'.$$

As above, a residual always designates the difference:

"residual" = observed value - computed value

The equations (22) yield eight residuals v_1, v_2 ... v_8, which are treated in the same manner as those of Section 2. We first compute the sum of their squares

$$(vv) = \sum_i v_i^2,$$

and the group standard deviation s_m, by

$$s_m = \sqrt{\frac{(vv)}{8-4}} = \sqrt{\frac{(vv)}{4}}.$$

It is to be noted that the denominator (8 - 4) corresponds to the difference: number of equations of condition - number of independent normal equations. The reader must always remember the presence of the term "independent". Thus one of the normal equations is not included in the counting. The number of independent normal equations is equal to the number of independent unknowns; the latter is equal to 4 because either

a) (1') is directly given in terms of the RK or,

b) the relation (26) A), expresses one of the unknowns in terms of the other three and the RK.

To calculate the standard deviations on individual weights, we have to make use of equations (28) and (29) for submultiples and equation (34) for the multiples.

In conformity with the theorem of propagation of variance, the calculation of the standard deviations is done as follows:

A. Call s_1, s_2, s_3, s_4, s_5 the standard deviations on S_1, S_2, S_3, S_4 and S_5, respectively; for instance, by (29) first line,

$$s_1^2 = (8^2 + 6^2 + 2^2 + 2^2 + 2^2)s_m^2,$$

$$s_1^2 = 112 s_m^2;$$

so that

$$\begin{aligned}
s_{(5)}^2 &= (1/2)^2\ s_M^2 + (1/28)^2 112\ s_m^2 = 0.25\ s_M^2 + 0.14_3\ s_m^2, \\
s_{(2)}^2 &= (1/5)^2\ s_M^2 + (1/35)^2 140\ s_m^2 = 0.04\ s_M^2 + 0.11_4\ s_m^2, \\
s_{(2')}^2 &= (1/5)^2\ s_M^2 + (1/35)^2 140\ s_m^2 = 0.04\ s_M^2 + 0.11_4\ s_m^2, \\
s_{(1)}^2 &= (1/10)^2\ s_M^2 + (1/10)^2 2836\ s_m^2 = 0.01\ s_M^2 + 0.14\ s_m^2, \\
s_{(1')}^2 &= (1/10)^2\ s_M^2 + (1/10)^2 3360\ s_m^2 = 0.01\ s_M^2 + 0.17\ s_m^2.
\end{aligned} \tag{36}$$

s_M is the standard deviation on the determination of M in terms of the IK. In the first step down from the kilogram, i.e. when M = 1000 g, the value of s_M is deduced from the operations by means of which the value of M had been established. Such is the case if M had been included in an intercomparison of kilograms (Section 2). In further steps downward, e.g. if M is nominally equal to 100 g, M represents (100') and therefore $s_M = s_{(1')} = s_{(100')}$.

B. From (34) and (35) we obtain:

$$
\begin{aligned}
s^2_{(5)} &= 25\ s^2_M + 210/49.s^2_m = 25\ s^2_M + 4.28_6\ s^2_m, \\
s^2_{(2)} &= 4\ s^2_M + 42/49.s^2_m = 4\ s^2_M + 0.85_7\ s^2_m, \\
s^2_{(2')} &= 4\ s^2_M + 42/49.s^2_m = 4\ s^2_M + 0.85_7\ s^2_m, \\
s^2_{(1)} &= s^2_M + 14/49.s^2_m = s^2_M + 0.28_6\ s^2_m.
\end{aligned}
\tag{37}
$$

Appendix 1

Intercomparison of equal masses

An intercomparison of five one kilogram masses gave the following equations of condition (2).

$$
\begin{array}{lllllll}
U & - V & & & & = -69.52 \times 10^{-6}\ \text{kg} \\
U & & - X & & & = -68.88 & '' \\
U & & & - Y & & = -63.60 & '' \\
U & & & & - Z & = -69.68 & '' \\
& V & - X & & & = +\ 0.64 & '' \\
& V & & - Y & & = +\ 5.80 & '' \\
& V & & & - Z & = -\ 0.48 & '' \\
& & X & - Y & & = +\ 4.68 & '' \\
& & X & & - Z & = -\ 1.28 & '' \\
& & & Y & - Z & = -\ 6.44 & ''
\end{array}
$$

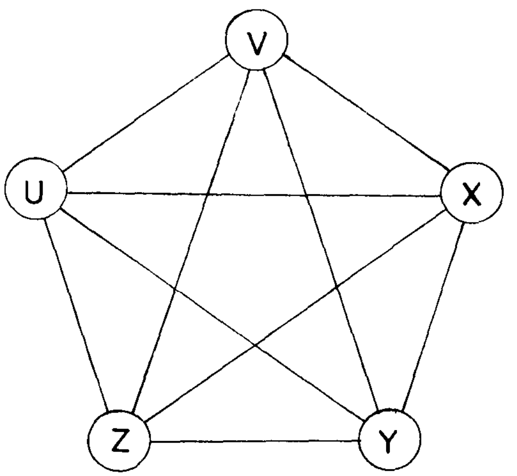

Diagram of intercomparisons of five weights in all possible combinations.

These equations are solved according to Table 10. The first number in each case is the observed value, the second (in brackets) is the computed value m' (11) and the third is the residual (12). The unit is 10^{-6} kg (milligram).

	U	V	X	Y	Z
-U	0	+69.52 (+69.43) + 0.09	+68.88 (+68.67) + 0.21	+63.60 (+63.68) - 0.08	+69.68 (+69.92) - 0.24
-V	-69.52 (-69.43) - 0.09	0	- 0.64 (- 0.76) + 0.12	- 5.80 (- 5.75) - 0.05	+ 0.84 (+ 0.49) - 0.01
-X	-68.88 (-68.67) - 0.21	+ 0.64 (+ 0.76) - 0.12	0	- 4.68 (- 4.99) + 0.31	+ 1.28 (+ 1.25) + 0.03
-Y	-63.60 (-63.68) + 0.08	+ 5.80 (+ 5.75) + 0.05	+ 4.68 (+ 4.99) - 0.31	0	+ 6.44 (+ 6.24) + 0.20
-Z	-69.98 (-69.92) + 0.24	- 0.48 (- 0.49) + 0.01	- 1.28 (- 1.25) - 0.03	- 6.44 (- 6.24) - 0.20	0
N -	-271.68	+75.48	+71.64	+46.68	+77.88
N/5 -	- 54.34	+15.09	+14.33	+ 9.34	+15.58

$$N_1 = -\ 271.62,\quad N_1/5 = -\ 54.34,\quad U = M - 54.34,$$
$$N_2 = +\ 75.48,\quad N_2/5 = +\ 15.09,\quad V = M + 15.09,$$
$$N_3 = +\ 71.64,\quad N_3/5 = +\ 14.33,\quad X = M + 14.33,$$
$$N_4 = +\ 46.68,\quad N_4/5 = +\ 9.34,\quad Y = M + 9.34,$$
$$N_5 = +\ 77.88,\quad N_5/5 = +\ 15.58,\quad Z = M + 15.58.$$

The sum of squares of residuals is computed as follows:

$$
\begin{aligned}
(v_1)^2 &= (-0.09)^2 = 0.0081 \\
(v_2)^2 &= (-0.21)^2 = 0.0441 \\
(v_3)^2 &= (+0.08)^2 = 0.0064 \\
(v_4)^2 &= (+0.24)^2 = 0.0576 \\
(v_5)^2 &= (-0.12)^2 = 0.0144 \\
(v_6)^2 &= (+0.05)^2 = 0.0025 \\
(v_7)^2 &= (+0.01)^2 = 0.0001 \\
(v_8)^2 &= (-0.31)^2 = 0.0961 \\
(v_9)^2 &= (-0.03)^2 = 0.0009 \\
(v_{10})^2 &= (-0.20)^2 = \underline{0.0040} \\
(vv) &= 0.2342;
\end{aligned}
$$

so that

$$s_m^2 = \frac{(vv)}{10-4} = \frac{0.2342}{6} = 0.0390,$$

$$s_m = 0.2.$$

By means of equations (7) the values of the unknowns U, V, X, Y, Z can be tied to any equation of definition. If, for instance, it is known that

$$Z = (1 + 0.50 \times 10^{-6}) \text{ kg},$$

then

$$M = (1 - 15.08 \times 10^{-6}) \text{ kg},$$

and

$$
\begin{aligned}
U = M + N_1/5 &= (1 - 69.42 \times 10^{-6}) \text{ kg}, \\
V = M + N_2/5 &= (1 + 0.01 \times 10^{-6}) \text{ kg}, \\
X = M + N_3/5 &= (1 + 0.75 \times 10^{-6}) \text{ kg}, \\
Y = M + N_4/5 &= (1 - 5.74 \times 10^{-6}) \text{ kg}.
\end{aligned}
$$

APPENDIX 2

Determination of submultiples

The algorithm that embodies the contents of (23), (24), (27), (28) is presented as follows:*

Solution

$$
\begin{array}{ccccc}
+m_1 & -m_1 & -m_1 & -m_1 & \\
+m_2 & -m_2 & -m_2 & & -m_2 \\
 & +m_3 & -m_3 & +m_3 & -m_3 \\
 & +m_4 & -m_4 & -m_4 & +m_4 \\
 & +m_5 & -m_5 & & \\
 & +m_6 & & -m_6 & -m_6 \\
 & & +m_7 & -m_7 & -m_7 \\
\hline
N_1 & N_2 & N_3 & N_4 & N_5 \\
\hline
\end{array}
$$

$$
\begin{array}{ccccc}
+7N_1 & +N_1 & +N_1 & +7N_1 & +7N_1 \\
 & +5N_2 & & & \\
 & & +5N_3 & & \\
-N_4 & -N_4 & -N_4 & +23N_4 & +3N_4 \\
+N_5 & & & +5N_5 & +25N_5 \\
\hline
S_1 & S_2 & S_3 & S_4 & S_5 \\
\hline
\end{array}
$$

$$
\begin{array}{ccccc}
S_1/28 & S_2/35 & S_3/35 & S_4/140 & S_5/140 \\
M/2 & M/5 & M/5 & M/10 & M/10 \\
\hline
(5) & (2) & (2') & (1) & (1')
\end{array}
$$

It is to be noted that quantities m_i, N_1, N_2, N_3, N_4, N_5, S_1, S_2, S_3, S_4, S_5 are "differences" (i.e. small quantities); the quantities M/2...M/10 represent the masses themselves and are of totally different order of magnitude.

---

* The most convenient way to fill the algorithm is to write it line per line rather than column per column.

In a calibration of submultiples of a kilogram the following results have been obtained. The symbols (5), (2), (2'), (1), (1') designate masses equal to 0.5, 0.2, 0.1, 0.1 kg respectively. They can also be replaced by the symbols (500), (200), (200'), (100), (100') if the masses are expressed in grams.

<u>Equations of condition</u>

$$
\begin{array}{lllllll}
+(5) & -(2) & -(2') & -(1) & & = m_1 & = -1.4 \text{ mg} \\
+(5) & -(2) & -(2') & & -(1') & = m_2 & = -0.6 \text{ "} \\
 & +(2) & -(2') & +(1) & -(1') & = m_3 & = +4.4 \text{ "} \\
 & +(2) & -(2') & -(1) & +(1') & = m_4 & = +2.2 \text{ "} \\
 & +(2) & -(2') & & & = m_5 & = +3.4 \text{ "} \\
 & +(2) & & -(1) & -(1') & = m_6 & = +3.2 \text{ "} \\
 & & +(2') & -(1) & -(1') & = m_7 & = 0.0 \text{ "} \\
 & & & +(1) & -(1') & = m_8 & = +1.4 \text{ "}
\end{array}
$$

The calculation is made according to the algorithm given above, the equation of definition being

$$M = (1 - 6.33 \times 10^{-6}) \text{ kg}$$

It is considered as consisting of two parts: 1°) the nominal value which is equal to 1 kg and 2°) the "excess" μ

$$\mu = -6.33 \times 10^{-6} \text{ kg} = -6.33 \text{ mg.}$$

The last line represents the excesses of the weights over their nominal values.

<u>Solution (Unit = 1 mg)</u>

- 1.4	+ 1.4	+ 1.4	+ 1.4	
- 0.6	+ 0.6	+ 0.6		+ 0.6
	+ 4.4	- 4.4	+ 4.4	- 4.4
	+ 2.2	- 2.2	- 2.2	+ 2.2
	+ 3.4	- 3.4		
	+ 3.2	-	- 3.2	- 3.2
		0.0	0.0	0.0
			+ 1.4	- 1.4
- 2.0	+ 15.2	- 8.0	+ 1.8	- 6.2

$+7x - 2.0$	$1x - 2.0$	$+1x - 2.0$	$7x - 2.0$	$7x - 2.0$
$-1x + 1.8$	$5x + 15.2$	$+5x - 8.0$	$23x + 1.8$	$3x + 1.8$
$+1x - 6.2$	$-1x + 1.8$	$-1x + 1.8$	$5x - 6.2$	$+25x - 6.2$
$- 14.0$	$- 2.0$	$- 2.0$	$- 14.0$	$- 14.0$
$- 1.8$	$+ 75.0$	$- 40.0$	$+ 41.4$	$+ 5.4$
$- 6.2$	$- 1.8$	$- 1.8$	$- 31.0$	-155.0
$S_1 = - 22.0$	$S_2 = + 72.2$	$S_3 = - 43.8$	$S_4 = - 3.6$	$S_5 = -163.6$
$- 0.786$	$+ 2.063$	$- 1.251$	$- 0.026$	-1.169
$- 3.165$	$- 1.266$	$- 1.266$	$- 0.633$	-0.633
$- 3.951$	$+ 0.797$	$- 2.517$	$- 0.659$	-1.802

The values of the compared masses are therefore equal to:

$$(5) = (0.5 - 3.951 \times 10^{-6}) \text{ kg},$$
$$(2) = (0.2 + 0.797 \times 10^{-6}) \text{ kg},$$
$$(2') = (0.2 - 2.517 \times 10^{-6}) \text{ kg},$$
$$(1) = (0.1 - 0.659 \times 10^{-6}) \text{ kg},$$
$$(1') = (0.1 - 1.802 \times 10^{-6}) \text{ kg}.$$

The resulting compensated values of the measured quantities m_i, the residuals and their squares are the following (in mg):

$m'_i = - 1.6$ mg	$v_1 = + 0.2$	$v_1^2 = 0.04$
$m'_2 = - 0.4$	$v_2 = - 0.2$	$v_2^2 = 0.04$
$m'_3 = + 4.5$	$v_3 = - 0.1$	$v_3^2 = 0.01$
$m'_4 = + 2.2$	$v_4 = 0.0$	$v_4^2 = 0.00$
$m'_5 = + 3.3$	$v_5 = + 0.1$	$v_5^2 = 0.01$
$m'_6 = + 3.3$	$v_6 = - 0.1$	$v_6^2 = 0.01$
$m'_7 = - 0.1$	$v_7 = + 0.1$	$v_7^2 = 0.01$
$m'_8 = + 1.1.$	$v_8 = + 0.3.$	$v_8^2 = 0.09.$

$$(vv) = 0.21$$

$$s_m^2 = (vv)/4 = 0.21/4 = 0.05.$$

The standard deviations on individual weights are computed by the formulae (36). If s_M (as given by the laboratory in which M has been calibrated against IK) is equal to $s_M = 8 \times 10^{-3}$ mg, these formulae lead to

$$s^2_{(5)} = 9.15 \times 10^{-3}\ \text{mg}^2, \quad s_{(5)} = 0.10\ \text{mg}$$
$$s^2_{(2)} = 6.02 \times 10^{-3}\ '' \ , \quad s_{(2)} = 0.08\ ''$$
$$s^2_{(2')} = 6.02 \times 10^{-3}\ '' \ , \quad s_{(2')} = 0.08\ ''$$
$$s^2_{(1)} = 7.23 \times 10^{-3}\ '' \ , \quad s_{(1)} = 0.08\ ''$$
$$s^2_{(1')} = 8.63 \times 10^{-3}\ '' \ , \quad s_{(1')} = 0.09\ ''$$

APPENDIX 3

Determination of multiples

The equations of condition are of the same form as in Appendix 2. The symbols (5), (2), (2'), (1), (1') designate now masses nominally equal to 5 kg, 2 kg, 2 kg, 1 kg, 1 kg, respectively

Equations of condition

$$
\begin{array}{rrrrrcll}
+(5) & -(2) & -(2') & -(1) & & = m_1 & = -25.0 & \text{mg} \\
+(5) & -(2) & -(2') & & -(1') & = m_2 & = -35.0 & '' \\
 & +(2) & -(2') & +(1) & -(1') & = m_3 & = -\ 5.8 & '' \\
 & +(2) & -(2') & -(1) & +(1') & = m_4 & = -\ 6.1 & '' \\
 & +(2) & -(2') & & & = m_5 & = -\ 0.3 & '' \\
 & +(2) & & -(1) & -(1') & = m_6 & = -\ 0.9 & '' \\
 & & +(2') & -(1) & -(1') & = m_7 & = -\ 0.8 & '' \\
 & & & +(1) & -(1') & = m_8 & = -\ 5.6 & ''
\end{array}
$$

The equation of definition is

$$M = (1') = (1 + 0.05 \times 10^{-6})\ \text{kg} = 1\ \text{kg} + \mu$$

$$\mu = +\ 0.05\ \text{mg.}$$

The algorithm is similar to the one given in Appendix 2.

Solution

$+ m_1$	$- m_1$	$- m_1$	$- m_1$	
$+ m_2$	$- m_2$	$- m_2$		$- m_2$
	$+ m_3$	$- m_3$	$- m_3$	$- m_3$
	$+ m_4$	$- m_4$	$- m_4$	$- m_4$
	$+ m_5$	$- m_5$		
	$+ m_6$		$- m_6$	$- m_6$
		$+ m_7$	$- m_7$	$- m_7$
			$+ m_8$	$- m_8$
N_1	N_2	N_3	N_4	N_5

	$12N_1$	$12N_1$		
	$6N_2$	$5N_2$		
	$5N_3$	$6N_3$		
$-N_4$	$2N_4$	$2N_4$	$+N_4$	
$-6N_5$			$-N_5$	
S_1	S_2	S_3	S_4	
$S_1/7$	$S_2/7$	$S_3/7$	$S_4/7$	
5M	2M	2M	M	M
(5)	(2)	(2')	(1)	(1')

With the values $m_1 \ldots\ldots m_8$ as given in the equations of condition we find:

Solution (Unit = 1 mg)

− 25.0	+ 25.0	+ 25.0	+ 25.0	
− 35.0	+ 35.0	+ 35.0		+ 35.0
	− 5.8	+ 5.8	− 5.8	+ 5.8
	+ 6.1	− 6.1	− 6.1	+ 6.1
	− 0.3	+ 0.3		
	− 0.9		+ 0.9	+ 0.9
		− 0.8	+ 0.8	+ 0.8
			− 5.6	+ 5.6
− 60.0	+ 59.1	+ 59.2	+ 9.2	+ 54.2
	−720.0	−720.0	+ 9.2	
	+364.6	+305.0	− 54.2	
	+296.0	+355.2		
− 9.2	+ 18.4	+ 18.4		
−325.2				
−334.4	− 51.0	− 50.9	− 45.0	
− 47.71	− 7.29	− 7.27	− 6.43	
* + 0.25	+ 0.10	+ 0.10	+ 0.05	+ 0.05
− 47.52	− 7.19	− 7.17	− 6.38	+ 0.05.

* See foot note next page.

The values of the compared weights are

$$(5) = 5 \text{ kg} - 47.52 \text{ mg}$$
$$(2) = 2 \text{ kg} - 17.19 \text{ ''}$$
$$(2') = 2 \text{ kg} - 17.17 \text{ ''}$$
$$(1) = 1 \text{ kg} - 6.38 \text{ ''}$$
$$(1') = 1 \text{ kg} - 0.05 \text{ ''}$$

The resulting residuals and their squares are (in mg):

$$v_1 = -25.0 + 26.8 = +1.8, \qquad v_1^2 = 3.24$$
$$v_2 = -35.0 + 33.2 = -1.8, \qquad v_2^2 = 3.24$$
$$v_3 = -5.8 + 6.4 = +0.6, \qquad v_3^2 = 0.36$$
$$v_4 = +6.1 - 6.4 = -0.3, \qquad v_4^2 = 0.09$$
$$v_5 = -0.3 + 0.0 = -0.3, \qquad v_5^2 = 0.09$$
$$v_6 = -0.9 + 0.9 = -0.0, \qquad v_6^2 = 0.00$$
$$v_7 = -0.8 + 0.8 = -0.0, \qquad v_7^2 = 0.00$$
$$v_8 = -5.6 + 6.4 = +0.8 \qquad v_8^2 = 0.64$$

Hence

$$(vv) = 7.66,$$
$$s_m^2 = \frac{(vv)}{8-4} = \frac{7.66}{4} = 1.92 \text{ (mg)}^2.$$

Assuming that $s_M^2 = 0.008 \text{ (mg)}^2$, $s_{(5)}^2$ becomes equal to

$$s_{(5)}^2 = 25 \times 0.008 + 4.28_6 \times 1.92 = 8.43 \text{ (mg)}^2,$$

and the final results are:

$$s_{(5)} = \sqrt{8.43} \text{ mg} = 2.90 \text{ mg}$$
$$s_{(2)} = \sqrt{1.68} \text{ mg} = 1.30 \text{ mg}$$
$$s_{(2')} = \sqrt{1.68} \text{ mg} = 1.30 \text{ mg}$$
$$s_{(1)} = \sqrt{0.56} \text{ mg} = 0.75 \text{ mg}.$$

* The successive entries in this line are equal to 5 , 2 , 2 , 1 , 1M, respectively.

Note:

The system of equations treated here may also be solved by following the algorithm given in Appendix 2. The masses (5), (2), (2'), (1), (1') would then be first expressed in terms of a provisional reference mass M* of nominal value equal to 10 kg:

$$M^* = (5) + (2) + (2') + (1).$$

This, according to (28), would lead to the equations

$$(5) = M^*/2 + 1/28.S_1^*$$

$$(2) = M^*/5 + 1/35.S_2^*$$

$$(2') = M^*/5 + 1/35.S_3^*$$

$$(1) = M^*/10 + 1/140.S_4^*$$

$$(1') = M^*/10 + 1/140.S_5^*$$

in which S_1^*, S_2^*, S_3^*, S_4^*, S_5^* are computed in terms of the observed quantities m_1, m_2, --- m_8 and expressed by equations (29).

The last of the equations above constitutes the link between the provisional reference mass M^* and the mass used in the equation of definition

$$M = (1').$$

By solving, (for M^*) the equation

$$M = M^*/10 + 1/140.S_5^*,$$

we obtain M^* in terms of M, i.e.

$$M^* = 10\ M - 1/14.S_5^*.$$

This is to be substituted in the expressions of (5), (2), (2'), (1).

For instance, the substitution leads to the following expression of (5):

$$(5) = 5M - 1/28.S_5^* + 1/28.S_1^*,$$

$$(5) = 5M - 1/28.(S_5^* - S_1^*).$$

By (29),

$$S_5^* - S_1^* = -4m_1 - 24m_2 - 20m_3 + 20m_4 - 28m_6 - 28m_7 - 20m_8,$$

and, therefore,

$$(5) = 5M - 1/7.(m_1 + 6m_2 + 5m_3 - 5m_4 + 7m_6 + 7m_7 + 5m_8).$$

This is identical to the first line of (34) so that the resulting expression for the variance $s^2_{(5)}$ will be identical to that of the first line of (37). The end results as presented in (37) are thus independent of the method of calculation. The reader must always remember that the symbols marked with an asterisk (*) refer to the provisional reference value

$$M^* = (5) + (2) + (2') + (1),$$

while M (without asterisk) refers to that weight, the mass of which is either postulated or is determined by an independent operation.

Final Remarks

The statistical methods described in this memorandum (this includes also the Appendices) are completely familiar to the staffs of all standardizing institutions of the world. Several other patterns of intercomparison are also used in the mass laboratory which are due to the legal existence in Canada of the non-metric systems of units of mass. The reader who may be interested in these patterns is invited to enter in contact with the laboratory's staff.

The purpose of the present memorandum, the study which should be preceded by that of the memorandum No. 6, is to make it more widely known how in spite of its simple appearance, the comparison of masses on a metrological level of precision is a complex operation. A reader, after having perused both memoranda will have a better appreciation of the effort the staff of a mass laboratory constantly puts into the problem of maintaining all reference and working sets of weights in metrologically satisfactory conditions.

These memoranda should also play the role of incentive for the users of high quality mass standards to perform themselves a) the operations of intercomparison of nominally equal masses (of various magnitudes) and b) the operations of standardization, following the pattern (5, 2, 2, 1, 1) or similar patterns, the most commonly used of which are analyzed in the memorandum No. 11. Other patterns may be found in literature or established by the metrologists themselves according to the balances and the weights which are available and the conditions in which the weighings are performed.

The authors express the wish that this memorandum will make all observers familiar with the calculation of standard errors. This calculation is the best method for making an observer fully aware of the accuracy he actually obtains in the comparisons of masses. It will also make him aware of the importance of limits of accuracy quoted in the certificates delivered by the standardizing laboratories and thus to appreciate correctly the accuracies of the operations based on these values.