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# Digital Payments in Firm Networks: Theory of Adoption and Quantum Algorithm

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# Abstract

We build a network formation game of firms with trade flows to study the adoption and usage of a new digital currency as an alternative to correspondent banking. We document endogenous heterogeneity and inefficiency in adoption outcomes and explain why higher usage may correspond to lower adoption. Next, we frame the model as a quadratic unconstrained binary optimization (QUBO) problem and apply it to data. Method-wise, QUBO presents an extension to the potential function approach and makes broadly defined network games applicable and empirically feasible, as we demonstrate with a quantum computer.



*Topics: Central bank research; Digital currencies and fintech; Digitalization; Economic models; Financial institutions; Payment clearing and settlement systems; Sectoral balance sheet*
*JEL codes: C6, C71, D4, D85, G, L22*



# Résumé

Nous construisons un jeu de formation de réseaux d’entreprises ayant des flux commerciaux pour étudier l’adoption et l’utilisation d’une nouvelle monnaie numérique comme solution de rechange à la correspondance bancaire. Nous illustrons l’hétérogénéité et l’inefficacité endogènes des résultats de l’adoption de cette nouvelle monnaie, et expliquons pourquoi un taux d’utilisation plus élevé peut correspondre à un taux d’adoption plus faible. Puis, nous façonnons le modèle comme un problème d’optimisation binaire quadratique sans contrainte (QUBO) et l’appliquons aux données. Du point de vue méthodologique, le problème QUBO se veut une extension de l’approche basée sur une fonction potentielle. Il rend les jeux de formation de réseaux vaguement définis applicables et empiriquement réalisables, comme nous le démontrons à l’aide d’un ordinateur quantique.



*Sujets : Recherches menées par les banques centrales; Monnaies numériques et technologies financières; Numérisation; Modèles économiques; Institutions financières; Systèmes de compensation et de règlement des paiements; Bilan sectoriel*
*Codes JEL : C6, C71, D4, D85, G, L22*

# 1 Introduction

The recent appeal toward decentralised finance (DeFi) and Central Bank Digital Currencies (CBDC) raises important questions about the role of decentralised and strategic payment arrangements in a global financial network. However, modelling of realistic contracts in networks is subject to several challenges, including lack of closed-form equilibrium solutions, exponentially growing network complexity, and limited algorithms for finding equilibria.

This paper develops a network formation game where heterogeneous firms cooperatively adopt and use cryptocurrency as an alternative to correspondent banking and cash.[1] This cooperative networks approach embraces the contract nature of interfirm relationships and the complexity of trade flows between firms. We find equilibrium adoption and usage decisions using theoretical tools and perform comparative statics exercises. Next, we propose a novel approach to solving general network-formation games as quadratic optimisation problems, which we test on the payments data using a quantum computer.

Our input to the literature is two-fold. First, we contribute to the literature on de-centralised markets by introducing contractual payment choices.[2] Intuitively, heterogeneity in preferences of firms (namely their desire for autonomy and trade arrangements) generate heterogeneity in the adoption and usage of digital currency. In addition, we show that heterogeneity and underadoption may arise even across symmetric firms due to the over-the-counter nature of interfirm payment contracts. Despite multiplicity of equilibria, we are able to characterise the stable networks up to stochastic dominance relationship to derive comparative statics results. Among them, we show that banks' strategic responses to digitisation of firms cause a non-trivial relationship between crypto adoption and usage. In particular, lower maintenance costs of cryptocurrency payment methods, such as

[1] The results apply more generally to other forms of money used as wholesale method of payment, including stable coins, private interfirm blockchains, and CBDC, under some assumptions.

[2] So far, the main focus of researchers has been on the digital substitutes to retail payments and the payment choices of individuals, while minimum attention has been given to interfirm payments. This is despite wholesale payments being a larger share of global financial flows than retail payments.

lighter policy burden, correspond to higher adoption and utilisation. However, lower transaction costs, such as reduced gas (transaction processing) fees, lower price volatility, and faster settlement time, may instead decrease adoption, while increasing the utilisation of digital currencies.[3]

Our main contribution is methodological. We present the network game as an unconstrained binary quadratic optimisation (QUBO) problem. The QUBO method serves as an extension to the potential function method used in game theory (Rosenthal (1973), Monderer and Shapley (1996)) and applied to networks (Bramoullé et al. (2014), Anshelevich et al. (2008), Tardos and Wexler (2007)). We show that more generally in network formation games (Myerson (1977), Jackson and Wolinsky (1996)), the equilibrium search is identical to solving a single maximisation problem, even when a potential function does not exist. This extension is achieved by expanding the arguments of the QUBO function beyond variables that constitute strategies.

QUBO representation makes a network problem feasible for quantum programming and other modern technologies. It also addresses two general concerns in the network literature: numerical complexity and multiplicity of equilibria.[4] Networks with strategic interactions are prone to exponential growth in dimensionality. In the case of networks with undirected unweighted links, finding a stable network formed by $N$ players would require searching through $2^{N(N-1)/2}$ different network candidates. This means that an exhaustive equilibrium search for only 17 players would demand more computational resources than there are atoms in the universe, which is not feasible on classical computers. Directed networks with weighted links or multiple layers are an even bigger challenge. While theoretical algorithms may be available for certain network games executed on classical machines (e.g., convergence in dominant strategies), solving general network models would require either

[3]The difference between adoption and usage was previously highlighted in retail payments by Li et al. (2020) and Koulayev et al. (2016) in the context of consumer-merchant-platform interactions and Alvarez and Lippi (2009) in the context of consumer cash inventory decision.

[4]Whereas the notion of pairwise stability can be purified to reduce the number of equilibrium outcomes (Bala and Goyal (2000), Bloch and Jackson (2006), Herings et al. (2009), Chen et al. (2010)) multiplicity of equilibria often remains present and is considered by many as an essential representation of reality.

using small networks, replacing cooperative link formation with an ultimatum rule (Bala and Goyal (2000)), or foregoing strategic behaviour altogether (such as in models of random graphs by Erdős et al. (1960) and Barabási and Albert (1999)).[5]

Quantum computing may help overcome these issues. Mathematically, quantum computers can solve some problems that classical machine cannot solve in reasonable time (such as prime factorization, Shor (1999)). Moreover, such "quantum advantage" has already been claimed in practice for a few less practical tasks (Arute et al. (2019); Madsen et al. (2022)). We apply a quantum computer for quadratic optimisation. Quantum computing uses heuristic algorithms to solve these types of problems, meaning that showing quantum advantage in this domain relies on empirical demonstrations.[6] Such empirical demonstrations have been claimed recently using D-Wave quantum annealing devices (King et al. (2024, 2023)). For network formation games, we demonstrate that quantum technology handles increasing network complexity asymptotically better than a classical search, which makes network applications for quantum promising. This is especially hopeful given how rapidly quantum annealers and quantum-inspired methods and hardware are improving.

To demonstrate our applications empirically, we calibrate the payments adoption game to the Canadian economy by using both sectoral input-output production matrices and interbank payments data. We therefore generate a more informative representation of unobserved interfirm networks than in recent papers on production networks (Carvalho and Tahbaz-Salehi (2019)). In particular, merging two network datasets allows for connected firms to exist within each sector. It also relates financial flows to production flows and delivers more realistic network shapes. We use the simulated interfirm trade network to study how adoption of digital payments may change financial flows in the classical banking system, and quantify the degree of inefficiency arising from network externalities when adoption is not coordinated (e.g., by a regulator).

[5] For extensive reviews of random network formation, see Wasserman et al. (1994) and Newman (2018); for comparison of such models with the cooperative network models, see Jackson (2010).

[6] This is analogous to how deep learning was shown to be useful empirically rather than through mathematical proofs.

In our model, we assume that each firm trades a continuum of heterogeneous products with a subset of other firms. Firms prefer some products to be traded via payment methods that maintain privacy and decentralisation (i.e., digital payments or possibly cash), while they prefer other products via methods that maintain safety of execution and convenience, as guaranteed by bank transfers. We focus on the payment decisions of firms, which consist of two parts. First, firms decide whether to adopt the digital method of payment in addition to the pre-existing bank-facilitated payment services and cash. As a result, the network of cryptocurrency adoption is formed endogenously. Second, firms decide what method to use for each type of good they trade. This allows the same firm to use multiple methods of payments depending on the privacy level of each transaction. In addition, the model is populated by banks. Banks are influenced by the network of trade flows but cannot change it. Nevertheless, banks can charge firms' payment fees as a response to digitalisation. This modelling approach allows us to capture four payment market specifics: cooperative payment arrangements, asynchronous decisions, crowding out of banking, and differences between adoption and usage.

The structure of the paper is as follows: Section 2 relates our work to the literature; Section 3 outlines the payment adoption model; Section 4 provides theoretical results; Section 5 presents empirical simulations; Section 6 explains how our network problem can be reformulated as a quadratic optimisation problem; Section 7 lays out the basics of quantum computing and uses it for simulations of the payment game; and Section 8 generalizes quadratic optimisation results to general network games.

## 2 Literature review

We contribute to several streams of literature. First, our paper is closely related to the research on network formation games (see Jackson (2010), Bramoullé et al. (2016), and Goyal (2009) for introductions). Primarily, we rely on Jackson and Wolinsky (1996) and Jackson and Watts (2002) for results on pairwise stability and dynamic formation of networks. In this literature, the question of technology adoption is often considered using diffusion models. See, for example, Morris

(2000), Jackson and Yariv (2007), Leister et al. (2022), and Grabisch et al. (2022). Our approach is an extension of this work because we allow each firm to make a cooperative adoption decision with each counterparty, rather than sticking to a unilateral choice. This leads to a larger number of possible network outcomes and numerical complexity. Method-wise, our paper is more aligned with the papers on interfirm networks with bilateral firm contracts. Among them, Goyal and Joshi (2003) and Goyal and Moraga-Gonzalez (2001) study production synergies between firms, Belleflamme and Bloch (2004) and Priazhkina and Page (2018)—market sharing agreements, Furusawa and Konishi (2007) and Goyal and Joshi (2006)—free trade agreements, Kranton and Minehart (2001) and Manea (2011)—buyer-seller bargaining, and Gale and Kariv (2007) and Babus (2016)—financial contracts.

We also contribute to the literature on payment networks. Such papers proliferated after the global financial crisis by focusing on interbank liquidity, although without explicitly modelling the firms on behalf of which payments are made. Attention has been given to key network players (Bech et al. (2010), Garratt and Zimmerman (2020), Denbee et al. (2021)), efficient clearing mechanism (Eisenberg and Noe (2001), D'Erasmo et al. (2022)), core-periphery property of the network structure (Craig and Von Peter (2014), Farboodi (2023)), financial contagion and systemic risk (Furfine (2003), Allen and Gale (2000), Acemoglu et al. (2012), Elliott et al. (2014), Elsinger et al. (2013)), recovering incomplete network data (Anand et al. (2018)), relationships with non-banks (Anderson et al. (2020)), and patterns in liquidity preferences for specific countries (Iori et al. (2008), Cocco et al. (2009), Afonso and Shin (2011), Martinez-Jaramillo et al. (2014), Bräuning and Fecht (2017)).

Following the rise of blockchain technology, only a few descriptive studies have appeared on cryptocurrency payment networks (Makarov and Schoar (2021)). However, the adoption of new payment methods has been studied more intensively in the non-networks payment literature (Kahn and Roberds (2009), Townsend (2020)). See, for instance, Chiu and Koeppl (2017), Schilling and Uhlig (2019), and Cong et al. (2021). Among these papers, the work of Fernández-Villaverde and Sanches (2019) on the competition between privately-issued currencies deserves particular attention. The authors embed a one-line network structure into a monetary search model; however, they include links defining agent migration patterns rather than trade contracts.

Finally, we model the method of payment choice of firms similarly to the retail model of Rochet and Tirole (2006). Also, the reaction of banks to the introduction of digital money in our paper echoes the literature on central bank digital currency, the safest version of digital money (Bech and Garratt (2017)). See, for instance, Keister and Monnet (2022), Andolfatto (2021), and Williamson (2022). In contrast to the retail CBDC case, we assume that banks respond to the new digital currency by changing fees for having wholesale payments infrastructure access rather than deposit services.

We also contribute more broadly to the growing literature on strategic behaviours in OTC financial markets with network structures: Gale and Shapley (1962), Freixas and Parigi (1998), Condorelli et al. (2017), Blume et al. (2009), Gofman (2017), Malamud and Rostek (2017), and Babus and Kondor (2018), Hendershott et al. (2020), Glode and Opp (2020), Colliard and Demange (2021).

Lastly, we are inspired by the literature on quantum computing, which promises significant technological advances (Bäumer et al. (2021); Yarkoni et al. (2021)), and which recently opened up to the field of economics. See Hull et al. (2020) for a general introduction, McMahon et al. (2022) on the ordering problem in payments, Skavysh et al. (2023) on Monte Carlo applications, and Fernández-Villaverde and Hull (2023) on dynamic programming in economics. Most closely to our work, Orús et al. (2019) make the first steps to simulate contagion of financial networks as in Elliott et al. (2014) using quantum annealing, although without network formation.

# 3 Model of cryptocurrency adoption in networks

## 3.1 Payment decisions and prices

Consider a set of N firms with established trade relationships captured by matrix $L$. If firm $i$ buys goods from firm $j$, $L_{ij} = 1$, otherwise $L_{ij} = 0$. Since our focus is on payment choice, we do not model the trade partner choice and take $L$ exogenously. Given $L$, each firm trades with $n_i = \sum_j (L_{ij} + L_{ji})$ other firms.[7]

Firms trade a continuum of heterogeneous goods. For each type of good, the trading pair can strategically select one method of payment in the set of available methods. Cash or bank transfers are available as status quo. A third method of payment, cryptocurrency, is available to firm pairs that first strategically adopt it. The focus of this paper is a network $G$ with its element $G_{ij}$ describing whether any two firms, $i$ and $j$, adopt the new method of payment, $G_{ij} = 1$, or not, $G_{ij} = 0$. Differently from $L$, $G$ has undirected connections (links), so $G_{ij} = G_{ji}$. $L$ and $G$ are related: cryptocurrency can be adopted only if the trade contact is established.

A firm acts either as a seller or a buyer in each transaction. We assume that the bargaining power is on the buyer's side, so the surplus of each payment is attributed to the firm sending it. This firm also decides on the method of payment for each transaction following the adoption decision.[8] Despite the unilateral decision on how each good is paid for, the decision to adopt or not adopt the new method of payment requires consent of both counterparties, as shown further.

Following Rochet and Tirole (2006), we assume that, depending on the good, firms may prefer one method of payment to the other. Specifically, a firm gets positive utility when paying in cash or cryptocurrency for privacy-sensitive goods. Otherwise, it gets disutility from paying for such goods via correspondent banking. We capture both by introducing a random variable, $p \sim U[-\bar{p}\pi, \bar{p}(1-\pi)]$, for marginal benefit of trading via bank transfer and not other methods. The fraction of privacy-sensitive goods is $\pi$, while the maximum disutility that firm $i$ gets from using a bank transfer for a sensitive good is $\bar{p}$.

Sender $i$ pays fee $f$ per bank transfer, cost $d$ per cash transaction, and cost

[7]For some of our results, it is sufficient to only specify the number of connections but not the network. In this case, we assume that $(n_1, ..., n_N)$ are selected such that the set of networks with such degrees is non-empty. For the symmetric case, $n_1 = ... = n_N$, it is always possible to find at least one network, which makes the problem well-defined. To see this, consider all nodes being ordered and located in a ring. Then make each node a trade partner with $n_i = n$ nodes that follow next in the ring.

[8]Fixed costs of accessing the banking payment system and cash are sunk and not considered.

$c$ per cryptocurrency payment. For the cryptocurrency channel to operate, there is also a maintenance (fixed) cost of $c_0$. We interpret $c_0$ as both the cost of hardware/software upgrades and training needed for the technology to start functioning, as well as the new regulatory and monitoring costs incurred by firms. For tractability, we assume that, in the business-to-business (B2B) context, cash is inferior to the crypto exchange once the channel is adopted.

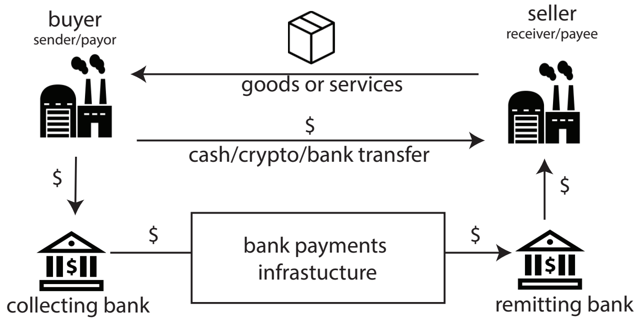


Figure 1: Schematic description of the good and payment transfers.

Establishing the cryptocurrency channel between firms $i$ and $j$ requires mutual consent.

Given the crypto channel exists, $G_{ij} = 1$, the probability of firms using bank transfer, and also the expected demand for it is

$$D_{ij}(f_i, c) = Prob(p - f_i \geq -c) = 1 - \pi - \frac{f_i - c}{\bar{p}}.$$

Without the crypto channel, $G_{ij} = 0$, the expected demand for banking services is

$$D_{ij}(f_i, d) = Prob(p - f_i \geq -d) = 1 - \pi - \frac{f_i - d}{\bar{p}}.$$

Banks are treated not as players of the cooperative game but rather as the second-movers once the network is established. Each firm $i$ uses services of a nearby monopolistic bank $k$. Bank $i$ cannot directly influence the network formation, but can change the marginal fee $f_i \in [0, \bar{f}]$ to firm $i$ for each banking transaction. Each payment made through bank $i$ produces marginal expense $e_i$ for the bank. Bank $i$ price differentiates among firms by choosing $f_i$ to maximise expected return $u_i^b$:

$$u_i^b(f_i) = (f_i - e)D_i(f_i), \tag{1}$$

where $D_i$ is the expected demand it faces

$$D_i(f_i) = \sum_j D_{ij}(f_i, d)(1 - G_{ij})L_{ij} + D_{ij}(f_i, c)G_{ij}L_{ij}.$$

To operate with the demand function in a more convenient way, we use notation for the relative node degree of firm $i$:

$$\delta_i = \frac{\sum_j G_{ij}L_{ij}}{n_i}.$$

The demand for banking services, $D_i$, is therefore inversely related to the adoption of cryptocurrency whenever digital payments are less costly than cash:

$$D_i(f_i) = \frac{n_i}{\bar{p}} \left(\bar{p}(1 - \pi) - f_i + d(1 - \delta_i) + c\delta_i\right).$$

The best response fee of each bank to a given network structure is anticipated by the firms and equal to:

$$f_i^* = argmax_{f_i} u_i^b(f_i) = \frac{1}{2} \left(e + \bar{p}(1 - \pi) + d(1 - \delta_i) + c\delta_i\right). \tag{2}$$

Intuitively, banks charge lower transaction fees to attract payments from a firm that establishes more cryptocurrency channels, assuming $d > c$.

The utility function of each firm is determined as the difference between expected benefit and costs paid for the good it trades. It also accounts for the fixed cost $c_0$ for each cryptocurrency channel established.[9]

$$u_i^f(\delta_i, f_i) = n_i(E[\mathbf{p} - f_i|\mathbf{p} \geq f_i - d](1 - \delta_i) + E[-d|\mathbf{p} < f_i - d](1 - \delta_i)$$
$$+E[\mathbf{p} - f_i|\mathbf{p} \geq f_i - c]\delta_i + E[-c|\mathbf{p} < f_i - c]\delta_i - c_0\delta_i).$$

## 3.2 Model interpretation

We can extend the model's notion of cryptocurrency to stable coins, central bank digital currencies, and even private payment ledgers based on distributed ledger

[9] We ignore other net benefits firms receive from trade, assuming they are additive to the payment costs and good-specific benefit $p$.

technologies. Depending on the payment option being considered, the interpretation of the product-specific payment benefit $p$ will change. The benefits of adopting a new digital method of payment may include avoiding exchange rate fees between fiat currencies (e.g., businesses in the emerging markets may adopt stable coins that are not volatile by design), avoiding costs during the exchange of cryptocurrencies (mainly among web3 companies and investors that use the crypto payment rail for everything from investments to payroll), having a non-destructible record of transactions on a public/private blockchain, and benefiting from price stability. For simplicity, we pick autonomy as the advantage of using digital money for privacy-sensitive goods.[10]

The usage of cash in the model also requires explanation, given that few B2B transactions involve cash. First, we include cash to reflect alternative payment arrangements, such as bilateral credit netting or postal payments, and as an outside option to discipline banks.[11] In addition, having cash allows for more heterogeneity of firms. Among others, criminals and tax evaders often use cash because of its anonymity (Rogoff (2017)), while small merchants normally keep some cash available for operations and use it to economise on banking fees (Baumol (1952), Tobin (1956)). Finally, the presence of cash in the model accounts for the cash paradox observed in many countries—cash being issued (as a ratio to GDP) keeps increasing despite digitisation, while cash usage in retail transactions continues to fall.[12]

[10]We refer to autonomy benefit as benefit from independence and lack of control from any middleman, whether that be the government, the financial system, or a company. We choose autonomy to echo the reasons why cryptocurrencies have been popularised originally (Nakamoto (2008)). Privacy in a form of undisclosed information, as in Garratt and Van Oordt (2021) or Kahn et al. (2005), is not a feature of traditional cryptocurrencies, as most transactions on blockchain can be traced to the users (see Makarov and Schoar (2021) among others for evidence). However, privacy can be achieved when firms use private payment ledgers, cryptocurrency exchanges, Layer 2 platforms, and sophisticated smart contracts, which involve randomness. Our model can be extended for the privacy feature in addition to autonomy if a bank's utility function is adjusted for the additional benefit a bank would receive from payment monitoring (e.g., as in Parlour et al. (2022)

[11]This is similar to Lagos and Zhang (2019), which looks at the economy with cash usage converging to zero but playing an essential role for prices and allocations.

[12]See Camera (2001), Khiaonarong and Humphrey (2019), and Jiang and Shao (2020).

### 3.3 Order of actions, equilibrium notion, and networks terminology

For the order of the game, firms first decide cooperatively on the formation of the payment network. Next, banks impose payment fees. Finally, firms decide which method of payment to use for each good. The equilibrium is formally defined as follows.

Adoption network $G$ and vector of fees $f$ constitute a *stable equilibrium* if there are no feasible deviations that make the deviating coalition better-off:
(I) any two trading firms, $L_{ij} = 1$, with formed cryptocurrency channel $G_{ij} = 1$ do not have incentives to remove it:

$$u_i^f(G_{ij} = 1, G_{-ij}, f_i^*(G_{ij} = 1, G_{-ij})) \geq u_i^f(G_{ij} = 0, G_{-ij}, f_i^*(G_{ij} = 0, G_{-ij}))$$

$$u_j^f(G_{ij} = 1, G_{-ij}, f_j^*(G_{ij} = 1, G_{-ij})) \geq u_j^f(G_{ij} = 0, G_{-ij}, f_j^*(G_{ij} = 0, G_{-ij}));$$

(II) any two trading firms, $L_{ij} = 1$, without formed cryptocurrency channel $G_{ij} = 0$ do not have incentives to form it because at least one of the firms would be strictly worse off, meaning one of the two conditions hold:

$$u_i^f(G_{ij} = 1, G_{-ij}, f_i^*(G_{ij} = 1, G_{-ij})) < u_i^f(G_{ij} = 0, G_{-ij}, f_i^*(G_{ij} = 0, G_{-ij}))$$

$$u_j^f(G_{ij} = 1, G_{-ij}, f_j^*(G_{ij} = 1, G_{-ij})) < u_j^f(G_{ij} = 0, G_{-ij}, f_j^*(G_{ij} = 0, G_{-ij}));$$

(III) any bank $k$ does not have incentives to change the equilibrium bank fee $f_k$:

$$f_k^*(G) = argmax_{f_k'} u_k^b(G, f_k').$$

For convenience, the game can be reformulated as a standard network formation game (see Jackson and Wolinsky (1996) and Bloch and Jackson (2006)). For this, we introduce a new utility function:

$$u_i(G) = \begin{cases} u_i^f(G, f_k^*(G)), \text{ if } \sum_j G_{ij}(1 - L_{ij})(1 - L_{ji}) = 0 \\ -u^{max}, \text{ if } \sum_j G_{ij}(1 - L_{ij})(1 - L_{ji}) > 0, \end{cases} \tag{3}$$

where extremely low payoff $-u^{max}$ is assigned to the infeasible links.

Then network $G$ is called *pairwise stable* if any two firms $i$ and $j$ do not have incentives to remove the link if they have it, $G_{ij} = 1$, meaning both conditions hold true:

$$u_i(G_{ij} = 1, G_{-ij}) \geq u_i(G_{ij} = 0, G_{-ij}) \tag{4}$$

$$u_j(G_{ij} = 1, G_{-ij}) \geq u_j(G_{ij} = 0, G_{-ij}); \tag{5}$$

and to form a link if they do not have it, $G_{ij} = 0$, meaning at least one condition holds true:

$$u_i(G_{ij} = 1, G_{-ij}) < u_i(G_{ij} = 0, G_{-ij}) \tag{6}$$

$$u_j(G_{ij} = 1, G_{-ij}) < u_j(G_{ij} = 0, G_{-ij}). \tag{7}$$

Pairwise stability is a static equilibrium notion; however, the process of reaching the pairwise stable equilibrium is often interpreted dynamically. For this, we introduce additional terminology used in the networks literature.

Consider deviations between any two *adjacent networks*—networks that differ by exactly one link. We say there is an *improving deviation* from $G$ to adjacent $G' = G + ij$, with a link between $i$ and $j$ being present in $G'$ but not $G$, if both $i$ and $j$ benefit from the formation of the new link: conditions (4) and (5) hold with at least one condition being strict. Similarly, there is an *improving deviation* from $G$ to adjacent $G' = G - ij$, with a link between $i$ and $j$ being present in $G$ but not $G'$, if severing the link makes either of them strictly better off: either (6) or (7) hold. In both cases, the players that strictly benefit from the deviation are called the *deviating coalition*.

Clearly, we only focus on single-player or bilateral deviations at a time, while multiple pairs can deviate. The presence of externalities in the game and the inability of each player to re-negotiate multiple links simultaneously makes it possible that after one improving deviation, there will be another improving deviation. As such, we also consider a sequence of improving deviations in-between adjacent networks $(G^{(1)}, ..., G^{(K)})$ called an *improvement path*. An improvement path $(G^{(1)}, ..., G^{(K)})$ is called an *improving cycle* if $G^{(1)} = G^{(k)}$.

Then, by definition, network $G$ is pairwise stable if there are no improvement paths from it to any other network. If such improvement paths exist, there is also an improvement path from $G$ to either some pairwise stable network or an improving cycle of networks (Jackson and Watts (2002)).

Finally, we call pairwise stable network $G^*$ *efficient* if it leads to the efficient allocation of payoffs between the players: $G^* = argmax_G \sum_i u_i(G)$. In general, not all pairwise stable networks are efficient (Jackson and Wolinsky (1996)).

# 4 Equilibrium and desired adoption and usage

## 4.1 Desired adoption and traded goods

In non-corner solutions, each firm settles with the limited usage of cryptocurrency. Deviation to a high usage would make the correspondent bank offer a sufficient discount on the payment fee and turn refusing bank services unprofitable. However, once a trade pair adopt cryptocurrency, the new method of payment will be used ex-post on all transactions for which it even slightly increases the marginal benefit, ignoring the sunk cost $c_0$. To guarantee limited usage of cryptocurrency in the equilibrium, limited adoption should take place initially, which means adoption with only a fraction of trade partners. We show that each bank $i$ aims for the *desired adoption*, $\delta_i^*$, defined as the proportion of $i$'s trade contracts that deliver zero marginal utility.

*Proposition* 1. Utility $u_i(\delta_i)$ is strictly concave with maximum at $\delta = \delta_i^*$,

$$\delta_i^* = \frac{1}{3} - \frac{1}{3}\frac{c-e}{d-c} + \frac{1}{3}\frac{3+\pi_i - 4c_0}{d-c}\bar{p}_i. \tag{8}$$

See Appendix A for the proof.

When $\delta_i^* n$ is an integer number, the preference of each firm in network $G$ is to form another link if the firm's node degree $\delta_i < \delta_i^*$ and to remove a link if $\delta_i > \delta_i^*$. When $\delta_i^* n$ is non-integer, the same statement can be made with regard to threshold $\bar{\delta}_i = \frac{[\delta_i^* n_i]}{n_i}$, where $[\delta_i n_i]$ is defined as a closest integer number to $\delta_i^* n_i$.[13]

[13]For simplicity, assume $\delta_i^*$ is not located exactly in the middle between two natural numbers. The result follows because the marginal utility of a firm is linear in the network degree $\delta_i^*$.

It is a function of the bank's profitability, share of privacy-sensitive goods, as well as fixed and variable costs of payments.

**Corollary 1.** *Ceteris paribus,*

- *higher proportion $\pi_i$ and higher valuation $\bar{p}_i$ of privacy-sensitive goods increase desired adoption of cryptocurrency: $\frac{\partial \delta_i^*}{\partial \pi_i} > 0$ and $\frac{\partial \delta_i^*}{\partial \bar{p}_i} > 0$;*
- *the preferences $u_i$ of firms $i$ with more trade partners $n_i$ are more sensitive to the changes in privacy frequency $\pi_i$ and valuation $\bar{p}_i$:*

$$\Delta\bar{p}_i : u_i(g, \bar{p}_i, \pi_i) = u_i(g + ij, \bar{p}_i + \Delta\bar{p}_i,\, \pi_i) \text{ is decreasing in } n_i,$$

$$\Delta\pi_i : u_i(g, \bar{p}_i, \pi_i) = u_i(g + ij, \bar{p}_i,\, \pi_i + \Delta\pi_i) \text{ is decreasing in } n_i.$$

The first result confirms the intuitive prediction that privacy needs and adoption are positively related. To understand the last result, notice that the desired adoption rate $\delta_i^*$ is independent of the firm's trade network $L$. However, $L$ still defines the maximum number of adopted cases $\delta_i^* n_i$. For instance, for a firm trading with $n_i$ other firms, the privacy-sensitivity $\bar{p}_i$ would need to increase by an increment of $\frac{3}{n_i}\frac{d-c}{3+\pi_i-4c_0}$ for the firm to be willing to make one more connection, which is decreasing in $n_i$.

## 4.2 Role of costs: Paradox of adoption vs usage

Because cryptocurrency serves as a disciplining device for banks, it can be adopted even when bank-facilitated payments are more efficient ($e < c$) and firms do not value privacy, as shown below.

**Corollary 2.** *Firm $i$ that is insensitive to privacy, $\bar{p}_i = 0$, benefits from adoption of cryptocurrency if and only if*

$$c \leq \frac{(n_i - 3)d + n_i e}{2n_i - 3}$$

As the density of the network increases, the adoption threshold for $c$ converges to a mid-point between the cost for cash $d$ and bank transfer $e$.

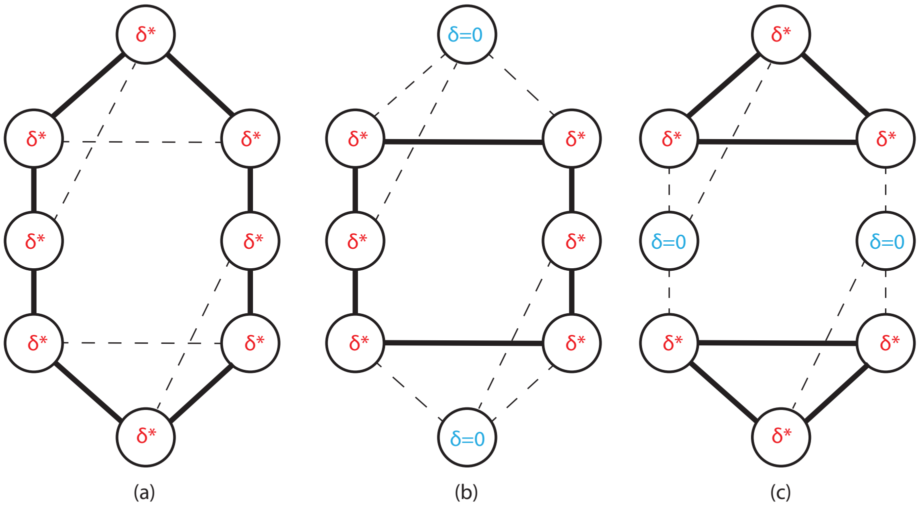


Figure 2: Multiplicity of equilibrium networks when $\delta^* = 2/3$, $n = 3$, $N = 8$: stable network (a) is efficient; stable networks (b) and (c) are not efficient.

The second result is that the effect of marginal cryptocurrency cost on adoption and usage can be different depending on the magnitude of fixed costs.

**Corollary 3.** *A decrease (increase) in the maintenance costs $c_0$ leads to an increase (decrease) in both desired adoption and usage,* $\frac{\partial \delta_i^*}{\partial c_0} < 0$, $\frac{\partial D_{ij}(\delta_i^*)}{\partial c_0} < 0$.
*A decrease (increase) in the marginal cost $c$ increases (decreases) desired adoption and usage* $\frac{\partial \delta_i^*}{\partial c} > 0$, $\frac{\partial D_{ij}(\delta_i^*)}{\partial c} > 0$, *only when the fixed cost $c_0$ is below threshold*

$$c_0 < \frac{d - e}{4\bar{p}_i} + \frac{3 + \pi_i}{4}; \tag{9}$$

*otherwise, a decrease (increase) in the cryptocurrency payments cost $c$ leads to a decrease (increase) in the desired adoption of the cryptocurrency and an increase (decrease) in the desired usage of it:* $\frac{\partial \delta_i^*}{\partial c} < 0$, $\frac{\partial D_{ij}(\delta_i^*)}{\partial c} > 0$.

This marginal cost paradox arises from the interaction of direct and indirect effects of marginal costs. The direct effect of lower cost $c$ is that firms have more incentives to adopt cryptocurrency due to lower per-unit fees. The indirect effect is that higher utilisation of cryptocurrency creates more competition for banks, which forces them to decrease bank fees. In this way, low $c$ discourages firms' adoption of cryptocurrency.[14] Fixed cost aside, direct effect always dominates indirect, which leads to higher crypto usage at lower $c$. Compared to a high maintenance cost $c_0$, the direct effect may not be large enough, so less crypto adoption would take place.

### 4.3 Efficiency, multiplicity, and underadoption

Firms do not necessarily reach desired adoption rates in equilibrium. Figure 2 depicts three equilibria for ex-ante identical firms, each trading with three other firms and willing to connect with up to two firms: $\delta_i^* = 2/3$ for $i = 1, ..., 8$. In equilibrium, heterogeneity arises due to coordination failure in networks, such that networks $(b)$ and $(c)$ are inefficient because two of the firms do not find reciprocity.[15] Apart from transferring privacy-sensitive goods in a costly way, these two firms also pay higher bank fees because their banks do not reduce monopoly rents. This example illustrates how inefficiency arises in adoption of cryptocurrency, with only under- and not overadoption being possible. This result holds more generally and is typical for many network games:

*Proposition* 2. If a pairwise stable network exists, the cryptocurrency adoption rate $\delta_i$ of each firm $i$ is less than or equal to $\bar{\delta}_i$.

See Appendix B for the proof.

### 4.4 Sequential adoption and classical algorithm

This section presents two results: the equilibrium set of the payment game is non-empty; and all pairwise stable networks can be found by executing an algorithm in dominated strategies that completes in a reasonable time when applied to small

[14] As in CBDC literature, digital payments help reduce bank rents: Andolfatto (2021), Chiu et al. (2023), Garratt and Zhu (2021).

[15] In contrast, the origin of equilibrium multiplicity in financial networks with cleared obligations takes place due to self-fulfilling chains of defaults (see Eisenberg and Noe (2001) and Jackson and Pernoud (2020)); in financial network formation games, group stability is required (Farboodi (2023)).

networks. In the sections related to quantum computing, we use the classical algorithm to find all pairwise stable equilibria in various small markets to evaluate its success in finding the same equilibria.

First, we consider a network formation process as a dynamic one in the fashion of (evolutionary) network formation games (Jackson and Watts (2002)). We assign probability likelihood $F(G^{(i)}, G^{(i+1)}) > 0$ to each pairwise improving deviation $(G^{(i)}, G^{(i+1)})$ along some improvement path $(G^{(1)}, ..., G^{(K)})$. In other words, we assign a probability to one link being either formed or removed in favour of the improving coalition. We assign zero probabilities to the deviations that are not coalition improving. If network $G^{(i)}$ is pairwise stable, we assign $F(G^{(i)}, G^{(i)}) = 1$. Then the probability distribution can be chosen to completely define the conditional probability of moving from network $G^{(i)}$ to another network, such that

$$\sum_j F(G^{(i)}, G^{(j)}) = 1. \tag{10}$$

In this manner, deviations can be considered to be sequential actions of firms to either adopt cryptocurrency or stop using it according to the random process $F$. We stick to the dynamic interpretation, as in Jackson and Watts (2002), because assigning probabilities to deviations is a convenient tool to characterise the improvement paths in the payment network formation game.

According to our definitions, the probability of reaching network $G^{(K)}$ from network $G^{(1)}$ along the selected improvement path $(G^{(1)}, ..., G^{(K)})$ is the product of probabilities

$$\Pi_{k=1}^{k=K} \ F(G^{(i)}, G^{(i+1)}) > 0 \tag{11}$$

We fix network $G^{(1)}$ to be the empty network and network $G^{(K)}$ to be one of the pairwise stable networks. We next construct improvement paths leading to each stable network $G^{(K)}$ by sequentially removing the links in backward induction.

- First, if network $G^{(K)}$ has degree distribution $(\delta_1, ..., \delta_N)$, build an improvement path to it from network $G^{(K-1)}$ with degrees $(\delta_1, ..., \delta_{i_1} - \frac{1}{n_{i1}}, ..., \delta_{i_2} - \frac{1}{n_{i_2}}, ..., \delta_N^k)$ for arbitrary selected players $i_1$ and $i_2$ connected in $G^{(K)}$. Because the link is absent in network $G^{(K-1)}$ but feasible, and the current node degrees are strictly below $\delta_{i_1}$ and $\delta_{i_2}$ (and thus strictly below $\bar{\delta}_{i_1}$ and $\bar{\delta}_{i_2}$ based on Proposition 2), two firms benefit from forming a link.

- Similarly, an improvement path from network $G^{(K-2)}$ to network $G^{(K-1)}$ is obtained by reducing one more link. Continuing the process of sequentially removing links, we arrive at the empty network $G^{(1)}$.

This proves that if an equilibrium exists with equilibrium degree distribution $(\delta_1, ..., \delta_N)$, there are $(\sum_i \delta_i n_i/2)!$ improvement paths to it from an empty network. Moreover, any network with node degrees element-wise less than or equal to $(\delta_1, ..., \delta_N)$ lies on some improvement path from the empty network $G^{(1)}$ to the pairwise stable network $G^{(K)}$. Lastly, we find all stable networks by starting from an empty network $G^{(1)}$ and sequentially adding links until there are no deviations and $\delta_i \leq \bar{\delta}_i$ for all $i$. Because the number of such outcomes is final and non-empty, the set of equilibrium networks is non-empty.

*Proposition* 3. The set of pairwise stable networks is non-empty.

Moreover, we can evaluate how fast the algorithm converges. The sequential adoption algorithm converges to a pairwise stable equilibrium within $\sum_{i=1}^N \bar{\delta}_i n_i/2 \leq N(N-1)/2$ steps, which is an upper limit on the number of links that can be added. Clearly, convergence to efficient pairwise stable networks is longer in this algorithm in terms of the number of operations.

## 4.5 Desired vs equilibrium adoption rates

So far, we focus on the desired adoption rate $\delta_i^*$ of each firm $i$.[16] For any assigned probabilities (11), we show that the equilibrium adoption rate is stochastically increasing in the desired adoption rate. This makes our comparative statics results for $\delta_i^*$ relevant for the game outcome.[17]

*Proposition* 4. For two payment games $A$ and $B$ with equal trade networks $L^A = L^B$, different desired adoption rates $\delta_A^* \leq \delta_B^*$, and any probability distributions

[16]The discrete version of it, $\hat{\delta}_i$, is, by definition, an increasing function of $\delta_i^*$.

[17]Generally, a different number of equilibrium networks in games A and B is the reason why the equilibrium adoption rate does not increase in the sense of point-wise stochastic dominance, and only in first-order stochastic dominance.

$F^A(\cdot,\cdot)$ and $F^B(\cdot,\cdot)$ defined over the improvement paths of games $A$ and $B$, realisation of the equilibrium adoption rate $\delta(\delta_A^*)$ is first-order stochastically dominant over adoption rate $\delta(\delta_B^*)$.

See Appendix C for the proof.

# 5 Empirical exercise: Canadian firms

To apply our model empirically, we first address the common problem of availability of interfirm data.[18] In the literature, interfirm networks are often analysed using sectoral input-output matrices (e.g., Acemoglu et al. (2012)). In this study, we go beyond the "one sector-one firm" assumption by combining Canadian input-output data with large-value payments data reported by the banks that clear the interfirm payments.[19] In particular, we simulate a network of trade relationships by assigning each firm two attributes: the economic sector it operates in and the corresponding bank that processes its payments. First, nodes of trade network $L$ are created in proportions resembling the size of sectors and bank shares. Each node in $L$ represents a firm. To create the first link, we randomly match two banks in the payment system according to the interbank probability matrix. We next assign probabilistically the link to two industries according to the input-output matrix (see Figure 3).[20] We then repeat the link-creation process until the average incoming degree of a node matches the exogenous moment. We rely on the

[18] For example, see supplier-buyer networks derived from Compustat reports in the United States, where firms are required to report sales to customers that account for 10% or more of annual sales (see Graham and De Paula (2020), Cohen and Frazzini (2008), and Atalay et al. (2011)).

[19] Payments made between firms that are serviced by the same bank are cleared by this bank internally and thus are not observed in the payment system data. To account for these payments, we fill in the diagonal of the interbank matrix by assuming that, for each bank, the proportion of received payments directed to itself is the same as the proportion of all payments directed to this bank in the financial system. Also, flows of small banks that indirectly participate in the payment system are absorbed into larger banks' flows due to data limitations (Chapman et al. (2011)).

[20] In simulations, two firms are allowed to have identical sectors and banks and can even be connected with each other.

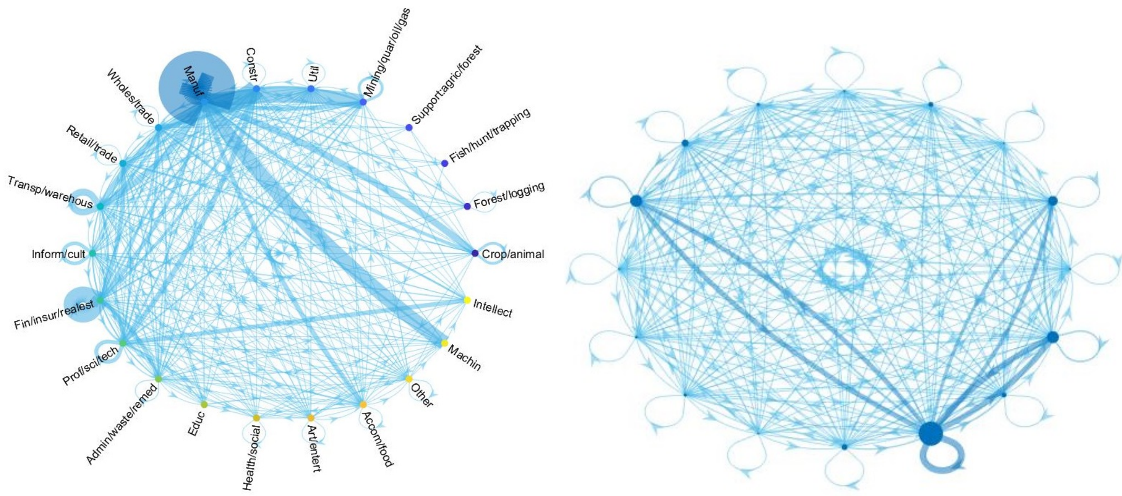


Figure 3: Left figure: observed input-output network for 22 economic sectors of Canada (source: Statistics Canada). Right figure: large-value payments flows between all participating banks; non-loops are observed in the Lynx payment system for 2022Q3 (source: Payments Canada and Bank of Canada), while loops are the authors' calculations. In the figure, each node size represents unweighted network degree and is irrelevant to the size of the bank.

assumption that the average indegree in Canadian interfirm networks is 2.5, which is in line with Canadian data observations.[21] Figure 4 depicts the resulting trade network.

We next use the results of Deloitte's Global Blockchain Survey to calibrate the preferences of firms for payment services.[22] In 2018–2021, the survey asked more than 1,400 senior executives and practitioners in 14 countries and regions about blockchain, digital assets, and distributed ledger technology. We are interested in the percentage of firms that identify blockchain as their top priority in the coming two years. We also focus on firms that identify payments as a blockchain use case. With the independence assumption, we use the product of the two likelihoods to find the desired cryptocurrency adoption rate $\delta^*$.

[21] It also stays within the range of 2–7 links per firm reported in Matous and Todo (2016, 2017) for Japanese firms, and Welburn et al. (2020) for U.S. firms.

[22] See Delloite Blockchain Survey Report.

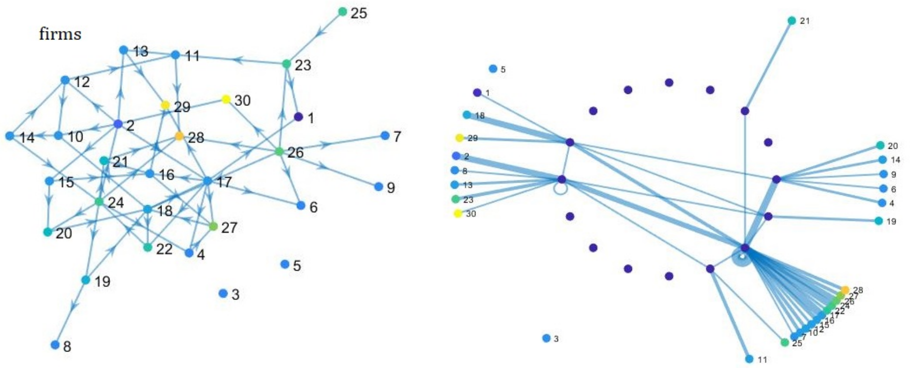


Figure 4: Simulated network with 30 firms (left: firms only, right: firms and corresponding banks in the payment system). Colours of firm nodes indicate industries as in Figure 3. Bank nodes are unlabelled and form a circle representing a large-value payment system.

Empirically, it is unclear whether there is a trend in the overall costs of using crypto for payments, given that transaction fees, price volatility, and transaction time provide conflicting evidence (see Table 1). So, we conduct a counterfactual analysis of a rather stylised nature, assuming the desired adoption $\delta^*$ of all firms increases from the 24.8% reported in the last survey to an arbitrary 35% (see Table 1). If privacy preferences $\pi$ and $\bar{p}$, fixed cost $c_0$, and cost advantage of banks $d-e$ are kept unchanged from the last data observation, according to equation (1), increase in demand is identical to the change in marginal cost from $c = d-\lambda(d-e)$ to $c = d - 1.32\lambda(d-e)$ for some $\lambda \in (0,1)$. This means that marginal crypto cost $c$ becomes 1.32 times further from the cost of using cash $d$ as from the cost of banking payments $e$.[23] For comparison, if $\lambda = 1$, the cost of crypto would become as cheap as sending a payment through an interbank network (e.g., SWIFT), while if $\lambda = 0$, crypto would be very costly at the margin and identical to transferring cash.

Finally, we run simulations according to our classical algorithm to find the equilibrium networks. Based on the counterfactual scenario, firms that are not coordinated on average adopt the new method of payment for only 24% of contracts,

[23]This result follows from the definition of $\lambda = \frac{d-c}{d-e}$, which is identical to $\frac{d-e-(3+\pi_i-4c_0)\bar{p}_i}{2-3\delta^*}$.

| | desired adoption $\delta^*$ | transaction fee | volatility of price (CV) | transaction time |
|---|---|---|---|---|
| | % | usd | % | minutes |
| Observed: 2018 | 12.9 | 6.68 | 0.23 | 6.39 |
| Observed: 2019 | 19.6 | 0.28 | 0.05 | 8.23 |
| Observed: 2020 | 16.5 | 0.67 | 0.17 | 8.95 |
| Observed: 2021 | 24.8 | 16.19 | 0.22 | 8.12 |
| Scenario: 2022 | 35.0 | – | – | – |

Table 1: Empirical observations about cryptocurrency adoption and costs and desired adoption rate in the counterfactual scenario.

while they would like to adopt it for 35% of contracts (see Figure 5). The "price of anarchy" (Koutsoupias and Papadimitriou (2009)) arises from the discrete nature of the adoption process, the network externalities, and a mismatch in incentives of firms in adoption timing.[24] The intuition of the last two effects is the most interesting: the responses of banks to the crypto-adoption process limit its progress among firms. With more adoption, banks are willing to provide discounts to firms on corresponding banking services. As a result, firms that adopt crypto first will have a higher likelihood of being matched with other adopters, which leaves late adopters unfavourable.

The position of firms in the trade network plays an important role for adoption and usage. Network externalities also mostly impact firms with fewer contracts (see Figure 6). However, high trading activity does not protect a firm from the negative impact of network externalities. There is a positive probability that even firms with 7–8 trade partners may not find any reciprocity of trade partners to adopt cryptocurrency if they are late making the adoption decision (because the monopoly rents of banks will be sufficiently reduced by then).

[24]On average, a firm would like to adopt cryptocurrency as a method of payment with less than one trading partner out of 2.5 average trading partners: $\frac{\sum n_i}{N}\delta_i = 0.875 < 1$. This means that due to the discrete nature of contracts, only 21 out of 30 firms are interested in adopting cryptocurrency, despite equal $\delta^*$.

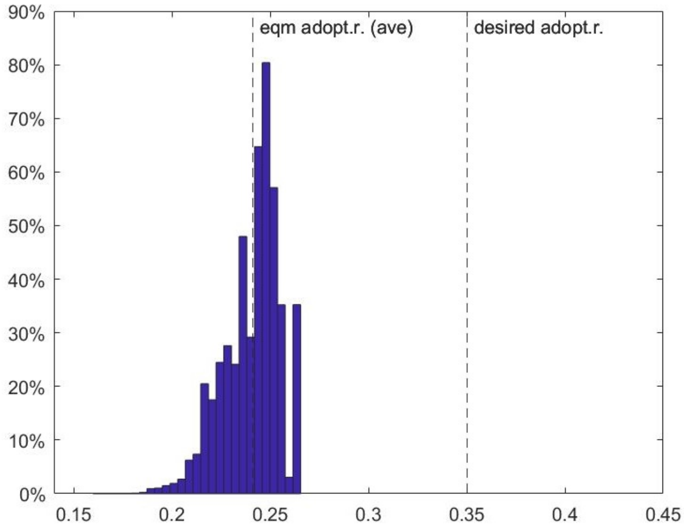


Figure 5: Desired adoption rate $\delta^*$ vs equilibrium adoption rate distribution.

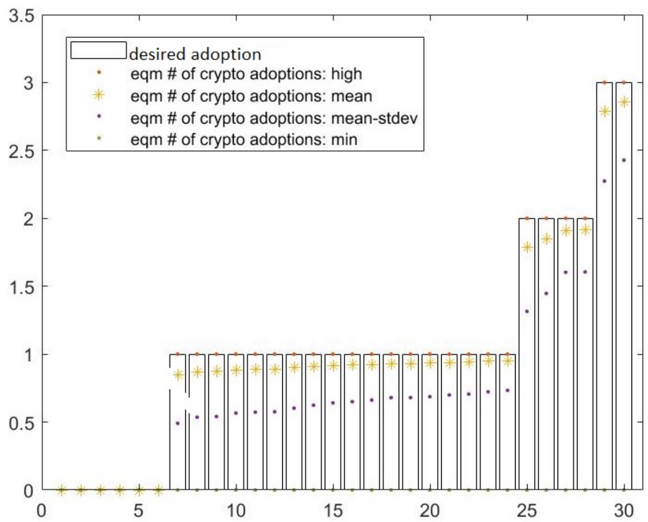


Figure 6: Equilibrium adoption rates.

Finally, cryptocurrency adoption is more likely to take place between well-connected firms. However, we do not find a similar prediction with regard to the sector of the firm and the bank that processes the payments. As Figure 7 shows, the new currency adoption decision is specific to the sector and the firm's bank.

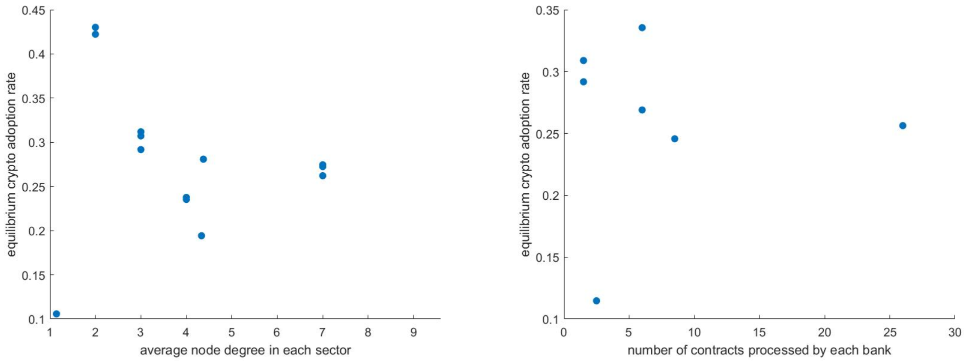


Figure 7: Average equilibrium adoption rate in relation to the centrality of economic sector and the bank that processes the payments.

However, simple degree of sector and popularity of the bank are not sufficient statistics to explain the variation. This highlights the importance of complex network topology for evaluating the impact on bank balance sheets and payment system.

# 6 Network payment game as quadratic optimisation

## 6.1 Quadratic optimisation on modern technology

In this section, we convert the original game in Section 3—namely its initial network matrix, utility functions, and network formation rules—into a quadratic function that maps from binary inputs into a single real-valued output. This makes the payments game feasible for modern technology, such as quantum computing. Having a unified framework for solving network formation games allows for various empirical applications and avoids numerical complexity when placing realistic behavioural assumptions in these models. In addition, the unique equation for a network formation game opens up the possibility of structural estimations of this game without the need to match many node-specific moments or check the behavioural choice of every single player, although this is beyond the scope of this paper.

Several quantum and quantum-inspired devices and algorithms for classical computers demonstrate potential in solving optimisation problems. A promising emerging technology in this domain is quantum computing, which has been explored extensively for such applications (Abbas et al. (2023)), including in finance and economics (Fernández-Villaverde and Hull (2023); McMahon et al. (2022); Orus et al. (2019)). For example, quantum annealers—specialized quantum computing devices designed for solving QUBO problems—have garnered significant attention (Yarkoni et al. (2022)). Recent studies highlight the efficiency of quantum annealers in these contexts. For example, King et al. (2024, 2023) have recently claimed an advantage to using quantum annealers over other known computing methods for solving optimisation problems in material science. Moreover, the push towards using quantum computing for optimisation challenges has spurred the creation of "quantum-inspired" innovations. These innovations use classical computing rules but incorporate ideas from quantum computing. This includes various algorithms such as tensor networks (Mugel et al. (2022)), and various new computing hardware such as in Kowalsky et al. (2022) or Meirzada et al. (2022). Although we do not investigate quantum-inspired methods in this paper, these technologies accept QUBO problems and can be used instead of the quantum annealer to possibly offer computational advantage for some problems, at least in practice.

## 6.2 Quadratic representation of the payment network game

For each network matrix of zeros and ones, $G$, we use its vector representation $g \in V_g$. Here $V_g$ denotes the space of all possible network configurations, with dimension $|V_g| = 2^{|B_g|}$, where $B_g = \{e_1, \ldots, e_{|B_g|}\}$ is a canonical basis of $V_g$, and $|B_g| = N(N-1)/2$ is the total number of links that can be formed between different pairs of nodes. In the canonical vector, each element corresponds to a network graph with a single link. Then any network vector $g$ is a linear combination of arrays in $B_g$.

We next show that it is possible to complement $g$ with a few more binary variables to form an argument vector for which the payment game is specified as binary quadratic optimisation.

**Theorem 5.** *There exists some number of additional variables, $m$, and a real-valued quadratic function*

$$H : x \in [0, 1]^{|B_g|+m} \to R,$$

*such that network $g$ is pairwise stable if and only if it delivers minimum to quadratic function $H$*

$$(g, v) = argmin_x H(x)$$

*together with some vector $v \in [0, 1]^m$.*

In our approach, quadratic form is used as an optimiser. This makes the mathematical problem well behaved and aligns all equilibrium network solutions on a single quadric hypersurface (such as ellipsoid or parabaloid), which are easy to handle computationally. This allows us to find all equilibrium networks by solving a single optimisation problem.

Our approach resembles the potential function technique (Rosenthal (1973), Monderer and Shapley (1996)) previously applied to network games (Bramoullé et al. (2014), Tardos and Wexler (2007)). In both methods, the extreme value of the optimisation function is achieved in the equilibrium network. However, differently from the potential function, an increase in $H$ does not imply an increase in marginal utilities, apart from when considered in equilibrium. Also, only a limited number of network games can be presented as potential games, with our approach being more general (see Section 8). In particular, when utility functions are quadratic in strategies, potential function can be considered a special case of $H$ in Theorem 5 given $m = 0$.

We prove Theorem 5 for our game by explicitly defining $x = (x^g, x^\delta, x^e, x^{sg}, z^1, z^2)$ and $H$. This specification is different than the generalized one in Section 8 to illustrate how the specifics of a utility function can be used to compose an efficient QUBO and improve the speed of funding an equilibrium.

- $x^g$: the first $|B_g|$ elements of $x$ model network $g$, which we want to be pairwise stable: if $H(x) = 0$, then $x^g = g$ is an equilibrium. If $x^g_j = 1$ for some $j = 1, ..., |B_g|$, link $e_j$ exists in the network. If $x^g_j = 0$, no such link exists.

- $x^\delta$: the sufficient statistic for the utility function of each player given network $g$. In our case, it is equal to the vector of node degrees:

$$x^\delta = (x^\delta_1, ..., x^\delta_N).$$

  If subvector $x^\delta_i \in \{0, 1\}^{n+1}$ has only the $k$'th element being 1 and other elements being 0, it assigns node degree $\frac{k-1}{n}$ to player $i$.[25]

- $x^e$ and $x^{sg}$: vectors that indicate whether each node $i$ is strictly above or at the desired adoption level $x^{sg}_i = 1$ if and only if $\delta_i(x^g) > \bar{\delta}_i$, $x^e_i = 1$ if and only if $\delta_i(x^g) = \bar{\delta}_i$, and zero otherwise.

- $z^1$ and $z^2$: slack variables that do not carry additional meaning and will be defined later.

We first model network formation rules. For each edge, assign label *left* to the node with the smaller index and label *right* to the node with the larger index. Define binary matrices $\nu^L$ and $\nu^R$, such that $\nu^L_{ij} = 1$, $\nu^R_{kj} = 1$ if and only if $i$ is the left and $k$ is the right node of edge $e_j$.

We record the incentives of left and right players to add or remove link $e_j$ by defining a function for the signs of marginal utilities—linear representations of vectors $x^{sg}$ and $x^e$:

$$x^{La}_j = \sum_i (1 - x^{sg}_{ij} - x^e_{ij})\nu^L_{ij}, \quad x^{Ra}_j = \sum_i (1 - x^{sg}_{ij} - x^e_{ij})\nu^R_{ij},$$

[25]In the optimisation problem, vector $x^\delta_i$ is not explicitly restricted to have a single 1 element and thus be a canonical vector. Instead, it can be any of $2^{n+1}$ binary combinations. When defining $H(x)$, we limit values of $x^\delta_i$ to be a canonical vector in the equilibrium by incorporating penalising terms.

$$x_j^{Lr} = \sum_i x_{ij}^{sg} \nu_{ij}^L, \quad x_j^{Rr} = \sum_i x_{ij}^{sg} \nu_{ij}^R.$$

Then we can present network formation rules with the quadratic function of $x$, which is positive whenever the network is unstable. This means that two nodes benefit from adding a link between them, or at least one node benefits from a link removal:

$$H^{NF} = \sum_j \left( x_j^{La} x_j^{Ra} + x_j^{Lr} + x_j^{Rr} \right).$$

Because variables $x^\delta$, $x^{sg}$, and $x^e$ are interdependent, we introduce function $H^U$ that is positive whenever the values of these variables are not consistent, and zero if the consistency is reached.

$$H^U(z^1, z^2, x^e, x^{sg}, x^\delta) = \sum_i \left( z_i^1 + \bar{\delta}_i n_i x_i^e + (\bar{\delta}_i n_i + 1) x_i^{sg} + z_i^2 - \eta_i x_i^\delta \right)^2$$
$$+ z_i^1 x_i^e + z_i^2 (1 - x_i^{sg}) + x_i^e x_i^{sg} \tag{12}$$

When slack variables $z^1 \in R$ and $z^2 \in R$ are selected to solve minimisation problem

$$\min_{z_i^1, z_i^2} H^U(z^1, z^2, x^e, x^{sg}, x^\delta)$$

subject to constraints

$$z_i^1 \in [0, \bar{\delta}_i n_i - 1] \tag{13}$$

$$z_i^2 \in [0, n_i(1 - 2\bar{\delta}_i)] \text{ if } \bar{\delta}_i < 1/2 \tag{14}$$

and $\eta_i$ being defined as vector

$$\eta_i = [0, 1, ..., n_i - 1, n_i],$$

we can easily show that the only way for conditions $H^U = 0$ and $\eta_i x_i^\delta = \bar{\delta}_i n_i$ to hold simultaneously is to have $z_i^1 = z_i^2 = x_i^{sg} = 0$ and $x_i^e = 1$, so consistency between $x^e$, $x^{sg}$ and $x^\delta$ is reached. When $\eta_i x_i^\delta > \bar{\delta}_i n_i$, then $x_i^{sg} = 1$, $x_i^e = 0$, and $z_i^1$ and $z_i^2$ are selected to make up the residual $z_i^1 + z_i^2 + 1 = \eta_i x_i^\delta - \bar{\delta}_i n_i$.[26] The conditions

[26] Assuming $x_i^e > 0$ when $\eta_i x_i^\delta > \bar{\delta}_i n_i$ leads either to the positive second line of the equation (12) and thus $H^U > z_i^1 x_i^e + z_i^2(1 - x_i^{sg}) + x_i^e x_i^{sg} > 0$, or the zero second line of the equation (12), $x_i^{sg} = z_i^1 = z_i^2 = 0$, the positive first line of (12), and thus $H^U > 0$.

in the second line of (12) are used to enforce when $z_i^1$ and $z_i^2$ can be non-zero: the first term specifies that $z_i^1 = 0$ when $x_i^e = 1$; the second term specifies that $z_i^2 \neq 0$ only when $x_i^{sg} = 1$; and the third term specifies that only either $x_i^{sg}$ or $x_i^e$ can be ones at any time.[27]

In conclusion, solving for equilibrium in the network formation game is identical to minimising $H$ function with respect to variables $x^g, x^\delta, x^e, x^{sg}, z^1, z^2$

$$H = H^U + H^{NF}$$

subject to constraints (13)-(14) and single selection constraint, $\sum_j x_{ij}^\delta = 1$ for each player $i$.

# 7 Applications using quantum computing

## 7.1 Introduction to quantum computing

Quantum computing applies the laws of quantum mechanics to perform computation. Quantum computing outperforms classical computing in various quantum algorithms—a situation known as "quantum advantage." For most of them, the advantage is proven theoretically (Jordan (2022); Nielsen and Chuang (2010)), but empirical demonstrations now also exist (Madsen et al. (2022)). Quantum advantage is expected to materialise further as quantum hardware improves to become fully fault-tolerant. In the meantime, Noisy Intermediate Scale Quantum (NISQ) processors are available. NISQ devices allow for early stage use case exploration and experimentation, and might soon provide quantum advantage for many computational tasks (Huang et al. (2022), Daley et al. (2022)). We rely on a NISQ device called the "quantum annealer" for finding equilibrium networks in the payments game.

In achieving quantum advantage, quantum algorithms most commonly take advantage of *superposition* and *entanglement*. A state of the quantum system is a

[27] The qubit usage is optimized by allowing for the lower region values $z_i^1$ to contribute to filling the upper region when $x_i^\delta \eta_i > \bar{\delta}_i$. Then only if $x_i^\delta > 2\bar{\delta}_i n_i$ are additional qubits required for $z_i^2$ to make up the remaining residual.

*superposition* of multiple other states if it is a linear combination of these states. For the simple two-state system called the "qubit" (a quantum analogue of the classical bit in the classical computer), the most general state can be expressed as a linear combination of the orthonormal basis states $|0\rangle$ and $|1\rangle$. Mathematically, the two basis states can be expressed as vectors $|0\rangle = [1, 0]$ and $|1\rangle = [0, 1]$ in the complex space $\mathbb{C}^2$. Therefore, the statement that a qubit in the state $|\psi\rangle$ is a superposition of states $|0\rangle$ and $|1\rangle$ is simply the statement that

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle, \; a_0, a_1 \in \mathbb{C}, \; |a_0|^2 + |a_1|^2 = 1. \tag{15}$$

This is unlike a classical bit that is always either in the state "0" *or* "1," but never both at the same time.

The notion of superposition is closely linked to the idea of measurement of a quantum system. While we mentioned that a qubit's state may be in $|0\rangle$ and $|1\rangle$ simultaneously, an observer will only record one of the two states, either $|0\rangle$ or $|1\rangle$, when the state is actually *measured*. The state could be, for instance, an electron with spin up or spin down, or a superconducting current flowing clockwise or counter-clockwise. In our case, a state will often indicate whether a network link is present between two nodes or not.[28]

For superposition to become useful in computation, we must extend this phenomenon from a single qubit to a collection of qubits. This can be achieved thanks to *quantum entanglement*. Two or more quantum systems are *entangled* if the properties of the composite quantum system cannot be described by considering the properties of each subsystem in isolation. Mathematically, a state in a composite Hilbert space $\mathcal{H}$ is entangled whenever it is not a tensor product of quantum states of Hilbert spaces composing it:

$$\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2 \text{ is entangled} \iff \psi \neq \phi_1 \otimes \phi_2 \text{ for any } \phi_1 \in \mathcal{H}_1, \; \phi_2 \in \mathcal{H}_2. \tag{16}$$

[28]How or why the superposition happens is not specified within the mathematics of quantum mechanics, although a plethora of philosophical explanations are proposed (Friebe et al. (2018); Jammer (1974)). However, the theory does provide the probability of the state ending up in any of its possible states. For instance, in (15) the probability of ending up in state $|0\rangle$ is the squared scalar product between vectors $\psi$ and $\langle 0|$, namely $|a_0^2|$, and the probability of ending up in state $|1\rangle$ is analogously $|a_1|^2$. Notice that quantum states are taken to be unit vectors in the first place so that the total probability is preserved $|a_0|^2 + |a_1|^2 = 1$.

In practice, this makes it possible to link together qubits such that the whole is greater than the sum of its parts. For instance, two qubits can be entangled such that measuring "0" on the first qubit will force the second qubit to be also "0."[29]

Because qubits cannot be considered separately, the dimension of the composite space of N qubits is

$$\dim(\mathcal{H}_1 \otimes ... \otimes \mathcal{H}_N) = \dim(\mathcal{H}_1) \cdot ... \cdot \dim(\mathcal{H}_N) = 2^N,$$

In practice, it means that with $N \geq 300$ qubits, the quantum computer can handle more computational states than the number of atoms in the universe. In contrast, classical bits of the same size will cover a composite space of size N only.

## 7.2 Quantum annealing

We use a D-Wave quantum annealer, which is designed to solve quadratic unconstrained binary optimisation (QUBO) problems.[30] Thus, the challenge of applying quantum computations to network formation games is in representing such games in terms of the quadratic function (often referred as *Hamiltonian*):

$$H(Q, x) = x'Qx\,, \tag{17}$$

where $Q$ is a triangular real-valued matrix and $x$ is an unknown binary vector.[31] In this case, solving the game would be equivalent to finding the *ground state* of the Hamiltonian $H(Q, x*)$ and all solutions:

$$x^*(Q) = argmin_x H(Q, x).$$

[29] This is an instantaneous effect that can happen over arbitrary distances between qubits. See, for instance, the recent physics experiments of The BIG Bell Test Collaboration (2018) and Li et al. (2022).

[30] We report results for D-Wave's quantum-classical hybrid solver Kerberos, based on the D-WAVE recommendation for problems with a high degree of connectivity. We obtain similar results when using hybrid solve LEAR. Currently, both of these hybrid solvers yield better performance than direct QPU (quantum programming unit) sampling for our applications. However, this may not be the case in the future as quantum hardware continues to improve.

[31] In practice, the Hamiltonian also includes another term corresponding to the transverse magnetic field, which is added to control the temperature of hardware.

This adiabatic quantum computation takes advantage of the *adiabatic theorem* (Born and Fock (1928)). In particular, the system starts in the ground state of some initial Hamiltonian, $H(Q_0, x)$, such that the ground state $x_0^*$ can be easily and reliably prepared using hardware. Then, if the coefficients of the Hamiltonian $Q_0$ are deformed into $Q$ adiabatically and slowly enough, the system will end up in the ground state of $H(Q, x^*)$.[32] In this context, *quantum annealing* is the physical process of implementing adiabatic quantum computation in real-life devices (Kadowaki and Nishimori (1998)).

In *adiabatic quantum computation*, the challenge is to reformulate a given problem as the Hamiltonian (Albash and Lidar (2018); Farhi et al. (2000)). In addition, because the current quantum hardware is noisy, the quantum optimiser may mistakenly find near-equilibrium $\hat{x}$, with $H(Q, \hat{x})$ being close to the ground state $H(Q, x^*)$. To eliminate the need to post-process the quantum outcomes and distinguish $x^*$ from $\hat{x}$, we also require a specific (zero) ground state Hamiltonian value for the equilibrium networks, $H(Q, x^*) = 0$, so that we can discard results with positive values of $H$ without checking the equilibrium conditions.

## 7.3 Applications of quantum computing to networks

### Full adoption scenario

We begin quantum applications by considering simple cases with only a few mutually-traded symmetric firms to document that quantum technology can handle well the exponentially growing complexity of network games. For comparison, we rely on the classical convergence algorithm in Section 4.4 to supply the equilibrium networks in each case.

We first focus on firms with high desired adoption $\delta^* > (N-1.5)/(N-1)$, which guarantees a unique pairwise stable (fully-connected) network and is therefore ideal for comparing search outcomes across different market sizes. We find that the

[32] Adiabatic means that no heat leaves or enters the system. In practice, one usually chooses the time of deformation to be $\mathcal{O}(\Delta^2)$ with $\Delta$ being the energy difference between the ground state and the second minimum value of $H(Q, x)$. In practice, the probability of ending in the ground state depends on how slowly $Q_0$ is deformed into $Q$ and on the difference in $\Delta$.

quantum computer handles well the growing network complexity. For instance, in the markets with eight firms, the quantum search delivers minimum Hamiltonian $H(Q,x) = 0$ in 2% of cases, in comparison to the likelihood of $10^{-8}$ if a network is selected at random. The distribution of the quadratic function $H$ for each network outcome reveals that despite finding 98% of networks with positive Hamiltonian, the quantum computer is more likely to find networks with only a few deviations away from the equilibrium (see Figure 9 for eight firms). For instance, networks with only one and two improving deviations are sampled about 10% and 28% of the time. This indicates that the nature of error in quantum search is hardware-based rather than algorithmic, and is likely to improve when the precision of quantum machines improve. Moreover, while most quantum search outcomes deliver $H > 0$, we should not interpret this fraction as an ultimate error rate. We observe the Hamiltonian generated by each network and thus can dismiss networks with $H > 0$ when selecting the output. Despite the quantum likelihood being confusing, it is still a useful statistic to understand the scaling properties of the quantum method. In particular, according to Figure 8, when the market size is expanded from 7 to 10 players, the quantum likelihood only slightly deteriorates from 10% to 1%, while the exhaustive likelihood decreases from the order of $10^{-6}$ to tiny $10^{-13}$, indicating exponential growth in the complexity of network problems.

**Pairwise matching scenario**

We next consider the case with low desired adoption $\delta^* = 1/(N-1)$, meaning each firm desires to form one connection. In this setup, the stable networks are given by the sets of (maximum) matching pairs. Thus, the number of stable states grows with the number of players in the network. For quantum, this leads to a sizeable increase in the ground state probability compared to the case of a single equilibrium network (e.g., 94% vs 1% for 8 firms, as shown in Figure 8), and even greater advantage relative to a random search. This case is especially promising for quantum given that finding all pairwise matching problems is unrealistic in the polynomial time on a classical machine.[33]

[33]Finding such pairs is a |#|P-Complete problem for non-planar graphs (Valiant (1979)).

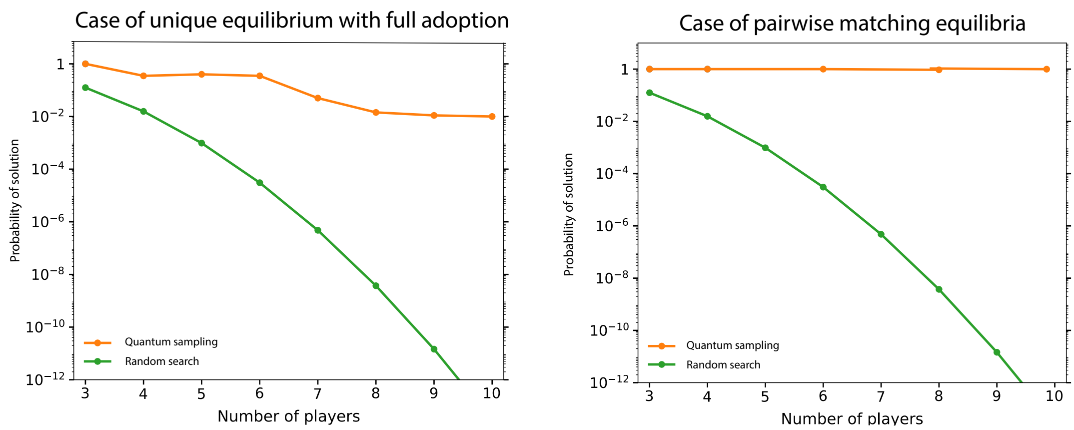


Figure 8: Probability of a network being an equilibrium in a uniform sampling vs observed likelihood of (ground state) equilibrium being found using a D-WAVE quantum computer.

### Non-monotonicity of equilibria search

We report two additional exercises in Table 2 to test scaling properties of quantum search. First, for five fully-connected symmetric firms, we sequentially increase the desired adoption rate $\bar{\delta}$ to show that the quantum likelihood is non-monotonic in the number of equilibria. In the next set of applications in Table 2, we keep $\bar{\delta}$ the same but increase the market size to demonstrate that quantum likelihood is non-monotone in the size of the network. In addition, there is a non-monotonicity with respect to the expected efficiency loss (from mis-coordination in networks) and heterogeneity among firms, which grows non-monotonically with the network size.[34]

### Data-driven application

We apply quantum algorithm using a D-Wave Quantum Annealer to the empirically-calibrated problem from Section 6.2. We find a stable network with 3.1% likelihood

[34]The time to run a single simulation is between 20s for a fully connected network with 4 nodes to 55s for a fully connected network with 10 nodes.

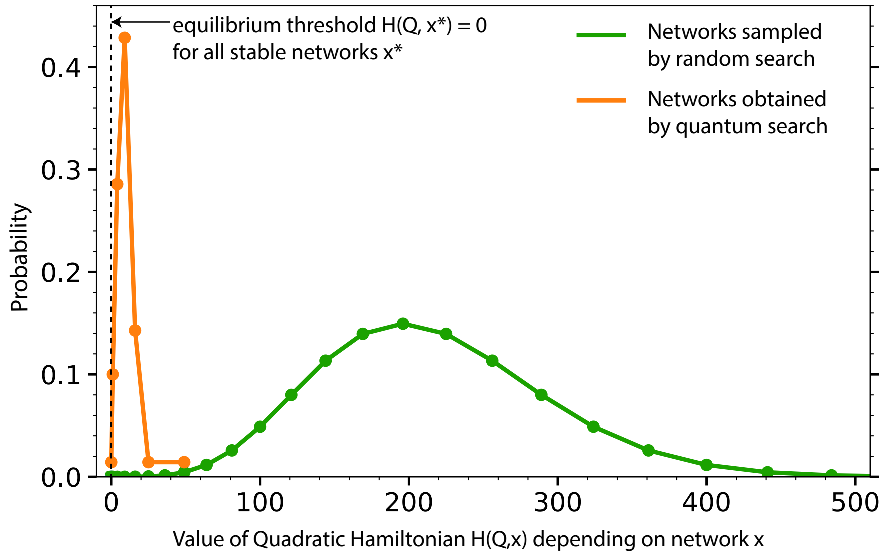


Figure 9: Empirical distribution of Hamiltonian value depending on the sampled network in the full adoption scenario.

| Number of firms N | Adopted contracts Desired | E(eqm) | SD(eqm) | Number of eqm networks | Equilibrium search Exhaustive likelihood | Quantum likelihood | Quantum 95% conf. int. |
|---|---|---|---|---|---|---|---|
| 5 | 1 | 0.80 | 0.40 | 15 | 0.015 | 0.83 | (0.74,0.93) |
| 5 | 2 | 1.72 | 0.60 | 37 | 0.036 | 0.89 | (0.86,0.93) |
| 5 | 3 | 2.74 | 0.60 | 35 | 0.034 | 0.55 | (0.50,0.61) |
| 4 | 1 | 1.00 | 0.00 | 3 | 0.047 | 0.26 | (0.14,0.38) |
| 7 | 2 | 1.26 | 0.19 | $> 250$ | $< 1.2 \times 10^{-4}$ | 0.98 | (0.97,0.99) |
| 10 | 3 | 1.94 | 0.12 | $> 500$ | $< 1.4 \times 10^{-11}$ | 0.83 | (0.78,0.88) |

Table 2: Results of equilibrium search on quantum annealer for small fully connected networks.

and a confidence interval of $[0.022, 0.041]$.[35] The found networks are representative and no bias is observed in which networks are selected. This confirms that quantum computing can be used successfully to generate predictions of similar network formation games.

# 8 Quadratic representation of network games

## 8.1 Decoding utility functions

In general network games, defining the utility function $u_i(g)$ of player $i$ requires defining $|V_g|$ numerical values. Equivalently, we can use sign matrices $S^+$ and $S^-$ of marginal utilities, with element $ij$ indicating whether player $i$ strictly benefits from adding or removing link $e_j$.[36]

$$s_{ij}^+(g) = H(u_i(g + e_j) - u_i(g)), \quad s_{ij}^-(g) = H(u_i(g - e_j) - u_i(g)), \tag{18}$$

where $H$ is the Heaviside function (1 if its argument is positive and 0 otherwise).

Future quantum computers are promised to deal efficiently with high dimensions of $|V_g|$. Currently, symmetry and sufficient statistics of utility functions can be used to reduce the computational complexity.

Example 1. Consider the special case where $u_i$ is linear in $g$. The payoff of player $i$ is a linear product of network vector $g$ and some numerical vector $u_i^v$:

$$u_i(g) = \langle g | u_i^v \rangle = \sum_{i=1}^{|B_g|} w_i \langle e_i | u_i^v \rangle. \tag{19}$$

Knowing values $\langle e_i | u_i^v \rangle$ is sufficient to define the payoffs. Because the number

[35] Out of 1,400 simulations, 44 networks are found. Based on classical simulations, the number of stable networks exceeds 48,000.

[36] The case of indifference is ignored for simplicity, assuming the addition and removal of a link does not leave deviating players indifferent. For more complicated utility functions, cases of zero value of $u_i(g + e_j) - u_i(g)$ should be treated separately by introducing another indicator function for the addition of a link. Introduction of such a function for removal is not necessary because according to the network formation rules, the deviating coalition must strictly benefit from a link removal.

of such values $|B_g|$ grows with the number of nodes $N$ as a quadratic polynomial, the reduction in the computational difficulty is significant compared to the exponentially growing number of payoff values $|V_g|$.[37]

Example 2. Similar logic can be applied if $u_i$ is a non-linear function of the linear sufficient statistic $u_i = u_i(\delta_i(g))$:

$$\delta_i(g) = \langle g|\delta_i^v\rangle = \sum_{i=1}^{|B_g|} w_i \langle e_i|\delta_i^v\rangle.$$

This again reduces the computational complexity of encoding (18) to $|B_g| < |V_g|$ numerical values, given a mapping vector between networks and states $\chi : (1, ..., |V_g|) \to (1, ..., |p|)$.

Finally, we can reduce complexity in a more general case when we reduce the values of utilities to only $|p|$ and not $|V_g|$ states, with each state linked to one or multiple networks. In this case, $s_{ij}^-(g)$ and $s_{ij}^+(g)$ can be defined as a linear combination of the canonical vector $(b_1^p, ..., b_{|p|}^p)$.

## 8.2 Assembling argument for optimisation function

To compose a QUBO problem (17) with Hamiltonian $H(x)$, we first define its argument $x$ as a combination of subvectors: $x = (x^g, x^\delta, x^d)$. Each element of $x$ corresponds to one qubit, which takes value 0 or 1 when measured.

- $x^g$: the first $|B_g|$ elements of $x$ model network $g$ that we want to be pairwise stable: if $H(x) = 0$, then $x^g = g$ is an equilibrium; if $x_j^g = 1$ for some $j = 1, ..., |B_g|$, link $e_j$ exists in the network, and if $x_j^g = 0$, no such link exists.
- $x^\delta$: binary vector of size $|p|$ determining the state of the network.
- $x^d = (x^{da}, x^{dr}) = (x_1^{da}, ..., x_{|B_g|}^{da}, x_1^{dr}, ..., x_{|B_g|}^{dr})$: the last $2|B_g|$ binary elements of $x$ record the presence of improving deviations by addition (da) and removal (dr) of links. Again, restrictions for such a deviation to be feasible in network $g$ will be imposed as penalising terms in $H(x)$.

[37] We abuse notation $u_i(g)$ when using the same symbol $u$ for the utility function in $u_i(G)$ to not overload the paper with notation.

## 8.3 Logical operations and mapping

Before describing the rest of QUBO formulation, it is first useful to review how to embed common logical operations as quadratic functions.

- **Logical operations AND and OR**

$$\begin{aligned} \text{AND}(a_1,\, a_2,\, \text{b}) &= -2(a_1 + a_2)b + a_1 a_2 + b\,, \\ \text{OR}(a_1,\, a_2,\, \text{b}) &= -2(a_1 + a_2)b + a_1 a_2 + a_1 + a_2 + b\,. \end{aligned} \tag{20}$$

  AND applied to arbitrary $a_1$, $a_2$, and $b$ returns 0 if and only if $b$ is the output of the logical operation "$a_1$ and $a_2$," and some positive number otherwise:

$$min_b \text{AND}(a_1,\, a_2,\, \text{b}) = \text{AND}(a_1,\, a_2,\, a_1 \;\&\; a_2) = 0.$$

  OR returns zero if and only if $b$ is the output of the logical operation "$a_1$ or $a_2$," and some positive number otherwise:

$$min_b \text{OR}(a_1,\, a_2,\, \text{b}) = \text{OR}(a_1,\, a_2,\, a_1 \parallel a_2) = 0.$$

- **Mapping a vector in the canonical basis**
  Another useful operation is mapping a natural number $k$ to the $k$-th element of the given canonical basis $a = (a_1, ..., a_{|V_a|})$. A canonical basis vector $a_k$ has only one unit entry at the $k$-th position, with all other entries being zero. Define the mapping in a functional form

$$\text{MAP}(k, a_j, |V_a|) = \left[ \left( \langle a_j | z \rangle - k \right)^2 + \lambda \left( \langle a_j | a_j \rangle - 1 \right)^2 \right] \,, \tag{21}$$

  where $z = (1, 2, \ldots, |V_a|)$ is a vector of ordered numbers up to the basis size. The mapping is implemented as a minimisation, such that the expression (21) achieves minimum whenever $k$ is mapped to $a_k$:

$$min_{a_j} \text{MAP}(k, a_j, |V_a|) = \text{MAP}(k, a_k, |V_a|) = 0.$$

  The first term is used to obtain the canonical basis vector $a_k$ and the second regularisation term enforces $||a_k|| = 1$; that is, there is only one non-zero entry in the mapped basis vector. Parameter $\lambda$ sets the magnitude of the regularisation term and must be tuned for optimal performance.

## 8.4 QUBO function

Finding all networks with number of deviations $d^*$ is equivalent to finding the minimum of $H(x)$:

$$H(x, d^*) = M(x^g, x^\delta) + E^{NF}(x^\delta, x^d) + FEAS(x^g, x^d) + D(x^d, d^*) \qquad (22)$$

- **Mapping state and network vectors**
  The first term of the Hamiltonian maps game state to the corresponding canonical vector as in (21).
  $$M(x^g, x^\delta) = \mathrm{MAP}(\langle x^g|\chi\rangle, x^\delta, p). \qquad (23)$$

- **Network formation rules**
  For each edge $e_j$ in the basis connecting nodes $i$ and $k$, assign label *left* to the node with smaller index $i, i < k$,, label *right* to the node with larger node index $k$, and define matrix elements $\nu^L_{ij} = 1$ and $\nu^R_{kj} = 1$.
  $$(x^{La}, x^{Lr}) = (x^{La}_1, ..., x^{La}_{|B_g|}, x^{Lr}_1, ..., x^{Lr}_{|B_g|}),$$
  $$(x^{Ra}, x^{Rr}) = (x^{Ra}_1, ..., x^{Ra}_{|B_g|}, x^{Rr}_1, ..., x^{Rr}_{|B_g|}),$$
  which record the incentives of "left" and "right" players to form or remove a link in $g$ if it exists.

  By definition of the sign matrices, the new variables are linear in vector $x^\delta$
  $$x^{La}_j = \sum_i s^+_{ij}(x^\delta)\nu^L_{ij}, \quad x^{Lr}_j = \sum_i s^-_{ij}(x^\delta)\nu^L_{ij},$$
  $$x^{Ra}_j = \sum_i s^+_{ij}(x^\delta)\nu^R_{ij}, \quad x^{Rr}_j = \sum_i s^-_{ij}(x^\delta)\nu^R_{ij}.$$
  Using new notation, implement the network formation rules in terms of the logical operations (20). This is done by introducing two quadratic penalty cost terms for addition and removal of each link $e_j$.
  $$E^{NF}(x^\delta, x^d) = \sum_j AND(x^{La}_j, x^{Ra}_j, x^{da}_j) + \sum_j OR(x^{Lr}_j, x^{Rr}_j, x^{dr}_j). \qquad (24)$$
  This term of Hamiltonian forces $x^{da}$ and $x^{dr}$ to be consistent with the marginal utilities, and thus network $x^g$.

- **Feasibility and existence of deviation**

  The Hamiltonian should be zero only if no deviations exist. For this, we introduce a penalty term that tracks whether the link exists (absent) and a coalition can benefit from removing (forming) the link.

$$FEAS(x^g, x^d) = \sum_j (1 - x^g{}_j) x_j^{da} + x_j^g x_j^{dr}. \tag{25}$$

- **Number of deviations**

  The last term indicates the number of improving deviations of the states of interest in the network,

$$D(x^d, d^*) = \sum_j (x_j^{da} + x_j^{dr} - d^*)^2. \tag{26}$$

Thus, minimisation of Hamiltonian (22) finds networks with any number of desired deviations $d^*$. When number of deviations $d^*$ is set to zero exogenously, we find pairwise stable networks. If the game does not have a pairwise stable equilibrium, we find small improving cycles by setting up $d^*$ to a given number of deviations in the optimisation of (22).

To check the existence of pairwise equilibria of any network formation game, verify whether quadratic equation $H((x^g, x^\delta, \{0\}^{2|B_g|}), 0) = 0$ has solutions in binary $x^g$ and $x^\delta$.

# 9 Conclusion

Our paper makes two major contributions. First, we propose a model for new payment method adoption in firm networks. Our results show that the cooperative bipartite nature of interfirm payment arrangements plays an important role and may lead to the underadoption of the new payment instrument, as well as heterogeneity in adoption decisions of ex-ante symmetric firm. The efficiency loss may be significant in certain trade networks. In the data-driven example, we show that if the desired adoption by each firm is 35% of all trade contracts, the average equilibrium adoption is only around 25%. This calls for further investigation of what can be done to control the adoption of new forms of money. For instance, a designer could impact the adoption by changing the property of the new currency while considering a trade-off between adoption and usage. According to our results, if the new payment method is made more attractive due to lower cost per transaction, it may surprisingly decrease the adoption of the new currency, while increasing the usage of it in equilibrium.

When more realistic assumptions are in place, numerical estimations are necessary for the effect of new payment methods. The quantitative approach may also be more valued by a data-focused regulator. Historically, the bottleneck in applying network formation games to data was the complexity of the possible equilibrium outcomes. We address this problem with our second contribution by showing how multiple networks' equilibria can be found efficiently using a single quadratic optimisation. We think the new methodology is especially valuable in light of the up-rise of quantum computing, which promises significant speed up in the nearest future. As an illustration, we execute the payment network on D-WAVE quantum annealer for the Canadian economy using input-output matrix and large-value payment system data to evaluate the adoption rate of DLT technology in cases when the benefit of transactions will increase enough for firms to get interested. Finally, although our payment model is a network formation game, we believe that our quadratic optimisation results can be extended to various forms of cooperative games to simplify equilibrium search and describe their properties.

# References

Abbas, A., Ambainis, A., Augustino, B., Bärtschi, A., Buhrman, H., Coffrin, C., Cortiana, G., Dunjko, V., Egger, D.J., Elmegreen, B.G., et al., 2023. Quantum optimization: Potential, challenges, and the path forward. arXiv preprint arXiv:2312.02279 .

Acemoglu, D., Carvalho, V.M., Ozdaglar, A., Tahbaz-Salehi, A., 2012. The network origins of aggregate fluctuations. Econometrica 80, 1977–2016.

Afonso, G., Shin, H.S., 2011. Precautionary demand and liquidity in payment systems. Journal of Money, Credit and Banking 43, 589–619.

Albash, T., Lidar, D.A., 2018. Adiabatic quantum computation. Reviews of Modern Physics 90, 015002. URL: https://link.aps.org/doi/10.1103/RevModPhys.90.015002, doi:10.1103/RevModPhys.90.015002.

Allen, F., Gale, D., 2000. Financial contagion. Journal of Political Economy 108, 1–33.

Alvarez, F., Lippi, F., 2009. Financial innovation and the transactions demand for cash. Econometrica 77, 363–402.

Anand, K., Van Lelyveld, I., Banai, Á., Friedrich, S., Garratt, R., Hałaj, G., Fique, J., Hansen, I., Jaramillo, S.M., Lee, H., et al., 2018. The missing links: A global study on uncovering financial network structures from partial data. Journal of Financial Stability 35, 107–119.

Anderson, H., Erol, S., Ordoñez, G., 2020. Interbank networks in the shadows of the federal reserve act. Technical Report. National Bureau of Economic Research.

Andolfatto, D., 2021. Assessing the impact of central bank digital currency on private banks. The Economic Journal 131, 525–540.

Anshelevich, E., Dasgupta, A., Kleinberg, J., Tardos, É., Wexler, T., Roughgarden, T., 2008. The price of stability for network design with fair cost allocation. SIAM Journal on Computing 38, 1602–1623.

Arute, F., Arya, K., Babbush, R., Bacon, D., Bardin, J.C., Barends, R., Biswas, R., Boixo, S., Brandao, F.G., Buell, D.A., et al., 2019. Quantum supremacy using a programmable superconducting processor. Nature 574, 505–510.

Atalay, E., Hortacsu, A., Roberts, J., Syverson, C., 2011. Network structure of production. Proceedings of the National Academy of Sciences 108, 5199–5202.

Babus, A., 2016. The formation of financial networks. The RAND Journal of Economics 47, 239–272.

Babus, A., Kondor, P., 2018. Trading and information diffusion in over-the-counter markets. Econometrica 86, 1727–1769.

Bala, V., Goyal, S., 2000. A noncooperative model of network formation. Econometrica 68, 1181–1229.

Barabási, A.L., Albert, R., 1999. Emergence of scaling in random networks. Science 286, 509–512.

Baumol, W.J., 1952. The transactions demand for cash: An inventory theoretic approach. The Quarterly Journal of Economics 66, 545–556.

Bech, M.L., Chapman, J.T., Garratt, R.J., 2010. Which bank is the "central" bank? Journal of Monetary Eonomics 57, 352–363.

Bech, M.L., Garratt, R., 2017. Central bank cryptocurrencies. BIS Quarterly Review September .

Belleflamme, P., Bloch, F., 2004. Market sharing agreements and collusive networks. International Economic Review 45, 387–411.

Bloch, F., Jackson, M.O., 2006. Definitions of equilibrium in network formation games. International Journal of Game Theory 34, 305–318.

Blume, L.E., Easley, D., Kleinberg, J., Tardos, É., 2009. Trading networks with price-setting agents. Games and Economic Behavior 1, 36–50.

Born, M., Fock, V., 1928. Beweis des adiabatensatzes. Zeitschrift für Physik 51, 165–180.

Bramoullé, Y., Galeotti, A., Rogers, B.W., 2016. The Oxford handbook of the economics of networks. Oxford University Press.

Bramoullé, Y., Kranton, R., D'amours, M., 2014. Strategic interaction and networks. American Economic Review 104, 898–930.

Bräuning, F., Fecht, F., 2017. Relationship lending in the interbank market and the price of liquidity. Review of Finance 21, 33–75.

Bäumer, E., Gisin, N., Tavakoli, A., 2021. Demonstrating the power of quantum computers, certification of highly entangled measurements and scalable quantum nonlocality. npj Quantum Information 7. URL: http://dx.doi.org/10.1038/s41534-021-00450-x, doi:10.1038/s41534-021-00450-x.

Camera, G., 2001. Dirty money. Journal of Monetary Economics 47, 377–415.

Carvalho, V.M., Tahbaz-Salehi, A., 2019. Production networks: A primer. Annual Review of Economics 11, 635–663.

Chapman, J., Embree, L., Roberts, T., Zhang, N., et al., 2011. Payment networks: A review of recent research. Bank of Canada Review 2010, 21–27.

Chen, H.L., Roughgarden, T., Valiant, G., 2010. Designing network protocols for good equilibria. SIAM Journal on Computing 39, 1799–1832.

Chiu, J., Davoodalhosseini, S.M., Hua Jiang, J., Zhu, Y., 2023. Bank market power and central bank digital currency: Theory and quantitative assessment. Journal of Political Economy .

Chiu, J., Koeppl, T.V., 2017. The economics of cryptocurrencies–bitcoin and beyond. Available at SSRN 3048124 .

Cocco, J.F., Gomes, F.J., Martins, N.C., 2009. Lending relationships in the interbank market. Journal of Financial Intermediation 18, 24–48.

Cohen, L., Frazzini, A., 2008. Economic links and predictable returns. The Journal of Finance 63, 1977–2011.

Colliard, J.E., Demange, G., 2021. Asset dissemination through dealer markets. Management Science 67, 6211–6234.

Condorelli, D., Galeotti, A., Renou, L., 2017. Bilateral trading in networks. The Review of Economic Studies 84, 82–105.

Cong, L.W., Li, Y., Wang, N., 2021. Tokenomics: Dynamic adoption and valuation. The Review of Financial Studies 34, 1105–1155.

Craig, B., Von Peter, G., 2014. Interbank tiering and money center banks. Journal of Financial Intermediation 23, 322–347.

Daley, A.J., Bloch, I., Kokail, C., Flannigan, S., Pearson, N., Troyer, M., Zoller, P., 2022. Practical quantum advantage in quantum simulation. Nature 607, 667–676.

Denbee, E., Julliard, C., Li, Y., Yuan, K., 2021. Network risk and key players: A structural analysis of interbank liquidity. Journal of Financial Economics 141, 831–859.

D'Erasmo, P., Erol, S., Ordoñez, G., 2022. Regulating clearing in networks. Available at SSRN 4273381 .

Eisenberg, L., Noe, T.H., 2001. Systemic risk in financial systems. Management Science 47, 236–249.

Elliott, M., Golub, B., Jackson, M.O., 2014. Financial networks and contagion. American Economic Review 104, 3115–53.

Elsinger, H., Lehar, A., Summer, M., 2013. Network models and systemic risk assessment. Handbook on Systemic Risk 1, 287–305.

Erdős, P., Rényi, A., et al., 1960. On the evolution of random graphs. Publication of the Mathematical Institute of the Hungarian Academy of Sciences 5, 17–60.

Farboodi, M., 2023. Intermediation and voluntary exposure to counterparty risk. Journal of Political Economy 131, 000–000.

Farhi, E., Goldstone, J., Gutmann, S., Sipser, M., 2000. Quantum computation by adiabatic evolution. arXiv:quant-ph/0001106.

Fernández-Villaverde, J., Hull, I.J., 2023. Dynamic programming on a quantum annealer: Solving the rbc model. Technical Report. National Bureau of Economic Research.

Fernández-Villaverde, J., Sanches, D., 2019. Can currency competition work? Journal of Monetary Economics 106, 1–15.

Freixas, X., Parigi, B., 1998. Contagion and efficiency in gross and net interbank payment systems. Journal of Financial Intermediation 7, 3–31.

Friebe, C., Kuhlmann, M., Lyre, H., Näger, P.M., Passon, O., Stöckler, M., 2018. The philosophy of quantum physics. Springer.

Furfine, C.H., 2003. Interbank exposures: Quantifying the risk of contagion. Journal of Money, Credit and Banking , 111–128.

Furusawa, T., Konishi, H., 2007. Free trade networks. Journal of International Economics 72, 310–335.

Gale, D., Shapley, L.S., 1962. College admissions and the stability of marriage. The American Mathematical Monthly 69, 9–15.

Gale, D.M., Kariv, S., 2007. Financial networks. American Economic Review 97, 99–103.

Garratt, R., Zhu, H., 2021. On interest-bearing central bank digital currency with heterogeneous banks. Available at SSRN 3802977 .

Garratt, R., Zimmerman, P., 2020. Centralized netting in financial networks. Journal of Banking & Finance 112, 105270.

Garratt, R.J., Van Oordt, M.R., 2021. Privacy as a public good: a case for electronic cash. Journal of Political Economy 129, 2157–2180.

Glode, V., Opp, C.C., 2020. Over-the-counter versus limit-order markets: The role of traders' expertise. The Review of Financial Studies 33, 866–915.

Gofman, M., 2017. Efficiency and stability of a financial architecture with too-interconnected-to-fail institutions. Journal of Financial Economics 124, 113–146.

Goyal, S., 2009. Connections: an introduction to the economics of networks .

Goyal, S., Joshi, S., 2003. Networks of collaboration in oligopoly. Games and Economic Behavior 43, 57–85.

Goyal, S., Joshi, S., 2006. Bilateralism and free trade. International Economic Review 47, 749–778.

Goyal, S., Moraga-Gonzalez, J.L., 2001. R&d networks. Rand Journal of Economics , 686–707.

Grabisch, M., Rusinowska, A., Venel, X., 2022. Diffusion in large networks. Journal of Economic Dynamics and Control , 104439.

Graham, B., De Paula, A., 2020. The Econometric Analysis of Network Data. Academic Press.

Hendershott, T., Li, D., Livdan, D., Schürhoff, N., 2020. Relationship trading in over-the-counter markets. The Journal of Finance 75, 683–734.

Herings, P.J.J., Mauleon, A., Vannetelbosch, V., 2009. Farsightedly stable networks. Games and Economic Behavior 67, 526–541.

Huang, H.Y., Broughton, M., Cotler, J., Chen, S., Li, J., Mohseni, M., Neven, H., Babbush, R., Kueng, R., Preskill, J., et al., 2022. Quantum advantage in learning from experiments. Science 376, 1182–1186.

Hull, I., Sattath, O., Diamanti, E., Wendin, G., 2020. Quantum technology for economists. Available at SSRN 3745608 .

Iori, G., De Masi, G., Precup, O.V., Gabbi, G., Caldarelli, G., 2008. A network analysis of the italian overnight money market. Journal of Economic Dynamics and Control 32, 259–278.

Jackson, M., Watts, A., 2002. The evolution of social and economic networks. Journal of Economic Theory 106, 265–295.

Jackson, M.O., 2010. Social and economic networks. Princeton university press.

Jackson, M.O., Pernoud, A., 2020. Credit freezes, equilibrium multiplicity, and optimal bailouts in financial networks. arXiv preprint arXiv:2012.12861 .

Jackson, M.O., Wolinsky, A., 1996. A strategic model of social and economic networks. Journal of Economic Theory 71, 44–74.

Jackson, M.O., Yariv, L., 2007. Diffusion of behavior and equilibrium properties in network games. American Economic Review 97, 92–98.

Jammer, M., 1974. Philosophy of quantum mechanics. the interpretations of quantum mechanics in historical perspective .

Jiang, J.H., Shao, E., 2020. The cash paradox. Review of Economic Dynamics 36, 177–197.

Jordan, S., 2022. Quantum algorithm zoo. URL: https://quantumalgorithmzoo.org/.

Kadowaki, T., Nishimori, H., 1998. Quantum annealing in the transverse ising model. Physical Review E 58, 5355.

Kahn, C.M., McAndrews, J., Roberds, W., 2005. Money is privacy. International Economic Review 46, 377–399.

Kahn, C.M., Roberds, W., 2009. Why pay? an introduction to payments economics. Journal of Financial Intermediation 18, 1–23.

Keister, T., Monnet, C., 2022. Central bank digital currency: Stability and information. Journal of Economic Dynamics and Control 142, 104501.

Khiaonarong, T., Humphrey, D., 2019. Cash use across countries and the demand for central bank digital currency. Journal of Payments Strategy & Systems 13, 32–46.

King, A.D., Nocera, A., Rams, M.M., Dziarmaga, J., Wiersema, R., Bernoudy, W., Raymond, J., Kaushal, N., Heinsdorf, N., Harris, R., et al., 2024. Computational supremacy in quantum simulation. arXiv preprint arXiv:2403.00910 .

King, A.D., Raymond, J., Lanting, T., Harris, R., Zucca, A., Altomare, F., Berkley, A.J., Boothby, K., Ejtemaee, S., Enderud, C., et al., 2023. Quantum critical dynamics in a 5,000-qubit programmable spin glass. Nature 617, 61–66.

Koulayev, S., Rysman, M., Schuh, S., Stavins, J., 2016. Explaining adoption and use of payment instruments by us consumers. The RAND Journal of Economics 47, 293–325.

Koutsoupias, E., Papadimitriou, C., 2009. Worst-case equilibria. Computer Science Review 3, 65–69.

Kowalsky, M., Albash, T., Hen, I., Lidar, D.A., 2022. 3-regular three-xorsat planted solutions benchmark of classical and quantum heuristic optimizers. Quantum Science and Technology 7, 025008.

Kranton, R.E., Minehart, D.F., 2001. A theory of buyer-seller networks. American Economic Review 91, 485–508.

Lagos, R., Zhang, S., 2019. On money as a medium of exchange in near-cashless credit economies. Technical Report. National Bureau of Economic Research.

Leister, C.M., Zenou, Y., Zhou, J., 2022. Social connectedness and local contagion. The Review of Economic Studies 89, 372–410.

Li, B., Cao, Y., Li, Y.H., Cai, W.Q., Liu, W.Y., Ren, J.G., Liao, S.K., Wu, H.N., Li, S.L., Li, L., et al., 2022. Quantum state transfer over 1200 km assisted by prior distributed entanglement. Physical Review Letters 128, 170501.

Li, B.G., McAndrews, J., Wang, Z., 2020. Two-sided market, r&d, and payments system evolution. Journal of Monetary Economics 115, 180–199.

Madsen, L.S., Laudenbach, F., Askarani, M.F., Rortais, F., Vincent, T., Bulmer, J.F., Miatto, F.M., Neuhaus, L., Helt, L.G., Collins, M.J., et al., 2022. Quantum computational advantage with a programmable photonic processor. Nature 606, 75–81.

Makarov, I., Schoar, A., 2021. Blockchain analysis of the bitcoin market. Technical Report. National Bureau of Economic Research.

Malamud, S., Rostek, M., 2017. Decentralized exchange. American Economic Review 107, 3320–3362.

Manea, M., 2011. Bargaining in stationary networks. American Economic Review 101, 2042–2080.

Martinez-Jaramillo, S., Alexandrova-Kabadjova, B., Bravo-Benitez, B., Solórzano-Margain, J.P., 2014. An empirical study of the mexican banking system's network and its implications for systemic risk. Journal of Economic Dynamics and Control 40, 242–265.

Matous, P., Todo, Y., 2016. Energy and resilience: the effects of endogenous interdependencies on trade network formation across space among major japanese firms. Network Science 4, 141–163.

Matous, P., Todo, Y., 2017. Analyzing the coevolution of interorganizational networks and organizational performance: Automakers' production networks in japan. Applied Network Science 2, 1–24.

McMahon, C., McGillivray, D., Desai, A., Rivadeneyra, F., Lam, J.P., Lo, T., Marsden, D., Skavysh, V., 2022. Improving the efficiency of payments systems using quantum computing. arXiv preprint arXiv:2209.15392 .

Meirzada, I., Kalinski, A., Furman, D., Armon, T., Vaknin, T., Primack, H., Tradonsky, C., Ben-Shlomi, R., 2022. Lightsolver–a new quantum-inspired solver cracks the 3-regular 3-xorsat challenge. arXiv preprint arXiv:2207.09517 .

Monderer, D., Shapley, L.S., 1996. Potential games. Games and Economic Behavior 14, 124–143.

Morris, S., 2000. Contagion. The Review of Economic Studies 67, 57–78.

Mugel, S., Kuchkovsky, C., Sanchez, E., Fernandez-Lorenzo, S., Luis-Hita, J., Lizaso, E., Orús, R., 2022. Dynamic portfolio optimization with real datasets using quantum processors and quantum-inspired tensor networks. Physical Review Research 4, 013006.

Myerson, R.B., 1977. Graphs and cooperation in games. Mathematics of Operations Research 2, 225–229.

Nakamoto, S., 2008. Bitcoin: A peer-to-peer electronic cash system. Decentralized Business Review .

Newman, M., 2018. Networks. Oxford university press.

Nielsen, M.A., Chuang, I.L., 2010. Quantum Computation and Quantum Information. Cambridge University Press.

Orús, R., Mugel, S., Lizaso, E., 2019. Forecasting financial crashes with quantum computing. Physical Review A 99, 060301.

Orus, R., Mugel, S., Lizaso, E., 2019. Quantum computing for finance: Overview and prospects. Reviews in Physics 4, 100028.

Parlour, C.A., Rajan, U., Zhu, H., 2022. When fintech competes for payment flows. The Review of Financial Studies 35, 4985–5024.

Priazhkina, S., Page, F.H., 2018. Sharing market access in buyer–seller networks. Journal of Economic Theory 175, 415–446.

Rochet, J.C., Tirole, J., 2006. Two-sided markets: a progress report. The RAND Journal of Economics 37, 645–667.

Rogoff, K.S., 2017. The curse of cash, in: The Curse of Cash. Princeton University Press.

Rosenthal, R.W., 1973. A class of games possessing pure-strategy nash equilibria. International Journal of Game Theory 2, 65–67.

Schilling, L., Uhlig, H., 2019. Some simple bitcoin economics. Journal of Monetary Economics 106, 16–26.

Shor, P.W., 1999. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. SIAM Review 41, 303–332.

Skavysh, V., Priazhkina, S., Guala, D., Bromley, T.R., 2023. Quantum monte carlo for economics: Stress testing and macroeconomic deep learning. Journal of Economic Dynamics and Control , 104680.

Tardos, E., Wexler, T., 2007. Network formation games and the potential function method. Algorithmic Game Theory , 487–516.

The BIG Bell Test Collaboration, 2018. Challenging local realism with human choices. Nature 557, 212–216.

Tobin, J., 1956. The interest-elasticity of transactions demand for cash. The Review of Economics and Statistics 38, 241–247.

Townsend, R.M., 2020. Distributed Ledgers: Design and Regulation of Financial Infrastructure and Payment Systems. MIT Press.

Valiant, L.G., 1979. The complexity of computing the permanent. Theoretical Computer Science 8, 189–201.

Wasserman, S., Faust, K., et al., 1994. Social network analysis: Methods and applications .

Welburn, J.W., Strong, A., Nekoul, F.E., Grana, J., Marcinek, K., Osoba, O., Koirala, N., Setodji, C.M., 2020. Systemic Risk in the Broad Economy: Interfirm Networks and Shocks in the US Economy. RAND Corporation.

Williamson, S., 2022. Central bank digital currency: Welfare and policy implications. Journal of Political Economy 130, 2829–2861.

Yarkoni, S., Raponi, E., Bäck, T., Schmitt, S., 2022. Quantum annealing for industry applications: Introduction and review. Reports on Progress in Physics 85, 104001.

Yarkoni, S., Raponi, E., Schmitt, S., Bäck, T., 2021. Quantum annealing for industry applications: Introduction and review. arXiv:2112.07491.

# Appendix A Proof of Proposition 1

For convenience, denote the expected privacy loss in both cases as

$$E_p^+(f_i - d) = E[p|p \geq f_i - d]$$

$$E_p^+(f_i - c) = E[p|p \geq f_i - c]$$

and $F$ the cumulative distribution function of $p$, to get a more intuitive version of the firm's payoff:

$$u_i^f(\delta_i, f_i) = n_i(E_p^+(f_i - d)(1 - \delta_i) + E_p^+(f_i - c)\delta_i + E[b] - f_i$$

$$+(f_i - d)F_p(f_i - d)(1 - \delta_i) + (f_i - c)F_p(f_i - c)\delta_i - c_0\delta_i)$$

Then the maximum utility is achieved whenever the first order conditions hold

$$-E_p^+(f - d) + E_p^+(f - c)$$

$$-(f - d)F(f - d) + (f - c)F(f - c) - c_0$$

$$-\frac{1}{2}(d - c)(F(f - d)(1 - \delta_i) + F(f - c)\delta_i - 1) = 0.$$

from which the statement of the theorem follows. Taking the second derivative proves strict concavity of the firm's preferences.

# Appendix B Proof of Proposition 2

If there is a candidate network with overadoption by firm $i$, $\delta_i^x > \delta_i^*$ and $\delta_i^x n_i - \delta_i^* n_i > 1$, firm $i$ can remove one of the links $(i, j)$ with $G_{ij} = 1$ without the consent of $j$, and increase its payoff, because

$$\frac{\partial u_i(\delta_i)}{\partial \delta_i} < 0 \text{ for } \delta_i \in (\delta_i^*, \delta_i^x).$$

That means that the only possibility is for equilibrium to be $\delta_i n_i < \delta_i^* n_i + 1$. Because $\delta_i n_i$ is by definition a natural number, and $\bar{\delta}_i$ is selected to maximise utility, the main result follows.

# Appendix C Proof of Proposition 4

First-order stochastic dominance is defined according to the inequality:

$$Pr(\delta(\delta_A^*) > x) \leq Pr(\delta(\delta_B^*) > x). \tag{27}$$

To prove that this inequality holds, define the set of all equilibrium networks in game $A$ as $(G_A^1, ..., G_A^i, ...)$ and the equilibrium adoption rate vectors corresponding to each network as $(\delta_A^1, ..., \delta_A^i, ...)$. Likewise, the set of all equilibrium networks in game $B$ is defined as $(G_B^1, ..., G_B^j, ...)$ and the equilibrium adoption rate vectors are $(\delta_B^1, ..., \delta_B^i, ...)$. All sets are finite but can be of different cardinality.

Then the left-hand size of (27) can be written as

$$Pr(\delta(\delta_A^*) > x) = \sum_i I(\delta_i^A > x) Pr(\delta(\delta_A^*) = \delta_i^A) \tag{28}$$

where $I(\cdot)$ is an indicator function.

The right-hand side can be defined in a similar way as a finite sum of probabilities

$$Pr(\delta(\delta_B^*) > x) = \sum_j I(\delta_j^B > x) Pr(\delta(\delta_B^*) = \delta_j^B). \tag{29}$$

In the previous section, we showed that an improvement path leads from at least one equilibrium network $G^i_A$ in game $A$ to each equilibrium network $G^j_B$ in game $B$. In this case, $F(G^i_A, G^i_B) > 0$. If there is no improvement path for any two equilibrium networks $G^i_A$ and $G^i_B$, by definition $F(G^i_A, G^i_B) = 0$.

Because $\sum_i Pr(\delta(\delta^*_A) = \delta^A_i) = 1$, we can further expand equation (29) using the law of total probabilities. For this, assume that the process randomly converges from an empty network to an equilibrium network $G^B_j$ in $B$ according to the convergence algorithm we developed, so only one improvement path is formed. On this path, there is one and only one network $G^A_i$ from the set of equilibria of game $A$. Because the probability of reaching network $G^A_i$ in this random convergence algorithm for $B$ is equivalent to the probability of reaching the same network in the algorithm for $A$, we get

$$\sum_i \sum_j I(\delta^B_j > x) Pr(\delta(\delta^*_B) = \delta^B_j | \delta(\delta^*_A) = \delta^A_i, F(G^A_i, G^B_j) > 0) Pr(\delta(\delta^*_A) = \delta^A_i) \tag{30}$$

Comparing (28) and (29), it is clear that the sufficient condition for (27) is

$$I(\delta^A_i > x) \leq \sum_j I(\delta^B_j > x) Pr(\delta(\delta^*_B) = \delta^B_j | \delta(\delta^*_A) = \delta^A_i, F(G^A_i, G^B_j) > 0). \tag{31}$$

Indicator $I(\delta^A_i > x)$ takes values 0 and 1. If $I(\delta^A_i > x) = 0$, inequality (31) holds because on the right hand side is a sum of non-negative variables. If $I(\delta^A_i > x) = 1$, by definition $\delta^A_i > x$. Also, probability $Pr(\delta(\delta^*_B) = \delta^B_j | \delta(\delta^*_A) = \delta^A_i, F(G^A_i, G^B_j) > 0)$ is positive only on the improvement paths. Because the adoption rate increases on the improvement path from $A$ equilibrium to $B$ equilibrium (as the convergence algorithm means sequential addition of links), there is a path from $G^A_i$ to at least one equilibrium network in $B$, which means the second half of inequality (31) equals 1. This proves that (31) holds, so (27) holds.

# Appendix D Resource estimation

Regarding computational requirements, the quantum algorithm in Section 8 requires $\mathcal{O}(N^2)$ qubits.[38] In addition, it needs $\mathcal{O}(N^2)$ classical calculations for creating sign matrices and the same level of effort for preparing the QUBO coefficient matrix for a quantum machine. As shown in Figure 10, the number of qubits needed varies with the number of players and links in the network, compared to the capacity of the D-WAVE quantum annealer. Our analysis shows that a network game with 40 mutually trading players, which translates to 780 links, can be managed by a quantum machine. In sparser networks, like the one in Section 2 where each firm trades with only three others, the model only needs to handle $3(N-1)/2$ links, making quantum simulations practical for networks with more than 500 firms. In the future, we expect quantum applications to expand

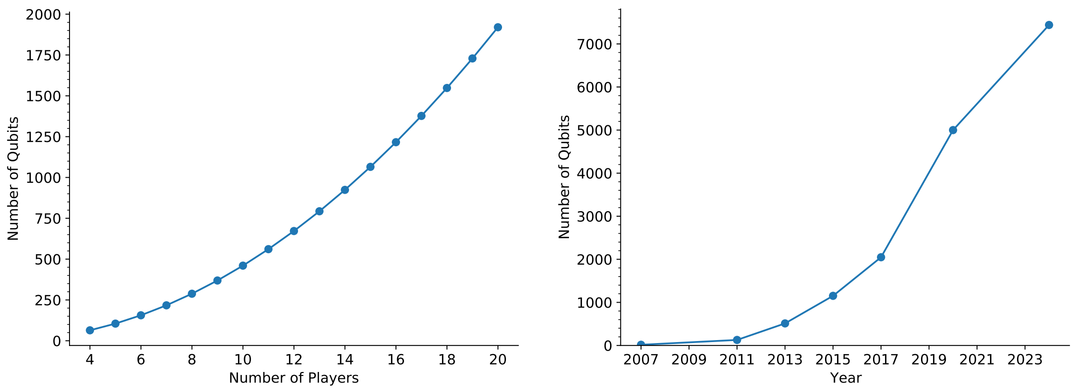


Figure 10: Left: Qubit scaling for the quantum sampling algorithm as a function of the number of players in the complete network. Right: The number of qubits on the best available D-Wave quantum annealer over time.

significantly. The rapid growth in qubit capacity, as illustrated in Figure 10, is a common trend in quantum computing. However, currently the main limitation in using a quantum annealer is due to the effectiveness of the quantum samplers rather than the problem size, given the majority of output does not deliver a zero Hamilton condition and only gets close to the actual solution. Continued progress

[38] In the optimised quantum algorithm in Section 6.2, the qubit count is capped at $2N^2$.

in quantum technologies is expected to ameliorate this problem.