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INDEXATION FOR TELECOMMUNICATIONS SERVICES:

A PROGRESS REPORT OF RECENT RESEARCH UNDERTAKEN BY THE CANADIAN DEPARTMENT OF COMMUNICATIONS

Alain de Fontenay

Department of Communications
Government of Canada
Ottawa, Canada

Forthcoming in Public Utilities (Ed. O. Anderson)

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A proper understanding of the economic accounts of any firm or sector such as the telecommunications sector is the key to reliable forecasts and incisive economic analysis. Central to the economic accounts construction is the index number problem which is the object of this paper. A diagrammatic treatment of both the traditional statistical approach and the modern economic one is presented. Duality theory is introduced to illustrate the index number theory of homogeneous processes. The proposed solutions to index numbers of non-homogeneous are presented and index number theory is shown not to be nested in the econometric methodology and the paper concludes with the rationale for using chain indexes.

INTRODUCTION

In any economic and forecasting analysis relating to a telecommunications sector, the analyst is almost always confronted with price and quantity series which are index numbers. This is due to the fact that these aggregates are used to describe highly complex entities which involve a great number of distinct elements. This complexity can be illustrated in terms of, say, the output quantity measure for a firm such as Bell Canada. Output is usually divided into seven components, one of which is local services; local services are then subdivided in four components which include "Contract Auxiliary". In "Contract Auxiliary", the company includes services mostly related to terminal equipment, and one such service is the Private Branch Exchange (PBX). However, a client who wants a PBX can choose from more than twenty distinct types of machines. Once a machine such as the 701 is selected, the client still has to make decisions with respect to over fifteen special features, such as attendant's position, digipulse,... The importance of accounting for the composition of aggregate measures becomes even more crucial whenever the relative share of the various components shift through time.

The Department of Communications is concerned with questions such as forecasting revenues and expenses, estimating demand elasticities, evaluating scale and scope effects,... The data bases on which those questions are evaluated are almost exclusively public. They are released by the carriers, most of which are privately owned, in the context of regulatory proceedings by bodies such as the Canadian Radio-Television and Telecommunications Commission (CRTC). The level of disaggregation in those accounts is very limited, and, more seriously, the public documentation on their construction is almost non-existant. For instance, Bell Canada's public economic accounts are probably the best and the most sophisticated of any company, yet it is telecommunications technology; the next section contains proposed solutions to index number theory in relation to non-homogeneous mappings. The lack of homogeneity renders the index path dependent, and the analysis consists in setting rules to select a path. In practice, all proposed approaches (Diewert, 1979) consist basically in using as reference a positive linear homogeneous function. In the quantity space, a natural candidate is the transformation function with which is associated the Malmquist index, while in the price space the obvious mapping is the cost function. The price index associated with the cost function is the Konus index. The Allen index is also introduced. The Malmquist approach yields a quantity index from which an implicit price index is derived; an explicit formulation of the latter is introduced. Similarly an explicit formulation is introduced for the Konus implicit price index. Finally, the diagrammatic analysis is used to compare Malmquist, Konus and Allen indexes.

The conclusion synthetizes the results of this paper and two theorems by Diewert (1979) to establish the main justification for the use of chain superlative indexes (this is not the only one, another excellent one having been developed by Diewert (1978); superlative indexes were defined by Diewert (1976)). It also provides a justification for the use of index numbers as complementary to the econometric methods. Contrary to general belief, it is shown that index number theory is not nested in econometric theory. It is indicated that they may in fact often provide a superior fit of the unspecified mapping under analysis than the fit which would be obtained through econometrics (but for the stochastic dimension). Nevertheless, they cannot be used to exclude the latter which can incorporate a formal stochastic formulation and provide information as to both the structure of the function and the presence of linear homogeneity and separability.

The results of this paper provide strong evidence as to the superiority of superlative indexes such as Torngvist and Fisher ideal indexes over common alternatives such as Laspeyres and Paasche indexes, their justification on economic grounds and the importance of chain indexation in domains such as the telecommunications

THE GEOMETRIC ANALYSIS OF INDEX NUMBERS

The Simple Statistical Index

In this section, we develop a simple diagrammatic approach to the construction of statistical indexes.

To begin, we consider a single commodity available in period t in a quantity x_t at a price p_t. The budget, denoted by $y_{t,t}$, corresponds to the expenditure on the commodity: $y_{t,t} = p_t.x_t$. Given new circumstances in the form of a price p_s, and possibly a new quantity x_s and/or a new budget $y_{s,s}$ such that $y_{s,s} = p_s.x_s$, we may define the budget implied by the price p_s and the quantity x_t, as $y_{s,t} = p_s.x_t$. Vice versa $y_{t,s} = p_t.x_s$. This is illustrated in Fig. 2.1.

not possible, from public records, to answer simple questions such as whether they are Laspeyres indexes based on a fixed 1967 basket rather than chained Laspeyres indexes with weights which are revised at each rate case,...

The concern with the economic accounts used in the economic analysis and the forecasting of key series is not solely a matter of data definition. Another major source of differences among carriers has been the methods used to aggregate the data. Certain carriers such as Bell Canada (and AT&T in the U.S.A.) use the Laspeyres approach while others, such as Teleglobe Canada, B.C. Tel. and AGT have opted in favour of the Tornqvist approximation to the Divisia index. In addition, we observe in many recent studies a mixture of indexes, with econometric models which utilize Tornqvist indexes of Laspeyres indexes. Yet, certain preliminary results appear to suggest that the difference it makes in the analysis can be substantial (Denny, Fontenay and Werner, 1980).

The last concern is the relative justification for alternative analytical methods such as econometrics and index number. In the analysis of the firm, some researchers are using almost exclusively econometric methods (Corbo et al., 1979) while others have adopted methods based on index numbers (Werner, 1979).

In this paper, our concern will be not with the problem of data definition but with the other problems which are all related to the index number problem. A diagrammatic analysis of Diewert (1979) is presented. In the next section, a simple diagrammatic analysis of index numbers, seen from a statistical and mechanical point of view is introduced. This analysis is rather limited. Even though it has received a formal axiomatic treatment (Eichhorn and Voeller, 1976), it has been shown by those same authors that the set of "tests" commonly required is inconsistent. The axiomatic analysis is not presented. Rather, since our problem is an economic problem, the third section resets the second section in an economic context. Such a context can be phrased in terms of a consumer who allocates his budget among goods and services to maximize his satisfaction. It will be phrased here mostly in terms of a producer who selects the output level at which he produces to maximize his profit and the input composition necessary to minimize cost. The analysis is used to reconsider the Laspeyres and Paasche indexes. They are shown to imply a Leontief-type of technology and to be "exact" economic indexes only if they are equal to one another.

Central to the index number problem is the factor reversal property first introduced by Fisher (1922). It led Divisia (1925) to propose a very simple definition for price and quantity indexes in continuous processes. Divisia's approach emphasizes the symmetry between prices and quantitities. This suggests that a natural approach to the economic analysis of index numbers is the theory of duality (Shephard, 1970). Starting with Darrough and Southey (1977), diagrammatic analysis of the indirect utility function, cost and transformation functions are introduced. Economic price and quantity indexes are defined and illustrated diagrammatically. Economic index number theory depends crucially upon positive linear homogeneity in the utility or production functions. The interdependence between linear homogeneity and path independence is illustrated.

Linear homogeneity has been fundamental to our analysis of index numbers. However, recent econometric studies of the telecommunications sector have strongly challenged its appropriateness to describe the

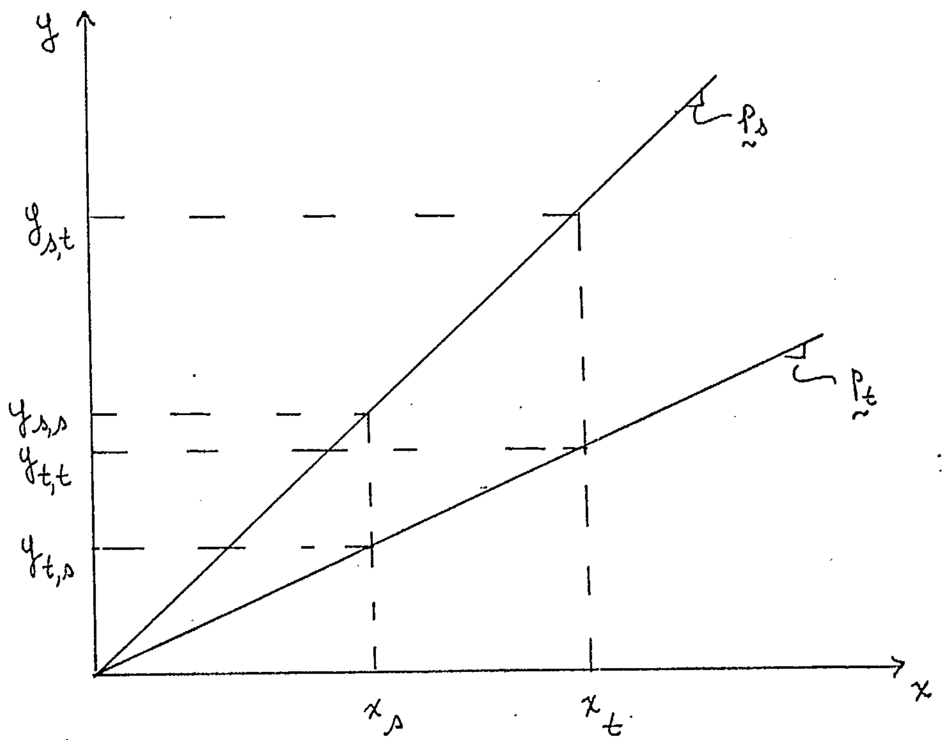

figure 2.1 Price Lines

Given that we measure the quantity x on the abscissa and the expenditure y on the ordinate, the price is given by the slope of the price line relating the expenditure to the quantity. The elementary price index, defined as the ratio of the new price to the old price, (p_s/p_t) can alternatively be read as $(y_{s,t}/y_{t,t})$ or as $(y_{s,s}/y_{t,s})$.

In practice a price index involves more than one commodity; hence, it must account not only for absolute price increases but also for changes in relative prices. Let's consider two commodities, quantities of which are denoted by x_1 and x_2 respectively. We denote the original equilibrium by $E_{t,t}$ where $E_{t,t}$ denotes the vector $\underset{\sim}{x}_t' = (x_{1,t}, x_{2,t})$, given the price vector $\underset{\sim}{p}_t' = (p_{1,t}, p_{2,t})$. To the equilibrium $E_{t,t}$ corresponds a budget $y_{t,t} = \underset{\sim}{p}_t' . \underset{\sim}{x}_t$. That situation can be illustrated simply by considering on the positive abscissa commodity 2 and on the positive ordinate commodity 1, as in Fig. 2.2. The budget line can be represented by taking one of the commodity, say commodity 1, as numéraire. Then, if $y^*_{t,t} = (y_{t,t}/p_{1,t})$, i.e. if $y^*_{t,t}$ denotes the expenditure, commodity being used as numéraire, the budget line is given by

$$x_{1,t} = y^*_{t,t} - (p_{2,t}/p_{1,t})x_{2,t}$$

with intercept $y^*_{t,t}$ and slope $\{-(p_{2,t}/p_{1,t})\}$

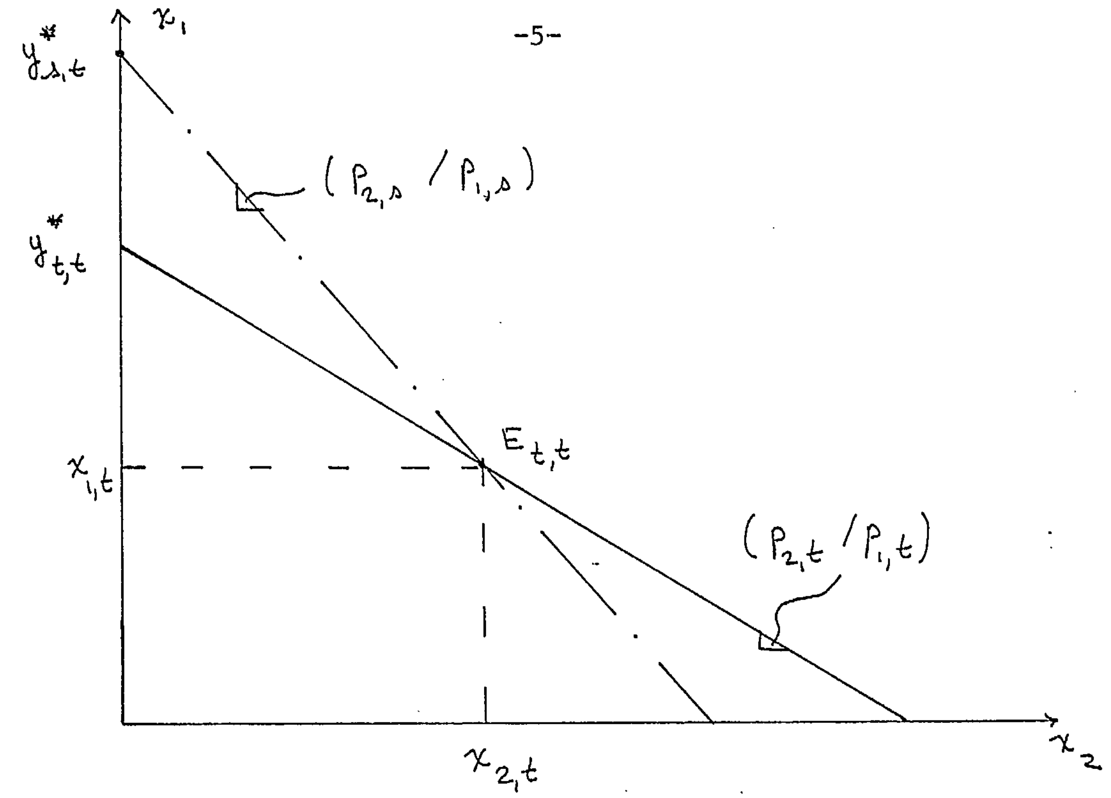

Fig 2.2 Budget Lines

Given the equilibrium $E_{t,t}$, a new price vector $\underset{\sim}{p}_s$ implies a new budget line. The new budget line differs from the old one if, and only if, the relative price ratio (p_2/p_1) changes. Assuming, for instance, that p_2 increases more rapidly than p_1 (or alternatively decreases more slowly), then the new budget line intercepts the ordinate in $y^*_{s,t}$ above $y^*_{t,t}$, reflecting the relative price change.

To construct a statistical price index, it is sufficient to combine the two approaches described above. This is done in Fig. 2.3. Let's read on the positive abscissa (ordinate) quantities of commodity 2(1) and on the negative abscissa (ordinate) expenditure on commodity 2(1).

Let the equilibrium be E. In terms of the relative price $(p_{2,t}/p_{1,t})$ E can be translated into the quantity of commodity 1, y^*_t corresponds in Fig. 2.3 to OA. It is translated by the price line $p_{1,t}$ into an expenditure OC. Equivalently, with commodity 2 as numéraire and a price line with slope $p_{2,t}$, E corresponds to a quantity OB, and an expenditure OD where OD = OC. E corresponds also to a point F on CD, representing the shares of the expenditure going to commodity 1 and commodity 2.

Given some new price vector $\underset{\sim}{p}_s$, with the whole budget allocated to commodity 1, total expenditures increase in the ratio OC'/OC). On the other hand, with the whole budget allocated to commodity 2, total expenditures increase in the ratio (OD'/OD). This reflects the fact that commodity 2 has become proportionately dearer. Clearly all price indexes are bounded from below by the smaller and from above

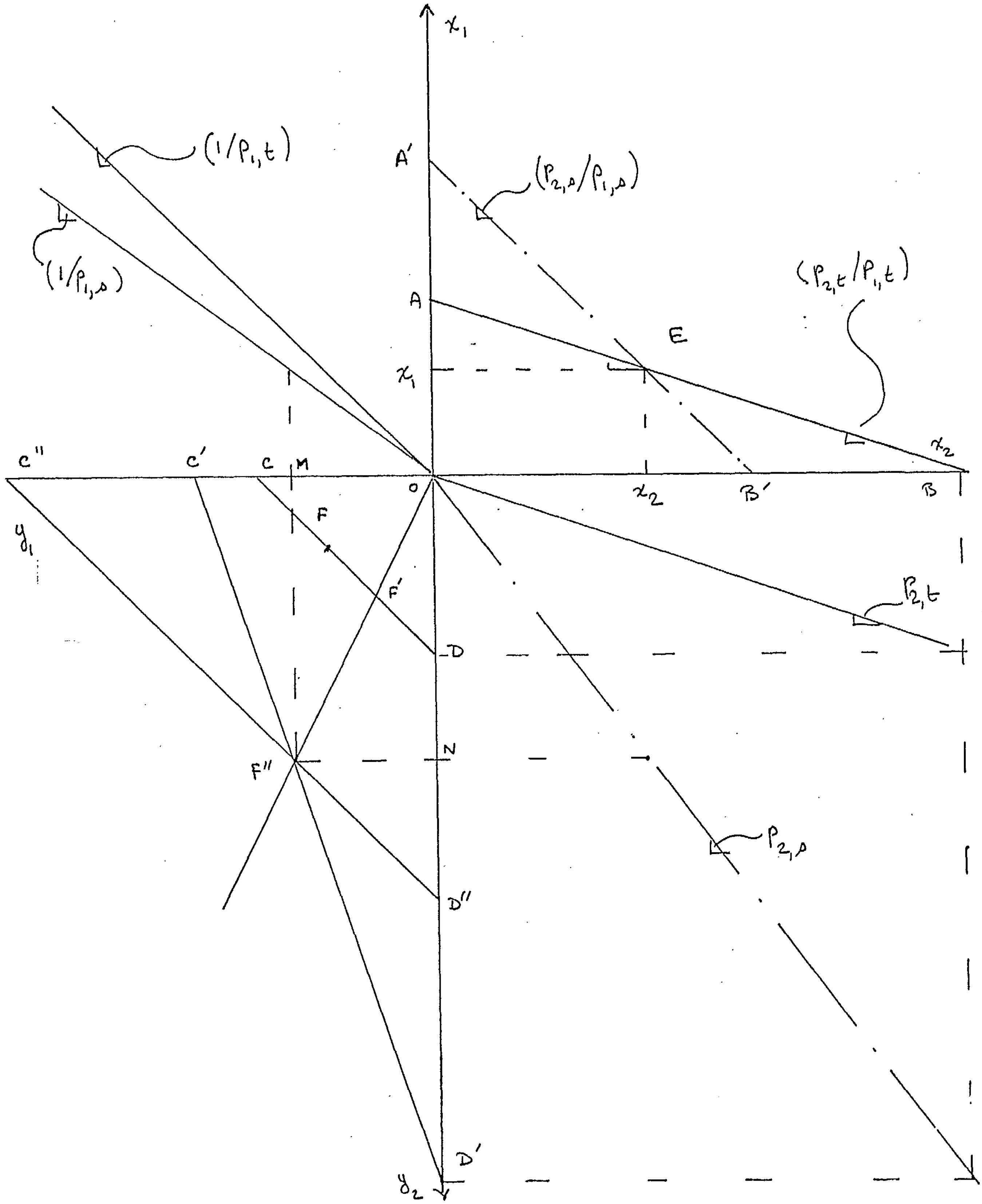

Figure 2.3 Statistical Index : Diagrammatic Representation

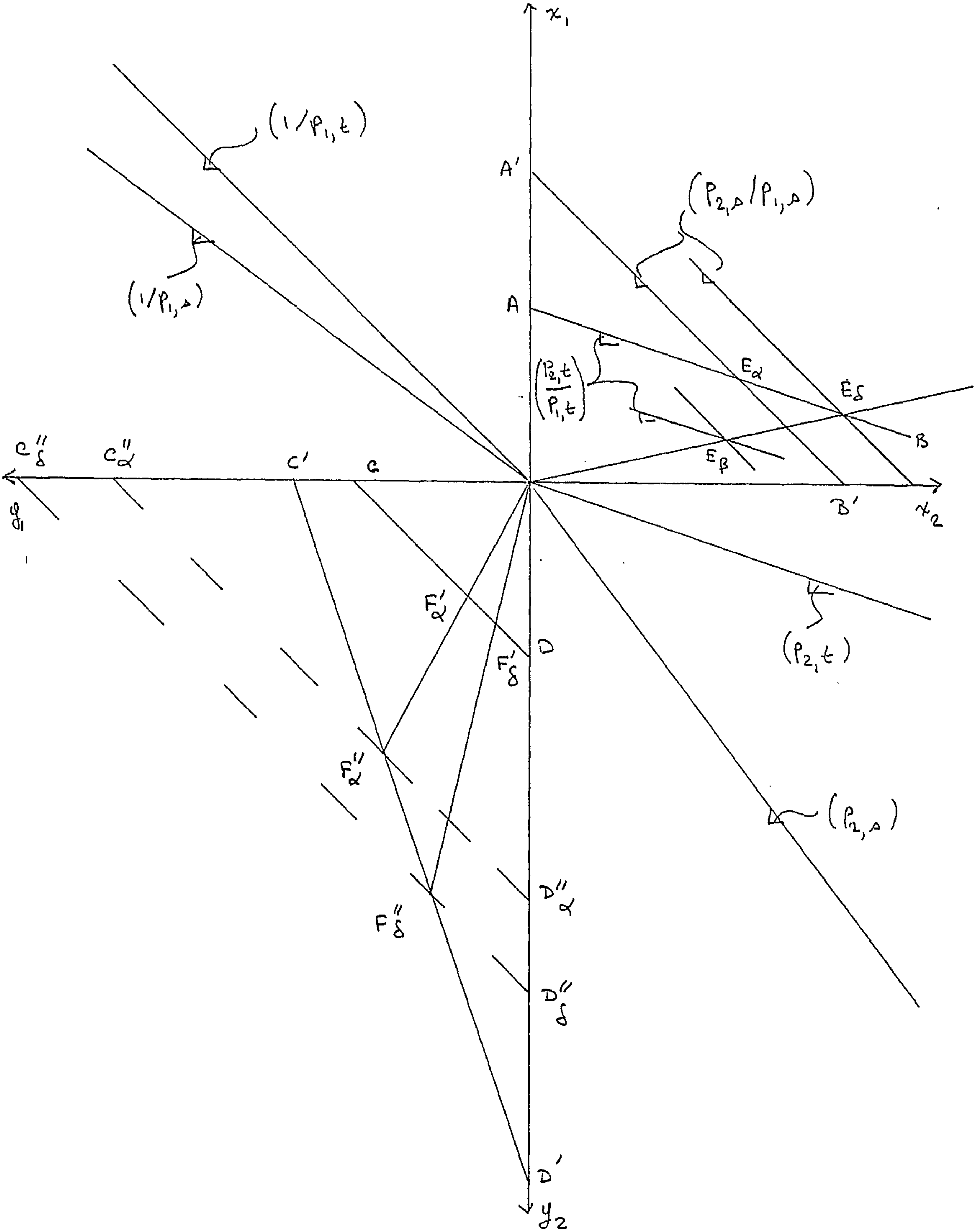

Figure 2.4 Statistical Index: Shift in Equilibrium

(ii) Edgeworth $$E_{s,t} = \sum_{i=1}^{n} \left\{ \frac{\{x_{i,t} + x_{i,s}\} p_{i,t}}{\sum_{i=1}^{n} \{x_{i,t} + x_{i,s}\} p_{i,t}} \right\} \left(\frac{p_{i,s}}{p_{i,t}} \right)$$

(iii) Fisher's Ideal $$F_{s,t} = \frac{(\sum_{i=1}^{n} p_{i,s} x_{i,t})(\sum_{i=1}^{n} p_{i,s} x_{i,s})}{(\sum_{i=1}^{n} p_{i,t} x_{i,t})(\sum_{i=1}^{n} p_{i,t} x_{i,s})}$$

(iv) Paasche $$P_{s,t} = \sum_{i=1}^{n} \left\{ \frac{p_{i,t} x_{i,s}}{\sum_{i=1}^{n} p_{i,t} x_{i,s}} \right\} \left(\frac{p_{i,s}}{p_{i,t}} \right)$$

Depending upon the direction of the inequality

$$(x_{2,s}/x_{1,s}) \overset{>}{\underset{<}{=}} (x_{2,t}/x_{1,t})$$

given t as the base period,

$$L_{s,t} \overset{<}{\underset{>}{=}} P_{s,t}$$

Just as Fisher's Ideal index, the Edgeworth index weights both the original equilibrium and the new one. Hence it is bounded by the Laspeyres and Paasche indexes. In addition, the Ideal index is generally smaller than the Edgeworth index since it is based on a geometric mean rather than an arithmetic one.

Price and Quantity Indexes

The price index was introduced by taking a quantity vector as given, x, and looking at the impact of a price change, i.e. by considering the quantity vector $\underset{\sim}{x}$ simultaneously under the price regimes $\underset{\sim}{p}_\alpha$ and $\underset{\sim}{p}_\beta$.

The same approach is adopted to define a statistical quantity index. The change being considered is a quantity change, and prices are taken as given; that is one considers the output vectors $\underset{\sim}{x}_\alpha$ and $\underset{\sim}{x}_\beta$ and the price vector $\underset{\sim}{p}$. In Fig. 2.5, the statistical quantity index is given by:

$$Q_s = \frac{OA'}{OA}$$

In practice, however, both quantities and prices change, i.e. there is a need to consider a move from the quantity vector $\underset{\sim}{x}_\alpha$ given the price vector $\underset{\sim}{p}_\alpha$ to the quantity vector $\underset{\sim}{x}_\beta$, given the price vector $\underset{\sim}{p}_\beta$. It generally matters whether one starts with a price index for the shift from $\underset{\sim}{p}_\alpha$ to $\underset{\sim}{p}_\beta$, given $\underset{\sim}{x}_\alpha$, to then consider the quantity index associated with the equilibrium points E_α and E_β, given $\underset{\sim}{p}_\beta$. The alternative consists in moving under price regime $\underset{\sim}{p}_\alpha$ from E_α to E_β and then considering the impact of the price change to $\underset{\sim}{p}_\beta$ given E_β. If the first strategy is considered, the price index is based upon $\underset{\sim}{x}_\alpha$, $\underset{\sim}{p}_\alpha$ and $\underset{\sim}{p}_\beta$, i.e. it will be a Laspeyres price index and the

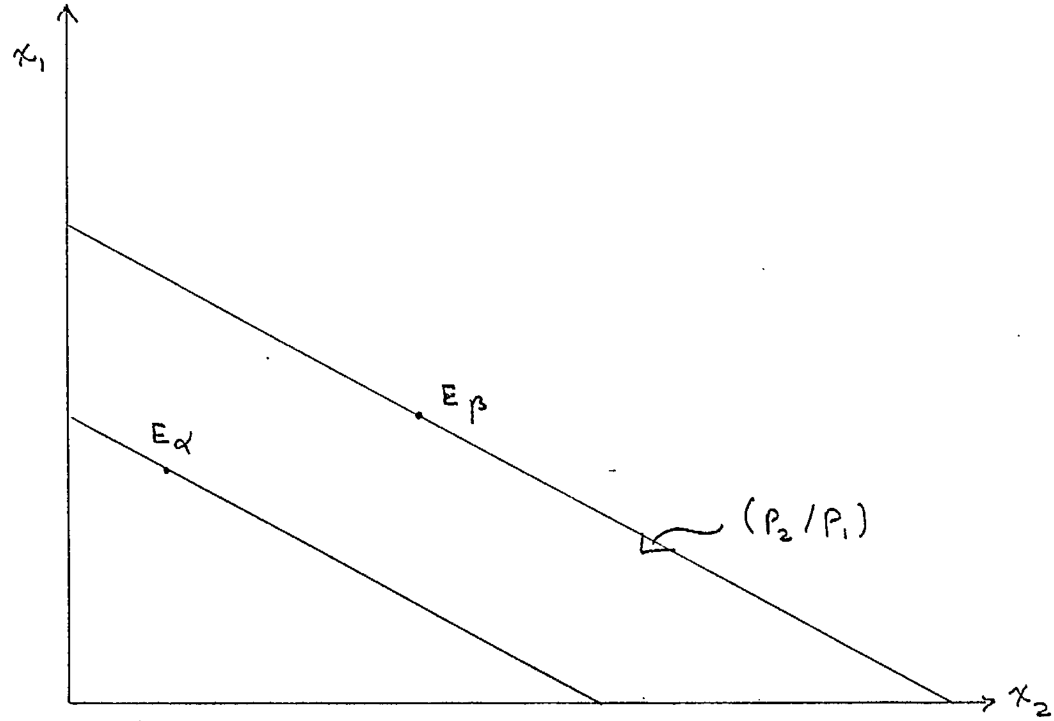

Figure 2.5 Statistical Quantity Index

corresponding quantity index will be based on $\underset{\sim}{p}_\beta$, $\underset{\sim}{x}_\alpha$ and $\underset{\sim}{x}_\beta$, i.e. it will be a Paasche quantity index. The other approach consists in beginning with a Laspeyres quantity index to use a Paasche price index. This illustrates a well known result of Fisher (1922).

Depending upon whether E_β corresponds to an increase or a decrease in the ratio (x_2/x_1), the Laspeyres price index is smaller or greater than the Paasche one. At the same time, the Laspeyres quantity index is greater or smaller than the Paasche one. This follows from our earlier result in the section on shift in equilibrium.

It seems logical to assume that most desirable indexes are bounded by Laspeyres and Paasche indexes. However, from a statistical point of view alone, it is not possible to establish how the ratio (x_2/x_1) changes; hence, how the Laspeyres and Paasche indexes are likely to bound the desirable indexes. In fact, it is not even possible to say what a desirable index should be. One approach has been to specify the criteria we would like an index to fulfill. This is the approach initiated by Fisher followed by numerous writers and formalized by Eichhorn and Voeller (1976). Eichhorn and Voeller show that some of the desired properties of index numbers are mutually exclusive. This is not the approach followed here. Rather we consider the economic context in which the index number problem appears.

THE ECONOMIC ANALYSIS OF INDEX NUMBERS UNDER HOMOTHETICITY

Introduction

In the statistical analysis of index numbers, given two price vectors, $\underset{\sim}{p}_t$ and $\underset{\sim}{p}_s$, and an output vector, $\underset{\sim}{x}_\alpha$, the construction of a price index raised no problem. It was furthermore possible to study without ambiguity the impact of a shift in the equilibrium on a statistical index. One could construct quantity indexes by symmetry with price indexes. This would imply that the quantities play the role the prices play and the prices play the role the quantities play respectively in price indexes. However, it is clear that we are limited by what can be said about a move from one price/quantity combination, $(\underset{\sim}{x}_\alpha, \underset{\sim}{p}_\alpha)$ to another, $(\underset{\sim}{x}_\beta, \underset{\sim}{p}_\beta)$.

In economics, those quantities are normally determined in terms of consumers who are demanding goods and services to maximize their satisfaction, or of producers who are producing goods and services to maximize their profit. In the following sections, we shall consider mostly the case of the producer, making reference only to the extent necessary to the cost of living index associated with the consumer.

The Construction of an Economic Index

Let's consider a firm which produces two commodities in quantities z_1 and z_2 using inputs x_1 and x_2 with the goal to maximize profit. With its resources, it faces a production possibility frontier F(z). It chooses to produce on it as far as possible, i.e. at the tangency of $F(\underset{\sim}{z})$ with the constraint with slope $(-p_2/p_1)$. From the producer's point of view, the resource constraint he faces determines the particular frontier F(z) on which he is. All output vectors z such that $F(\underset{\sim}{z}) = F(\underset{\sim}{z}_\alpha)$ require the same resources to be produced, and the producer selects a particular output vector $\underset{\sim}{z}_\alpha$ in terms of the price vector $\underset{\sim}{p}_\alpha$ so as to maximize his profit. In fact, his profit motive leads him to associate one particular output vector with each price vector $\underset{\sim}{p}$. If the price vector passes from $\underset{\sim}{p}_\alpha$ to $\underset{\sim}{p}_\beta$, the producer will move along his production possibility frontier from E_α to E_β such that $F(\underset{\sim}{z}_\alpha) = F(\underset{\sim}{z}_\beta)$ as shown in Figs. 3.1 and 3.2. Adopting the approach of the preceding section, the expenditure $C_\alpha D_\alpha$ corresponds to E_α with the price vector $\underset{\sim}{p}_\alpha$ and OC_α is the budget. With a move in the price vector from $\underset{\sim}{p}_\alpha$ to $\underset{\sim}{p}_\beta$, and with the same real constraint represented by the aggregate output level $F(\underset{\sim}{z}_\alpha)$, the profit maximizer increases the production of the commodity which becomes relatively more expensive, commodity 2. He simultaneously decreases the production of the commodity which becomes relatively cheaper, commodity 1. The equilibrium moves from E_α to E_β. E_β determines with the price vector $\underset{\sim}{p}_\beta$ the new expenditure line $C_\beta D_\beta$. Substitution in the output composition is represented by a movement from E_α to E_β, where $F(\underset{\sim}{z}_\beta) = F(\underset{\sim}{z}_\alpha)$. The economic analysis tells us that, since the resource requirement is not changed (i.e. other things equal) on the aggregate, the output is not changed and the quantity index is unity. If Q denotes the index,

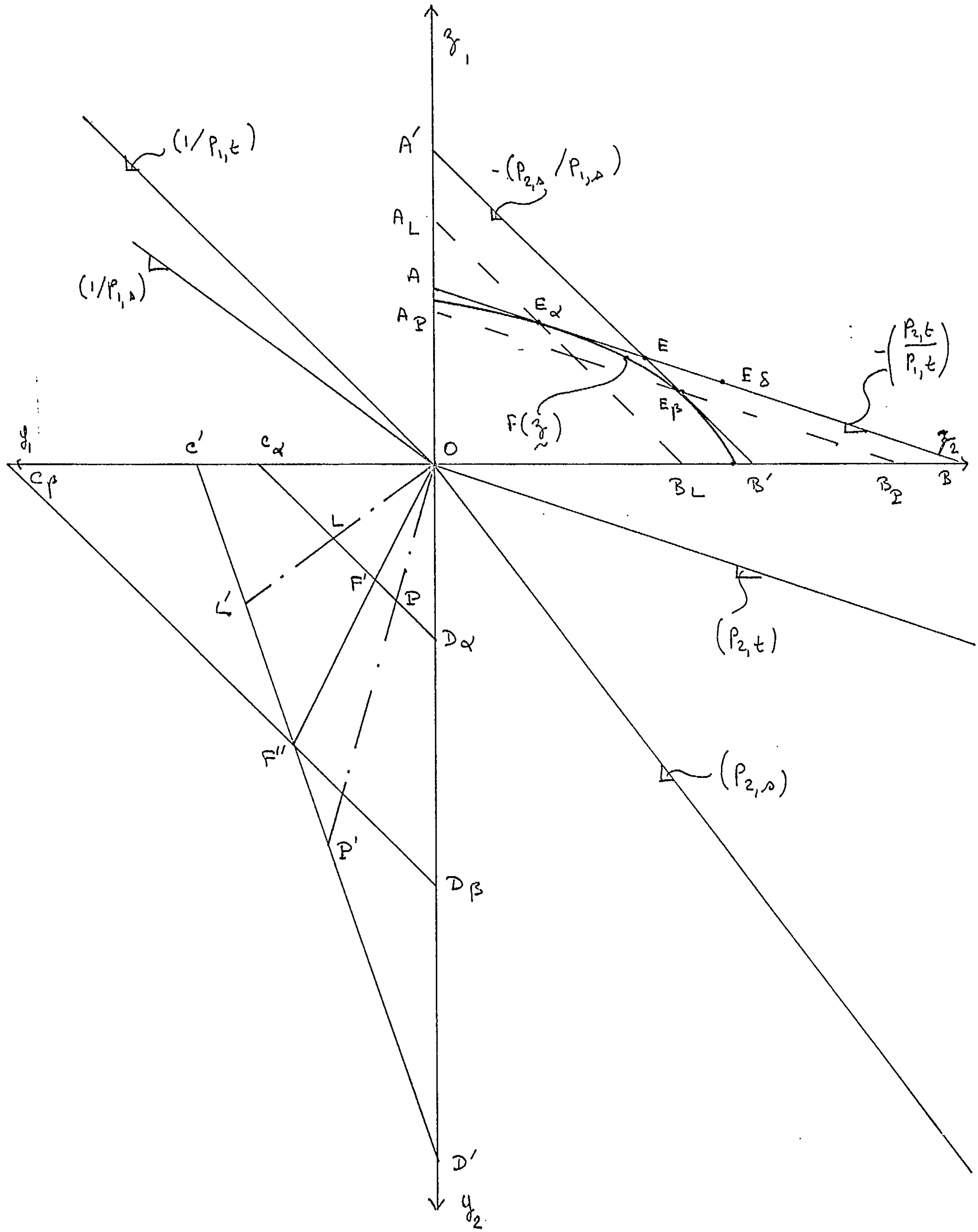

Figure 3.1 The Output Deflater and the Laspeyres and the Paasche Indexes.

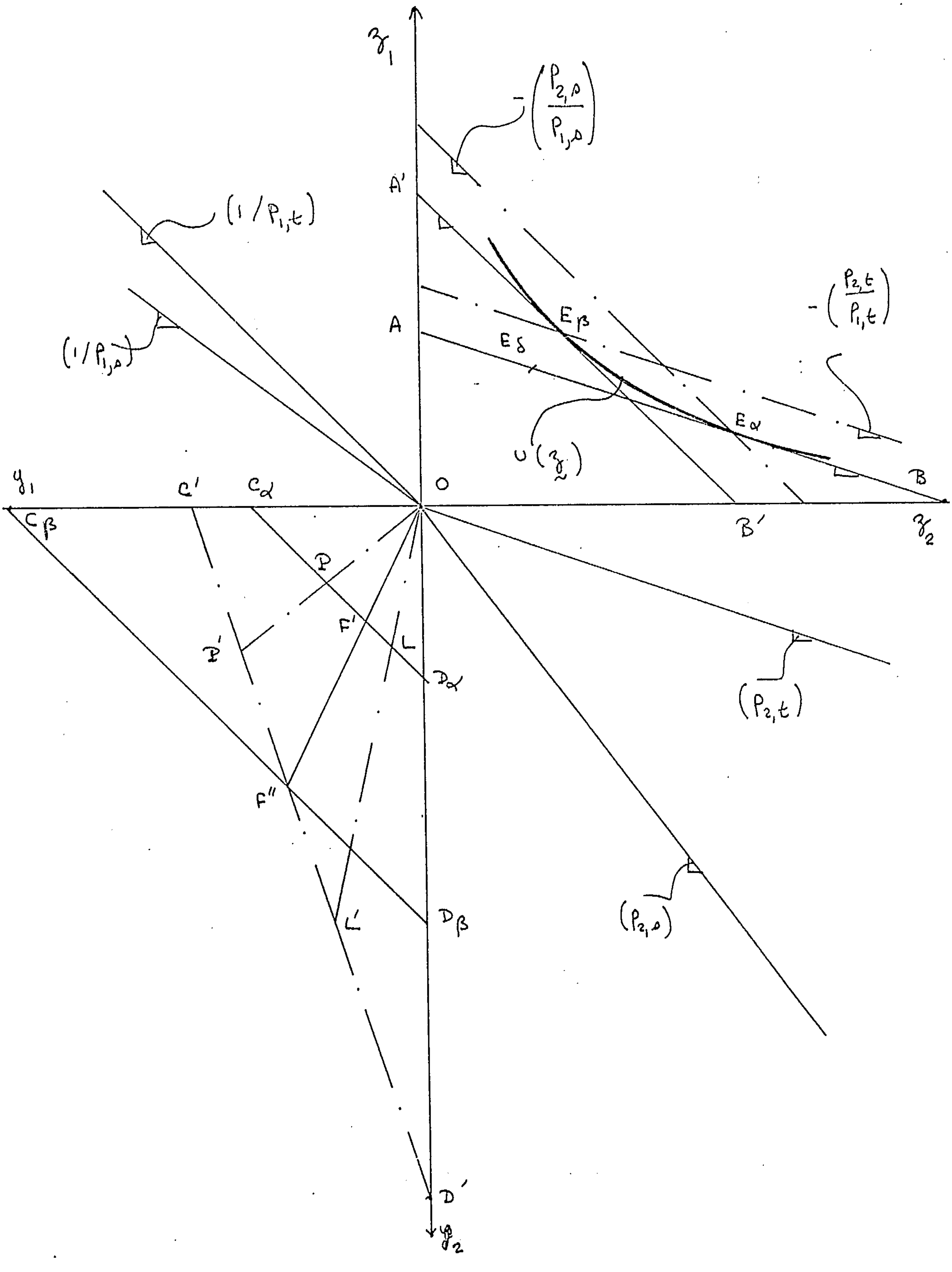

Figure 3.2 The Cost-of-Living Index and the Laspeyres and the Paasche Indexes

$$Q = \frac{F(\underset{\sim}{z}_\beta)}{F(\underset{\sim}{z}_\alpha)} = 1$$

and the whole change is a price change, given by

$$P = \frac{OC_\beta}{OC_\alpha} = \frac{OF'}{OF}$$

The same approach can be followed to develop a cost-of-living index. The only difference is that the economic agents involved are now consumers who maximize utility in terms of a preference mapping. This is illustrated in Fig. 3.2.

The Economic Index and the Laspeyres and Paasche Indexes

The Laspeyres index corresponds to the statistical index defined in terms of the original equilibrium E_α. The Paasche index corresponds to the statistical index defined in terms of the new equilibrium E_β. From results of the preceding section and considering the output deflator, since the production of the commodity, the price of which has increased the most (Fig. 3.1), has increased

$$(p_{1,\beta}/p_{1,\alpha}) < \underline{P}_L < \underline{P} < \underline{P}_P < (p_{2,\beta}/p_{2,\alpha})$$

where $\underline{P}_L$ = (OL'/OL) is the Laspeyres index

$\underline{P}_P$ = (OP'/OP) is the Paasche index

$\underline{P}$ = (OF'/OF0 is the exact economic index.

In terms of the cost-of-living index, since the utility maximizer has the opposite reaction to a change in price, the relatively greater price increase of commodity 2 incites the consumers to substitute commodity 1 in place of commodity 2. The equilibrium point, under utility maximization, moves from E_α under price regime $\underset{\sim}{p}_\alpha$, to E_β under price regime $\underset{\sim}{p}_\beta$, and the ratio (x_1/x_2) increases. The relationship between the Laspeyres and the Paasche indexes is reversed (Fig. 3.2):

$$\frac{p_{1,\beta}}{p_{1,\alpha}} < \underline{P}_P < \underline{P}_C < \underline{P}_L < \frac{p_{2,\beta}}{p_{2,\alpha}}$$

An index number formula is said to be exact if it corresponds to the economic index one would obtain for a given utility or production-type of mapping. In other words, to find out for what type of production mapping the Laspeyres index formula is exact, we need to find the type of mapping such that $\underline{P}_L$ is indeed the economic price index. From Fig. 3.1, it can easily be seen that as the curvature of the production possibility frontier increases about E_α, a given price change from $\underset{\sim}{p}_\alpha$ to $\underset{\sim}{p}_\beta$ implies a smaller substitution between commodities 1 and 2, hence E_β is closer to E_α. At the limit, when the production possibility frontier has a right angle kink at E_α, E_δ and E_β become one and the same, for all price regimes $\underset{\sim}{p}_\alpha$ and $\underset{\sim}{p}_\beta$. It follows that a Laspeyres index is exact for a technology which does not allow substitution between outputs, i.e. a Leontief-type fixed

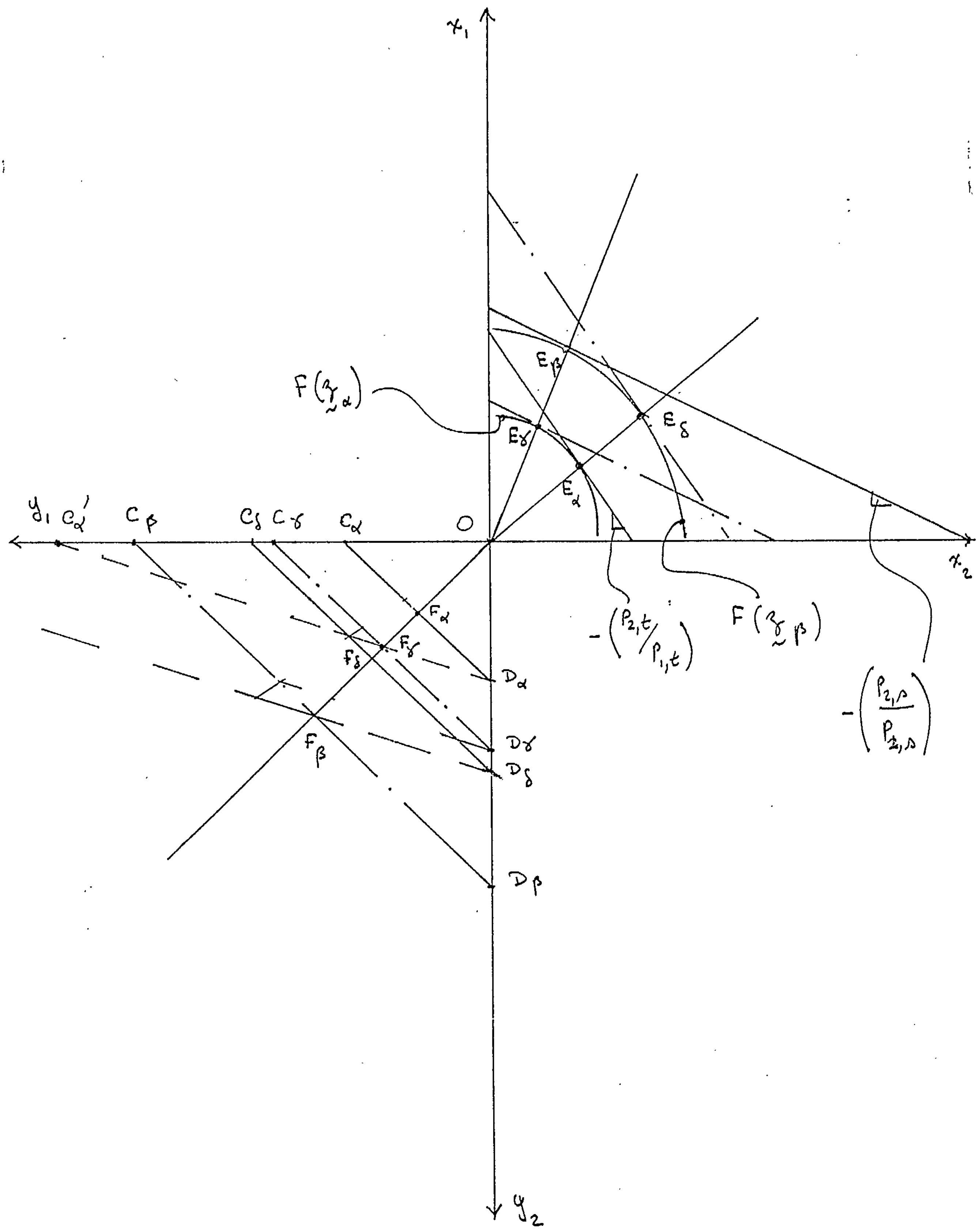

Figure 3.3 Price and Quantity Indexes

coefficient production function (Diewert, 1976).

Following the same argument, we observe that the Paasche price index is exact in terms of the same Leontief-type fixed coefficient production function. Furthermore, if the technology is such that the Paasche index formula is exact, then the Laspeyres index formula is also exact and both must yield the same answer, and vice versa:

$$\text{either } \underline{P}_P = \underline{P} \quad \text{or} \quad \underline{P}_L = P$$

$$\text{implies } \underline{P}_P = \underline{P} = \underline{P}_L$$

Price and Quantity Indexes

Earlier, we had considered a move from E_β to E_α on the assumption that $F(\underset{\sim}{z}_\beta) = F(\underset{\sim}{z}_\alpha)$. We had defined a quantity index

$$Q = \frac{F(\underset{\sim}{z}_\beta)}{F(\underset{\sim}{z}_\alpha)} = 1$$

Assuming that the production possibility function F(z) is homothetic, or even positive linear homogeneous, the same definition can be used unambiguously to define the quantity index even if E_β is not on the same isocurve as E_α. Let the line originating from the origin and passing through E_α intercept the production possibility frontier $F(\underset{\sim}{z}_\beta)$ in E_δ (Fig. 3.3) and let that line originating from the origin which passes through E_β intercept $F(\underset{\sim}{z}_\alpha)$ in E_γ such that $F(\underset{\sim}{z}_\alpha) = F(\underset{\sim}{z}_\gamma)$ and $F(\underset{\sim}{z}_\alpha) = F(\underset{\sim}{z}_\gamma)$. Then the economic quantity index Q is given by

$$Q = \frac{F(\underset{\sim}{z}_\beta)}{F(\underset{\sim}{z}_\alpha)} = \frac{OE_\beta}{OE_\gamma} = \frac{OE_\delta}{OE_\alpha} = \frac{OA_\beta}{OA_\gamma} = \frac{OA_\delta}{OA_\alpha}$$

because of the homotheticity of F.

It is easy to see that the price index is left unaffected; if it is taken in terms of the move from E_α to E_γ as $\underset{\sim}{p}_\alpha$ becomes $\underset{\sim}{p}_\beta$, then

$$\underline{P} = \frac{OF_\gamma}{OF_\alpha} = \frac{OC_\gamma}{OC_\alpha}$$

On the other hand, if $\underline{P}$ is measured in terms of the movement from E_δ to E_β, then

$$\underline{P} = \frac{OF_\beta}{OF_\delta} = \frac{OC_\beta}{OC_\delta}$$

Since OC_δ is the total expenditure given $\underset{\sim}{p}_\alpha$ and $F(\underset{\sim}{z}_\beta)$, then $OC_\delta = Q.OC_\alpha$, and since OC_γ is the total expenditure given $\underset{\sim}{p}_\beta$ and $F(\underset{\sim}{x}_\alpha)$, then $OC_\gamma = OC_\beta/Q$, i.e.

$$\underline{P} = \frac{OC_\gamma}{OC_\alpha} = \frac{OC_\beta/Q}{OC_\alpha} = \frac{OC_\beta}{OC_\delta} = \underline{P}$$

By specifying E_δ and E_γ, we have specified, as with the statistical index, paths to go from E_α to E_β. In fact, homotheticity ensures that the price and the quantity indexes thus obtained are path-independent, i.e. that they are the same regardless of the path followed.

We shall analyze later the problem associated with the absence of homotheticity, and the problem of path dependence which has been wrongly associated with Divisia indexes (Usher, 1974).

ECONOMIC INDEX NUMBERS AND DUALITY THEORIES

Introduction

In the preceding section, a diagrammatic analysis of economic index numbers was introduced, under the hypothesis that the underlying mapping (production or utility) was homothetic. Two problems, however, could not be easily treated in that framework. On the one hand, the analysis failed to emphasize the symmetrical treatment of prices and quantities characteristic of index number theory, and, on the other hand, it did not provide any insight into the homotheticity problem.

The symmetrical treatment of prices and quantities dates back to Fisher (1922) who introduced the factor reversal property. If $P_{t,s}$ and $Q_{t,s}$ are respectively the price and quantity indexes, the factor reversal property requires $P_{t,s}$ and $Q_{t,s}$ to be such that

$$P_{t,s} \cdot Q_{t,s} = \frac{\sum_{i=1}^{n} p_{i,s} q_{i,s}}{\sum_{i=1}^{n} p_{i,t} q_{i,t}}$$

It is furthermore understood that, if $P_{t,s} = f(p_{i,s}, p_{i,t}, q_{i,s}, q_{i,t};\ i=1, 2, \ldots, n)$, then $Q_{t,s} = f(q_{i,s}, q_{i,t}, p_{i,s}, p_{i,t};\ i=1, 2, \ldots, n)$, where f( ) denotes the index formula, a formula which utilizes only price and quantity data. Since f( ) is common to both $Q_{t,s}$ can be recovered from $P_{t,s}$'s formula by substituting the p_i for the q_i and vice versa. The factor reversal test was instrumental to Divisia (1925) proposing a very intuitive approach to the definition of index numbers symmetrical in their treatment of prices and quantities.

A formulation closely related to Fisher's factor reversal test is to require that a price index P_t and a quantity index Q_t, if they exist, be such that

$$P_t \cdot Q_t = \sum_{i=1}^{n} p_{i,t} q_{i,t}$$

where $\{\sum_{i=1}^{n} p_{i,t} q_{i,t}\}$ is an aggregate expenditure.

Total differentiation yields

$$P_t dQ_t + Q_t dP_t = \sum_{i=1}^{n} p_{i,t} dq_{i,t} + \sum_{i=1}^{n} q_{i,t} dp_{i,t}$$

Dividing through by the total expenditure and simplifying, we derive

$$\frac{dQ_t}{Q_t} + \frac{dP_t}{P_t} = \sum_{i=1}^{n} \left\{ \frac{p_{i,t}q_{i,t}}{\sum_{i=1}^{n} p_{i,t}q_{i,t}} \right\} \left(\frac{dq_{i,t}}{q_{i,t}} \right) + \sum_{i=1}^{n} \left\{ \frac{p_{i,t}q_{i,t}}{\sum_{i=1}^{n} p_{i,t}q_{i,t}} \right\} \left(\frac{dp_{i,t}}{p_{i,t}} \right)$$

It can be seen that the two components on the right are respectively weighted averages of the proportional change in quantity $(dq_{i,t}/q_{i,t})$ and of the proportional change in prices. The ith weight is the share of the expenditure allocated to the ith commodity. The Divisia index then simply consists in defining Q_t and P_t such that

$$\frac{dQ_t}{Q_t} = \sum_{i=1}^{n} \left\{ \frac{p_{i,t}q_{i,t}}{\sum_{i=1}^{n} p_{i,t}q_{i,t}} \right\} \left(\frac{dq_{i,t}}{q_{i,t}} \right)$$

$$\frac{dP_t}{P_t} = \sum_{i=1}^{n} \left\{ \frac{p_{i,t}q_{i,t}}{\sum_{i=1}^{n} p_{i,t}q_{i,t}} \right\} \left(\frac{dp_{i,t}}{p_{i,t}} \right)$$

The prices and quantities are treated symmetrically.

This is not the sole reason for developing the Divisia index, nevertheless, it will suffice at this stage.

In the following section, the symmetrical treatment of prices and quantities is approached from the economy theory point of view. The natural approach, in the economic context, is to use the duality theory developed by Shephard (1970) in which the solution to a consumer or producer's optimization can be expressed alternatively in terms of quantities or in terms of prices. A diagrammatic approach to duality, first introduced by Darrough and Southey (1977), is introduced and expanded to present the cost function. Then the quantity index is represented as a movement on the production (or utility) function, while the price index is represented as a movement on the cost function.

One of the reasons for the development of the Divisia index was to account for the path followed through both price and quantity spaces through time. In terms of the economic analysis of index numbers, any exact index number implies some assumption regarding the path. This is formalized in the Divisia analysis. It is important that between any two successive equilibria, since the actual path is not generally known, the index formula selected be independent of the path of integration. However, this is possible only when the production (or utility) mapping is positive linear homogeneous (Hulten, 1973).

Economic Price and Quantity Indexes

Even though the following analysis could be carried out in terms of utility maximizing consumers or profit maximizing producers, it is the latter that is presented here. Let the technology be represented by

$$z = F(\underset{\sim}{x})$$

where z denotes the output
$\underset{\sim}{x}$ an (nxl) input vector

Cost minimization by the producer, in terms of the output level z and the (nxl) input price vector $\underset{\sim}{r}$, will lead to the selection of the input vector $\underset{\sim}{x}$ which involves the lowest cost. When n=2, this is illustrated in Fig. 4.1. The output level z and the price vector $\underset{\sim}{r}$ define a budget line AB which is tangent in E to $F(\underset{\sim}{x})$, determining the quantities x_1 and x_2. However, with cost minimization by the producer, there is an alternative interpretation of E. There is an isoquant $F(\underset{\sim}{x})$ which passes through any point $(\underset{\sim}{x})$ in the output space. That isoquant will have slope (r_2/r_1) at E. Given, in addition, the total expenditure on the inputs, we are able to derive uniquely r_1 and r_2. It follows that the level of output z, under cost minimization, can be derived in two ways. Either it can be derived directly from the input vector through the production function or it can be arrived at indirectly from the input price vector through a new function, the indirect production function. To define the indirect production function, we begin by normalizing the input price vector by the expenditure M to obtain the normalized price vector $\underset{\sim}{\rho}$:

$$\underset{\sim}{\rho} = (1/M)\underset{\sim}{r}$$

Then the indirect production function is defined as

$$G(\underset{\sim}{\rho}) = \max_{\underset{\sim}{x}} F(\underset{\sim}{x});\ \underset{\sim}{\rho}'.\underset{\sim}{x}<1\}$$

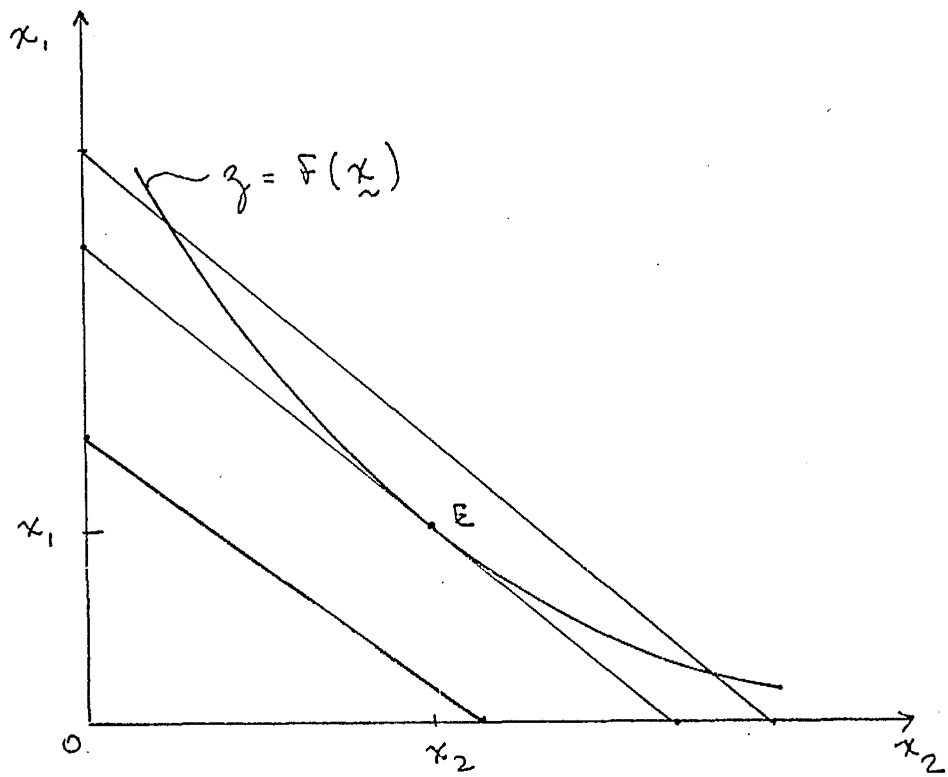

figure 4.1 Equilibrium under Cost Minimization.

$G(\rho)$ is the maximum output level the producer can produce given both the price vector r and the budget constraint $M = 1$. The production fraction F completely determines the indirect product G and, given that both F and G satisfy some regularity conditions, G determines a reconstructed production function associated with F (Blackorby, Primont, and Russell, 1978).

$$\dot{F}(x) = \min_{\rho}\{G(\rho);\ \rho'.x<1\}$$

In Fig. 4.2, quadrant I, Fig. 4.1 is repeated. The budget line AB corresponds to the equation

$$x_1 + (p_2/p_1)x_2 = (1/p_1)$$

i.e. OA corresponds to the total expenditure evaluated in terms of commodity 1 as numéraire and OB corresponds to the same expenditure evaluated in units of commodity 2. If $\bar{x}_1$ is the quantity which corresponds to OA and $\bar{x}_2$ is that which corresponds to OB, then

$$\bar{x}_1 = 1/p_1$$

$$\bar{x}_2 = 1/p_2$$

In quadrants II and IV, we can translate the quantity into the price in terms of the budget $M = 1$ by plotting unit hyperbolae. We can measure p_1 and p_2 on the axis of quadrant III, and we have mapped the original equilibrium E in the quantity space, corresponding to x_1 and x_2 onto the equilibrium J in the normalized price space, corresponding to p_1 and p_2.

The operation can be repeated for every point on the isoquant $z = F(x)$ in the quantity space to determine a new isoquant $G(\rho)$ in the normalized price space.

What matters in terms of the index number problem is that, given z, i.e. given one isoquant of the indirect production function in the normalized price space, the value of G on that isoquant is constant. It equals z, hence it equals z times the expenditure where, here, $M = 1$.

The function which determines the expenditure level M can be defined in the price space. It is the cost function which is defined as

$$C(r,z) = \min_{x} r'.x;\ F(x)>z\}$$

By construction,

$$C(\rho,z) = 1$$

hence,

$$C(\rho,z) = (1/z)G(\rho)$$

However, the cost function is also by construction positive linear homogeneous and

$$C(r,z) = (M/z)G(\rho)$$

This equality implies that the whole cost mapping is a positive

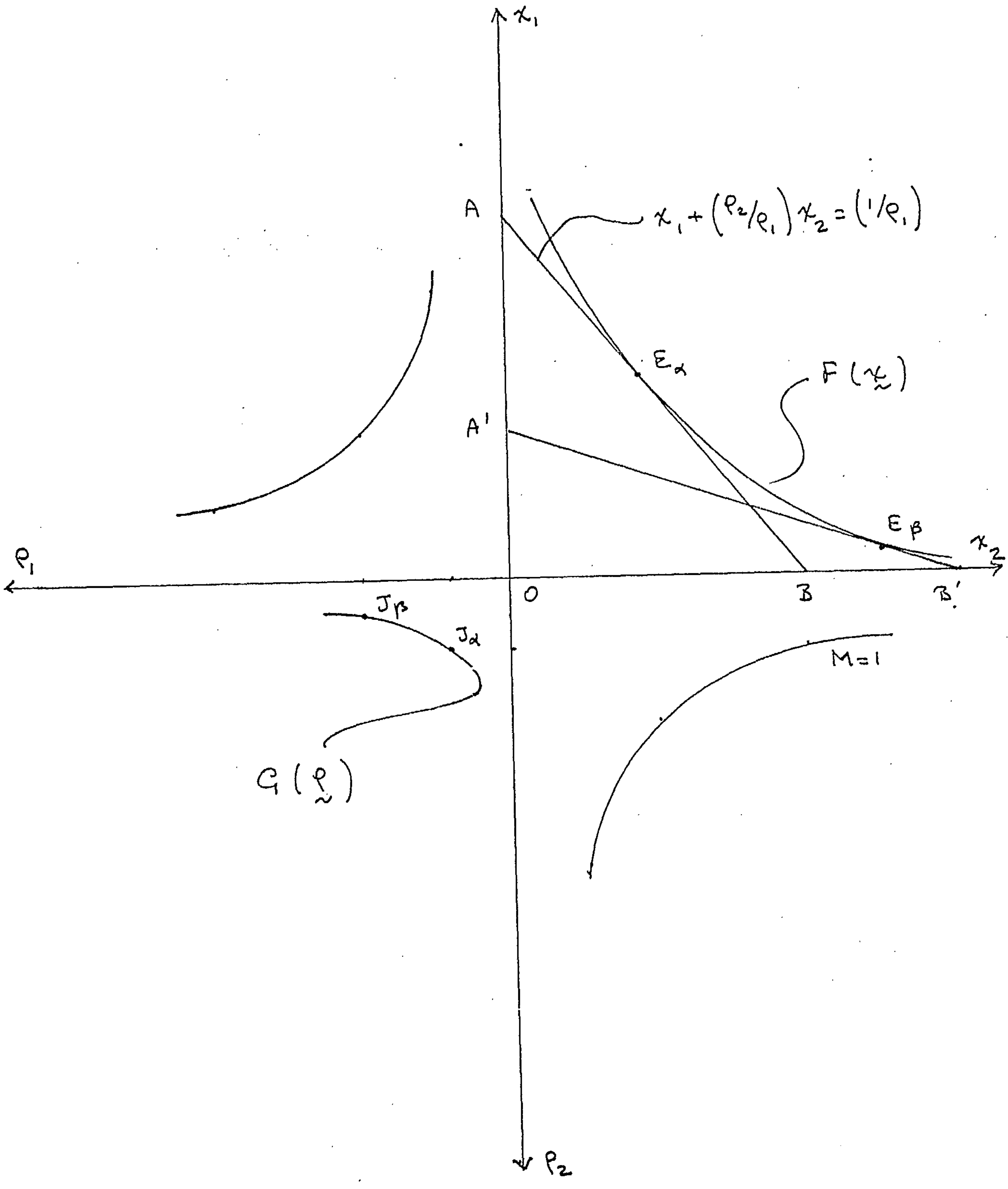

Figure 4.2 Indirect Production Function

linear homogeneous expansion by M of the indirect production isoquant $G(\rho)$ rescaled by $(1/z)$. Darrough and Southey's approach has been extended to yield the cost function in the third quadrant.

The price index can be constructed by the strategy adopted to construct the economic quantity index. If the output level z is given, $F(\underset{\sim}{x})$ can be seen as the input aggregator function. A shift in the relative input prices implies a shift in the input vector from $\underset{\sim}{x}_t$ to $\underset{\sim}{x}_s$ with $F(\underset{\sim}{x}_s) = F(\underset{\sim}{x}_t)$, i.e. such that, where $Q_{s,t}$ is the index,

$$Q_{s,t} = F(\underset{\sim}{x}_s)/F(\underset{\sim}{x}_t) = 1$$

The factor reversal test implies that

$$P_{s,t} = \underset{\sim}{r}_s'.\underset{\sim}{x}_s/\underset{\sim}{r}_t.\underset{\sim}{x}_t$$

Alternatively in the absence of change in the input aggregator function, the shift in expenditure fully reflects a shift in price. Since optimization implies

$$\underset{\sim}{r}'.\underset{\sim}{x} = C(\underset{\sim}{r},z)$$

the definition of the price index $P_{s,t}$ is given by

$$P_{s,t} = \frac{C(\underset{\sim}{r}_s,z)}{C(\underset{\sim}{r}_t,z)}$$

$\underset{\sim}{r}_s$ and $\underset{\sim}{r}_t$ determine two distinct cost isoquants and the price index is represented as in Fig. 4.3, by their relative distance from the origin (OB/OA). Evidently, this is due to the positive linear homogeneity property of the cost function in terms of the price vector.

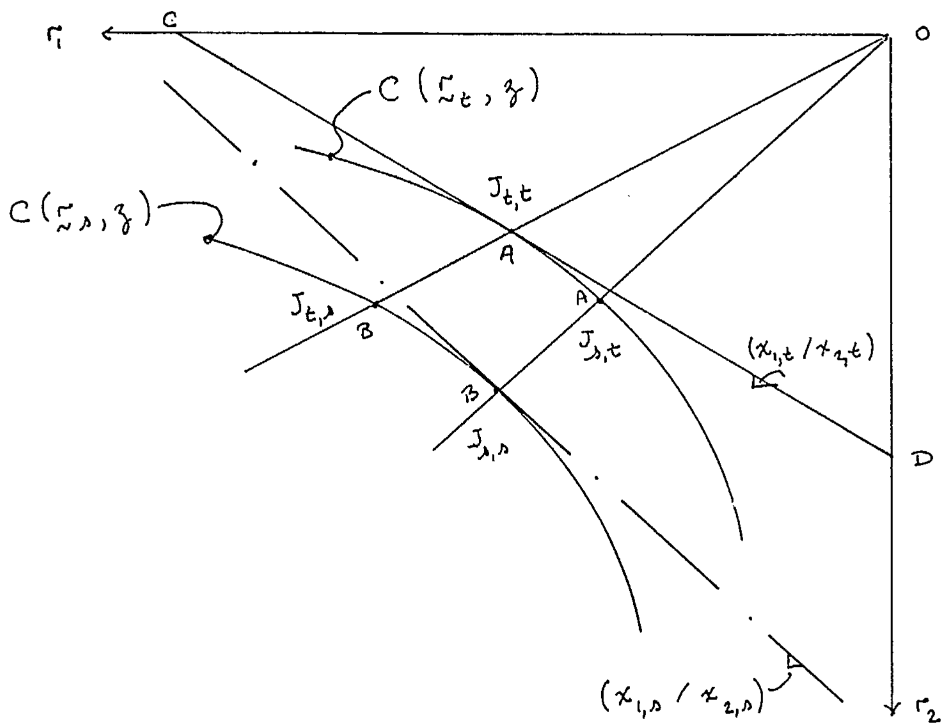

Figure 4.3 Economic Price Index $P_{s,t}$

The above result depends on $F(\underset{\sim}{x}_s) = F(\underset{\sim}{x}_t) = z$. In practice, it is almost always true that $F(\underset{\sim}{x}_s) \neq F(\underset{\sim}{x}_t)$, i.e. that the aggregate quantity of inputs used to produce the output shifts. Then $Q_{s,t} \neq 1$.

In Fig. 4.4, the construction of the indirect production frontier corresponding to the output levels $F(\underset{\sim}{x}_t)$ and $F(\underset{\sim}{x}_s)$ is illustrated. By the rescaling of all inputs by $\lambda = F(\underset{\sim}{x}_s)/F(\underset{\sim}{x}_t)$, there is a point $E_{t,s}$ on $F(\underset{\sim}{x}_s)$ which corresponds in its input composition to the original equilibrium $F(\underset{\sim}{x}_t)$.

Linear homogeneity also implies that the slope of the isoquant is the same in $E_{t,s}$ as in $E_{t,t}$. To construct the indirect production function, we are using $M = 1$ as a constraint. If $\underset{\sim}{r}_t$ is the price regime in $E_{t,t}$ consistent with the unit expenditure level, as the relative prices are unchanged in $E_{t,s}$, the new price regime consistent with $M = 1$ in $E_{t,s}$ must be

$$\underset{\sim}{r}_{t,s} = (1/\lambda)\underset{\sim}{r}_t$$

This only reflects the fact that, given a fixed expenditure level, an expansion by λ in quantities must be accompanied by a contraction by $(1/\lambda)$ in prices.

If $J_{t,t}$ and $J_{t,s}$ are the mappings of $E_{t,t}$ and $E_{t,s}$ respectively in the price space, this implies both that while

$$OE_{t,s} = \lambda OE_{t,t}$$

$$\lambda OJ_{t,s} = OJ_{t,t}$$

and that the slope of the indirect production function in both $J_{t,t}$ and $J_{t,s}$ must be $(x_2/x_1)_t$. That of the production function is $(r_2/r_1)_t$ in both $E_{t,s}$ and $E_{t,t}$.

Two cost functions, corresponding respectively to $F(\underset{\sim}{x}_t)$ and $F(\underset{\sim}{x}_s)$, can thus be constructed in the price space. However, the linear homogeneity of F implies that the second cost mapping $C(\underset{\sim}{r},z_s)$ corresponds to the first cost function, $c(\underset{\sim}{r},z_t)$, rescaled by the factor $(1/\lambda)$. It is a matter of indifference whether the price index is estimated in terms of $C(\underset{\sim}{r},z_t)$ or $C(\underset{\sim}{r},z_s)$:

$$\begin{aligned} P_{s,t} &= \frac{C(\underset{\sim}{r}_s,z_s)}{C(\underset{\sim}{r}_t,z_s)} \\ &= \frac{(1/\lambda)C(\underset{\sim}{r}_s,z_t)}{(1/\lambda)C(\underset{\sim}{r}_t,z_t)} \\ &= \frac{C(\underset{\sim}{r}_s,z_t)}{C(\underset{\sim}{r}_t,z_t)} \end{aligned}$$

where the quantity index is given by

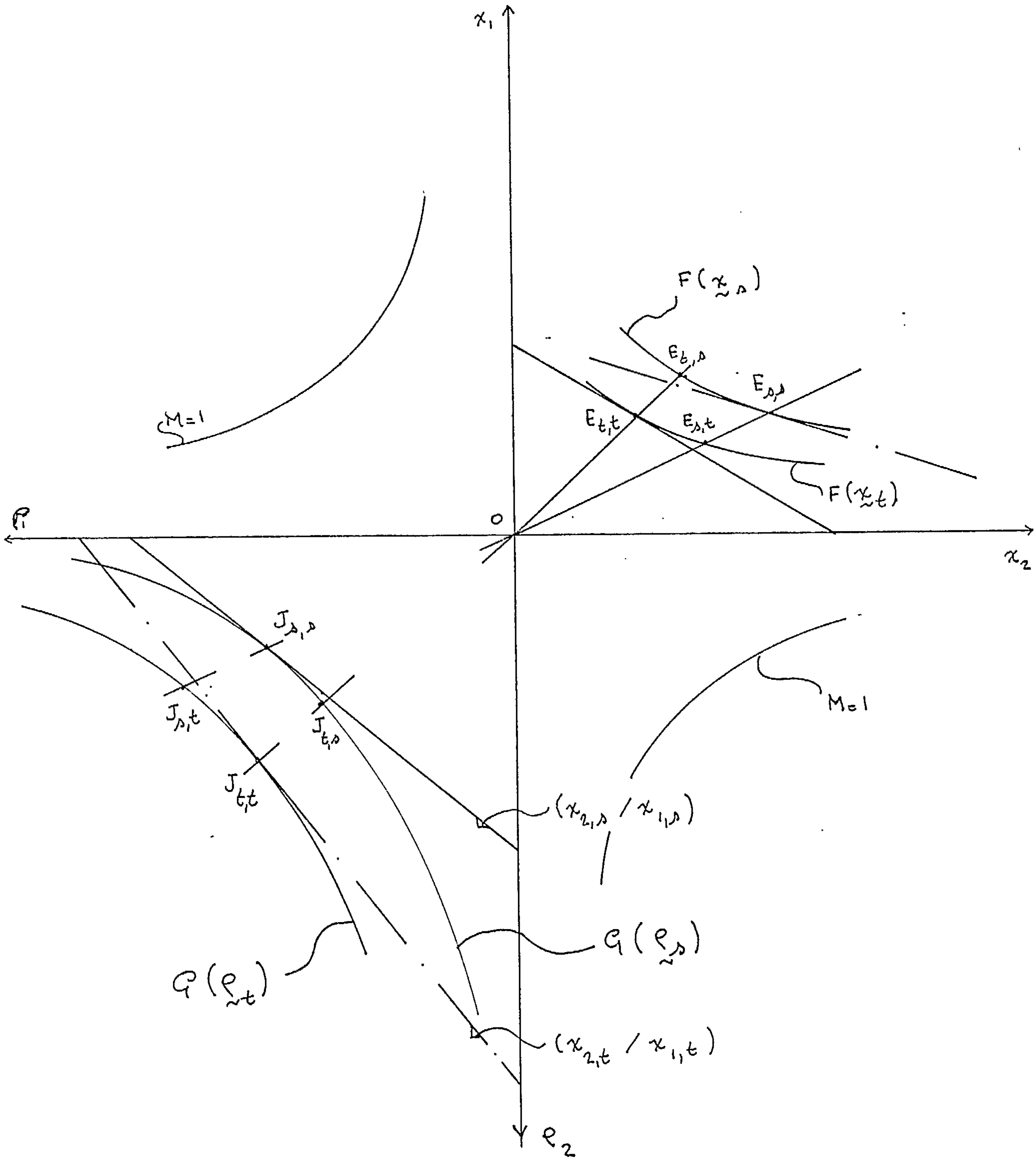

Figure 4.4 Indirect Production Frontiers

$$Q_{s,t} = \frac{F(\underset{\sim}{x}_s)}{F(\underset{\sim}{x}_t)}$$

The linear homogeneity implies that the factor reversal property is met since, as $C(\underset{\sim}{r},z_s) = \lambda C(\underset{\sim}{r},z_t) = F(\underset{\sim}{x}_s)C(\underset{\sim}{r},1)$,

$$P_{s,t} \cdot Q_{s,t} = \frac{C(\underset{\sim}{r}_s,1)}{C(\underset{\sim}{r}_t,1)} \cdot \frac{F(\underset{\sim}{x}_s)}{F(\underset{\sim}{x}_t)}$$

$$= \frac{C\{\underset{\sim}{r}_s,F(\underset{\sim}{x}_s)\}}{C\{\underset{\sim}{r}_t,F(\underset{\sim}{x}_t)\}} = \frac{\underset{\sim}{r}_s' \cdot \underset{\sim}{x}_s}{\underset{\sim}{r}_t' \cdot \underset{\sim}{x}_t}$$

INDEX NUMBERS AND NON-HOMOTHETIC MAPPINGS

Introduction

Until now, the assumption of positive linear homogeneity (or homotheticity) for the mapping considered, be it a utility function or a production function, has been central to the analysis. Recent econometric analyses of the telecommunications sector, however, raise serious doubts as to the validity of that assumption (Corb. and Smith, 1979; Corbo et al., 1979; Denny, Fuss and Everson, 1979; Bernstein, 1980; Breslaw and Smith, 1980).

In this section, solutions proposed to resolve the index number problem given non-homogeneous technologies are reviewed (Diewert, 1979). Before we go further, however, it must be pointed out that such solutions do not resolve the problem; they are no more than rules to define the path and as such, they do not remove the ambiguity created by non-homogeneity. Furthermore, because those rules are not analytical, the choice among them is arbitrary. Full information about the process does not even resolve such an ambiguity. The set of proposed solutions has one common factor: the first step consists in specifying some positive linear homogeneous mapping in terms of which the index can be defined following the procedure introduced earlier. As, with linear homogeneity, the factor reversal property disappears, only one of the price and quantity indexes can be computed directly; the other is derived implicitly. Even though this renders the analysis of the implicit index more difficult, the problem can be bypassed by following the logic of earlier results regarding linear homogeneous mappings. This transforms the "implicit" index into an "explicit" index defined with respect to the positive linear homogeneous function originally selected. This analysis shows that such implicit indexes are equivalent, seen in the context of neoclassical analysis, to hypotheses which are not only consist nt with standard optimizing behaviour but also at least as restrictive.

It was indicated that the first step consists in specifying a positive linear function. In the quantity space, the natural candidate, is the transformation function with which is associated the Malmquist index. In the price space, a natural candidate is the cost function with which is associated the Konus index. Finally, the cost function, given the price vector, can be been as a positive linear homogeneous function in the quantity space, and this is the point of view from which the Allen quantity index is defined.

Without linear homogeneity, these functions are defined in terms of a reference isoquant of the direct or indirect mapping, i.e. in terms of an isoquant in the quantity space for Malmquist indexes and one in the price space for both Konus and Allen indexes. Which reference isoquant is selected is arbitrary, however, two possible criteria are particularly attractive. It could be chosen to pass through the original equilibrium or through the final equilibrium in which case generalized Laspeyres and Paasche formulations are obtained. In both situations, it is possible to compare indexes obtained with Konus, Allen and Malmquist rules unambiguously. These results are illustrated diagrammatically.

Malmquist Quantity and Price Indexes

The positive linear homogeneous mapping, as was indicated in the introduction, is the transformation function, defined in terms of some arbitrary isoquant of the production function, say $F(\underset{\sim}{x}_o)$, in the input space as

$$D\{F(\underset{\sim}{x}_o),\underset{\sim}{x}\} \equiv \max_{\lambda}\{\lambda;\ F(\underset{\sim}{x}/\lambda)>F(\underset{\sim}{x}_o),\ \lambda>o\}$$

The transformation function is a distance function since it specifies how far on a ray from the origin a point is from the reference level $F(\underset{\sim}{x}_o)$. Then the Malmquist quantity index (Diewert, 1979) is defined as the ratio of the relative distance of the new equilibrium to the reference output level in terms of the relative distance of the original equilibrium to the same reference output level. This is illustrated in Fig. 5.1 where the original equilibrium is denoted by $E_{t,t}$, the ray from the origin passing through $E_{t,t}$ intersects the reference level $F(\underset{\sim}{x}_o)$ in $E_{t,o}$, the final equilibrium is $E_{s,s}$, and the ray from the origin passing through $E_{s,s}$ intersects $F(\underset{\sim}{x}_o)$ in $E_{s,o}$. Then the Malmquist quantity index Q_M, given the reference output level $F(\underset{\sim}{x}_o)$, will be given by $\frac{OE_{s,s}}{OE_{s,o}}/\frac{OE_{t,t}}{OE_{t,o}}$. In general, the Malmquist is given by

$$Q_M(\underset{\sim}{x}_s,\underset{\sim}{x}_t,\underset{\sim}{x}_o) = \frac{D\{\underset{\sim}{x}_s,F(\underset{\sim}{x}_o)\}}{D\{\underset{\sim}{x}_t,F(\underset{\sim}{x}_o)\}}$$

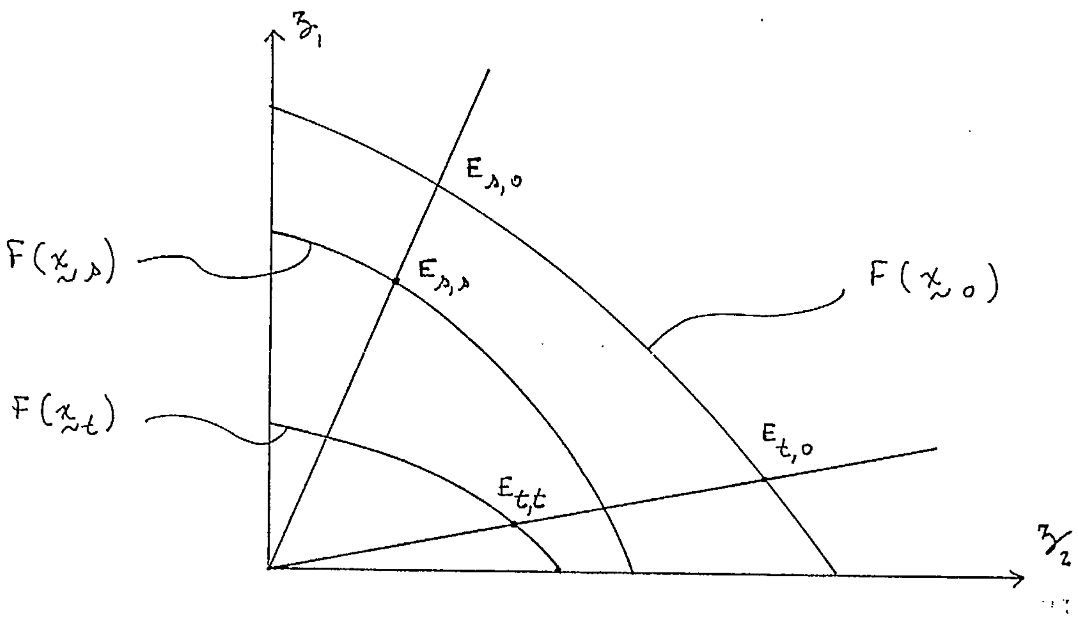

Figure 5.1 Malmquist Quantity Index

An alternative and complementary interpretation of the transformation function is obtained by noting that $D\{\underset{\sim}{x},F(\underset{\sim}{x}_o)\}$ is a possible linear homogeneous function of $\underset{\sim}{x}$; it corresponds to the reference isoquant $F(\underset{\sim}{x}_o)$ rescaled by the factor $\gamma = \{1/F(\underset{\sim}{x}_o)\}$ and expanded by the factor λ. This is illustrated in Fig. 5.2 where the isoquant $D\{\underset{\sim}{x}_s,F(\underset{\sim}{x}_o)\}$ intersects the ray $OE_{t,t}$ in $E_{t,s}$. The Malmquist quantity index is then given by

$$Q_M = \frac{OE_{s,s}}{OE_{s,o}} \quad \frac{OE_{t,t}}{OE_{t,o}}$$

and, by the homogeneity of O, we can deduce that

$$Q_M = \frac{OE_{t,s}}{OE_{t,o}} \quad \frac{OE_{t,t}}{OE_{t,o}}$$

$$= \frac{OE_{t,s}}{OE_{t,t}}$$

$$= \frac{OE_{s,t}}{OE_{t,t}}$$

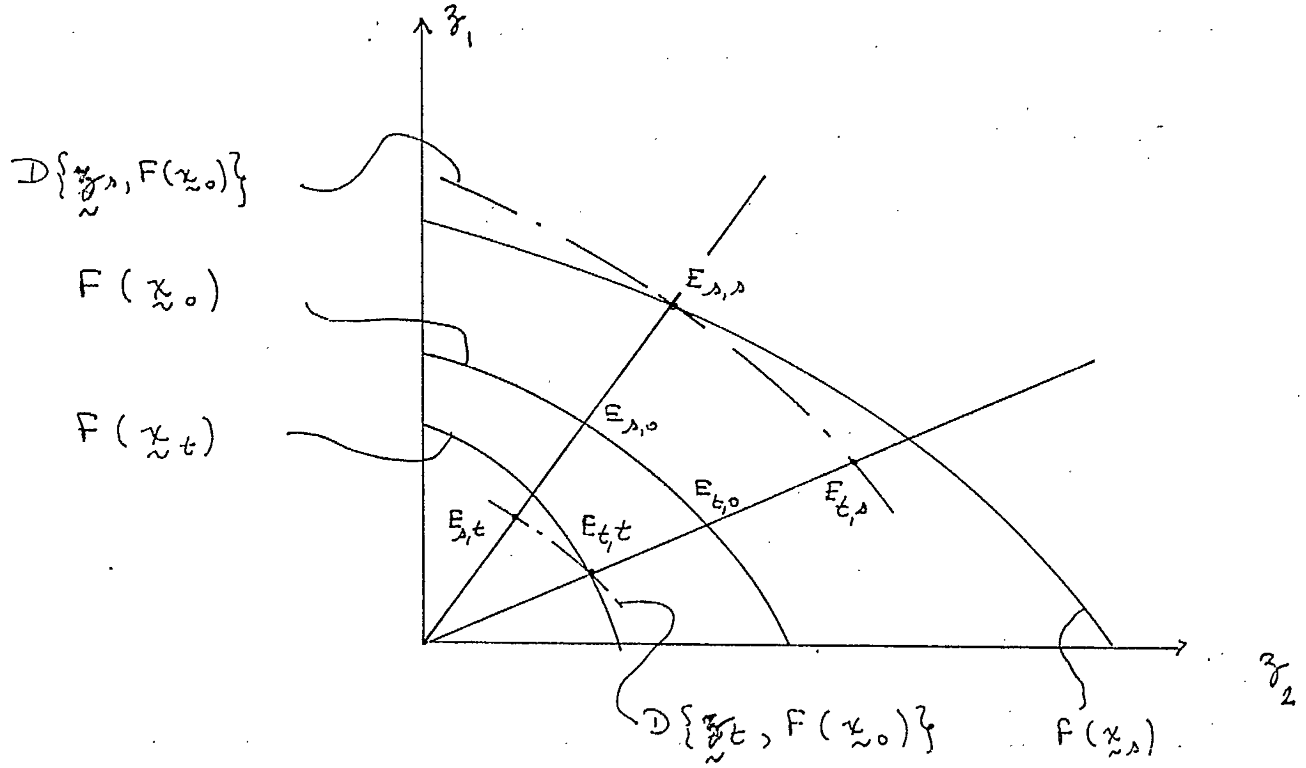

Figure 5.2 Malmquist Quantity Index and the Transformation function $D\{\underset{\sim}{x}, F(\underset{\sim}{x}_o)\}$

The Malmquist quantity index is seen to be equivalent to the quantity index for positive linear homogeneous technologies introduced earlier. Now, however, the transformation function has the role of a pseudo-production function. The two yield identically the same answer whenever the production function is positive linear homogeneous since then D is a rescaling of F:

$$D\{\underset{\sim}{x},F(\underset{\sim}{x}_o)\} = F(\underset{\sim}{x})/F(\underset{\sim}{x}_o)$$

The advantage of this alternative interpretation of the Malmquist quantity index now becomes evident as we consider the implicit price index it implies. The implicit Malmquist price index is defined residually by the imposition of the factor reversal property as:

$$\tilde{P}_M(\underset{\sim}{r}_s,\underset{\sim}{r}_r,\underset{\sim}{x}_s,\underset{\sim}{x}_t,\underset{\sim}{x}_o) = \left\{\frac{\underset{\sim}{r}_s'\cdot\underset{\sim}{x}_s}{\underset{\sim}{r}_s'\cdot\underset{\sim}{x}_t}\right\} \Big/ Q_M(\underset{\sim}{x}_s,\underset{\sim}{x}_t,\underset{\sim}{x}_o)$$

Using the residual representation, there does not seem to exist any intuitive interpretation of $\tilde{P}_M$. However, a direct derivation of $\tilde{P}_M$ is now presented.

$D\{\underset{\sim}{x}_o,F(\underset{\sim}{x}_o)\}$ is the $F(\underset{\sim}{x}_o)$ isoquant rescaled by the factor $\gamma = \{1/F(\underset{\sim}{x}_o)\}$. Since $D\{\underset{\sim}{x},F(\underset{\sim}{x}_o)\}$ is a positive linear homogeneous function of $\underset{\sim}{x}$, as we saw earlier, the quantity index is obtained by $D\{\underset{\sim}{x}_s,F(\underset{\sim}{x}_o)\}/D\{\underset{\sim}{x}_t,F(\underset{\sim}{x}_o)\}$.

To the isoquant $F(\underset{\sim}{x}_o)$ in the quantity space corresponds an isoquant of the indirect production function $G(\underset{\sim}{\rho}) = F(\underset{\sim}{x}_o)$ in the price space. From the isoquant $G(\rho)$, a cost mapping can be constructed in the price space. In fact, using earlier results on positive linear homogeneous production functions, the transformation function $D\{\underset{\sim}{x},F(\underset{\sim}{x}_o)\}$ can be used to construct families of pseudo-cost functions which will only differ from one another by a scalar. Those cost functions will be of the form $C(\underset{\sim}{r}^o,D\{\underset{\sim}{x},F(\underset{\sim}{x}_o)\})$ where $\underset{\sim}{r}^o$ denotes the pseudo-price vector implied by the expenditure M and tangency to $D\{\underset{\sim}{x},F(\underset{\sim}{x}_o)\}$. As both they and the transformation functions are positive linear homogeneous by construction, the implicit Malmquist price index which corresponds to the Malmquist quantity index can alternatively be expressed as the following explicit index:

$$D_M(\underset{\sim}{r}_s,\underset{\sim}{r}_t,\underset{\sim}{x}_s,\underset{\sim}{x}_t,\underset{\sim}{x}_o) = \frac{C(\underset{\sim}{r}_s^o,D\{\underset{\sim}{x},F(\underset{\sim}{x}_o)\})}{C(\underset{\sim}{r}_t^o,D\{\underset{\sim}{x},F(\underset{\sim}{x}_o)\})}$$

where the quantity vector $\underset{\sim}{x}$ in the transformation function could be $\underset{\sim}{x}_t$, $\underset{\sim}{x}_s$, $\underset{\sim}{x}_o$ or any other quantity vector.

The Malmquist index is the index which is based upon the transformation function which is itself interpreted as a pseudo-production function and a positive linear homogeneous substitute for the production function F(x). It is illustrated in Fig. 5.3. To the isoquants $F(\underset{\sim}{x}_t)$ and $F(\tilde{\underset{\sim}{x}}_s)$ have been substituted the isoquants

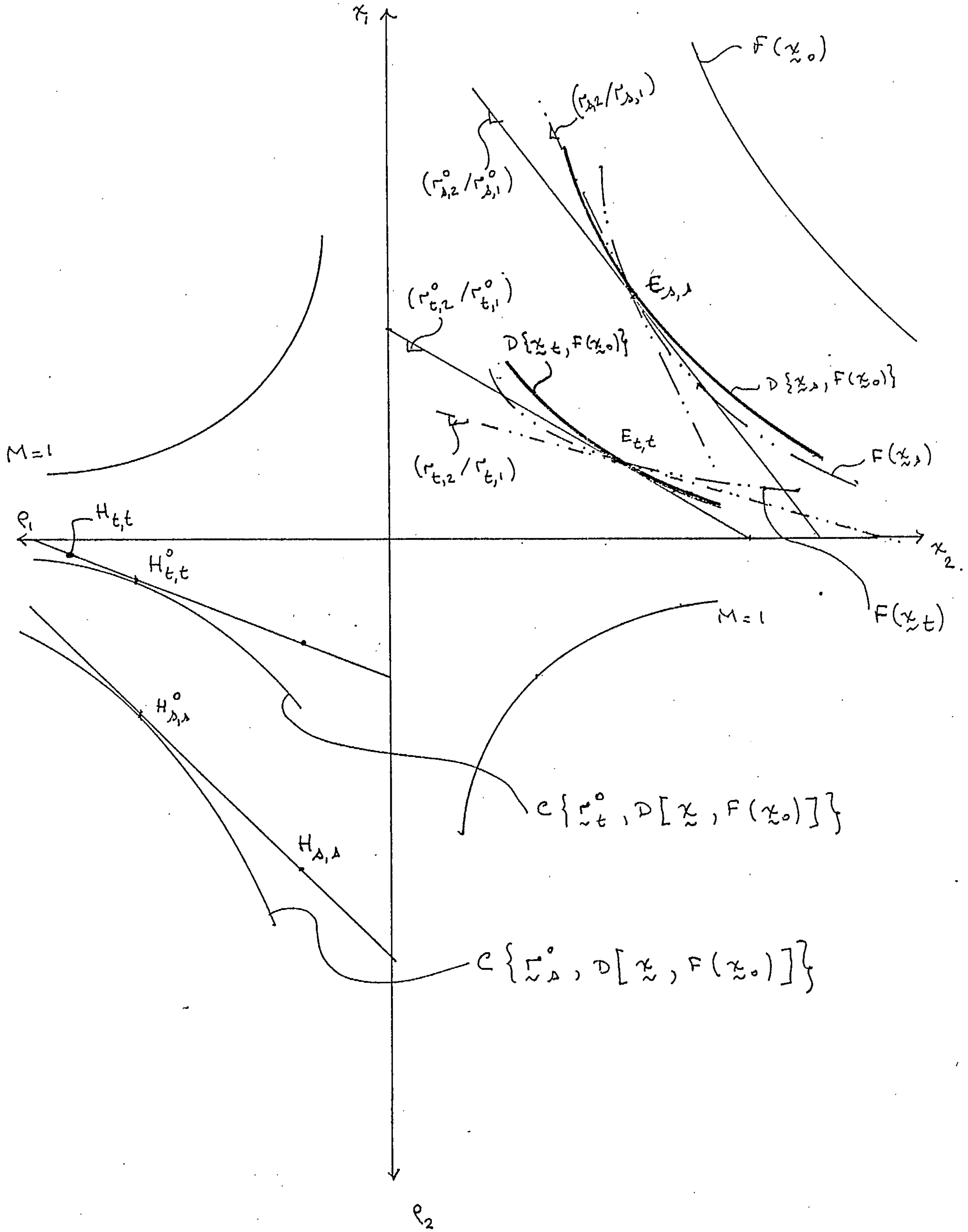

Figure 5.3 Malmquist Price and Quantity Indexes

$D\{\tilde{x}_t, F(\tilde{x}_o)\}$ and $D\{\tilde{x}_s, F(\tilde{x}_o)\}$; to the input price vectors $\tilde{r}$ have been substituted pseudo-input price vectors $\tilde{r}^o$; finally to the cost function $C\{\tilde{r}, F(\tilde{x})\}$ has been substituted the pseudo-cost function $C(\tilde{r}^o, D\{\tilde{x}, F(\tilde{x}_o)\})$. The implications of this analysis is important since, while it is true that the Malmquist quantity index does not require the cost minimization assumption (Diewert, 1979), the preceding analysis shows that it is not consistent with every behavioural assumption; in fact, it is inconsistent with cost minimization. The Malmquist index is consistant with only one behaviour which is not optimal, given a neoclassical analysis.

First of all, it is seen that a change in the slope of $F(\tilde{x}_t)$ and/or $F(\tilde{x}_s)$, due to, say, technological progress, does not modity P_M, i.e. that properly speaking, both Q_M and P_M are independent of $\tilde{r}_t$ and $\tilde{r}_s$.

Konus and Allen Indexes

For an existing approach to defining an index, some positive linear homogeneous reference mapping is needed to construct price and quantity indexes. The alternatives considered were to work either with the function which is definitionally positive linear homogeneous in the quantity space, the transformation function, or with the function which is definitionally positive linear homogeneous in the price space, the cost function. As in the case of the transformation function used to construct the Malmquist index, the cost function is dependent upon the output level, and there is no unambiguous choice of a reference level. Let $F(\tilde{x}_o)$ be the reference output level, then, as illustrated in Fig. 5.4, the Konus price index, P_k, is defined as (Diewert, 1979)

$$P_k(\tilde{r}_s, \tilde{r}_t, \tilde{k}_o) = \frac{C\{\tilde{r}_s, F(\tilde{x}_o)\}}{C\{\tilde{r}_t, F(\tilde{x}_o)\}}$$

If F is indeed positive linear homogeneous, then it is the price index defined earlier.

The quantity index is also obtained residually as an implicit Konus quantity index through the imposed factor reversal property:

$$\tilde{Q}_k(\tilde{x}_s, \tilde{x}_t, \tilde{r}_s, \tilde{r}_t, \tilde{x}_o) = \left\{ \frac{\tilde{r}_s^1 \cdot \tilde{x}_s}{\tilde{r}_t^1 \cdot \tilde{x}_t} \right\} \Big/ P_k(\tilde{r}_s, \tilde{r}_t, \tilde{x}_o)$$

As with the Malmquist indexes, it is useful and possible to find the corresponding explicit form of $\tilde{Q}_k$. In fact, the same approach can be adopted, the role played by the price space and the quantity space being reversed. P_k is derived from the cost function $C\{\tilde{r}, F(\tilde{x}_o)\}$, which is itself derived from the indirect production isoquant $G(\tilde{\rho}) = F(\tilde{x}_o)$ which was obtained from $F(\tilde{x}_o)$.

We know from price and quantity indexes derived from positive linear homogeneous production function that the factor reversal property implies

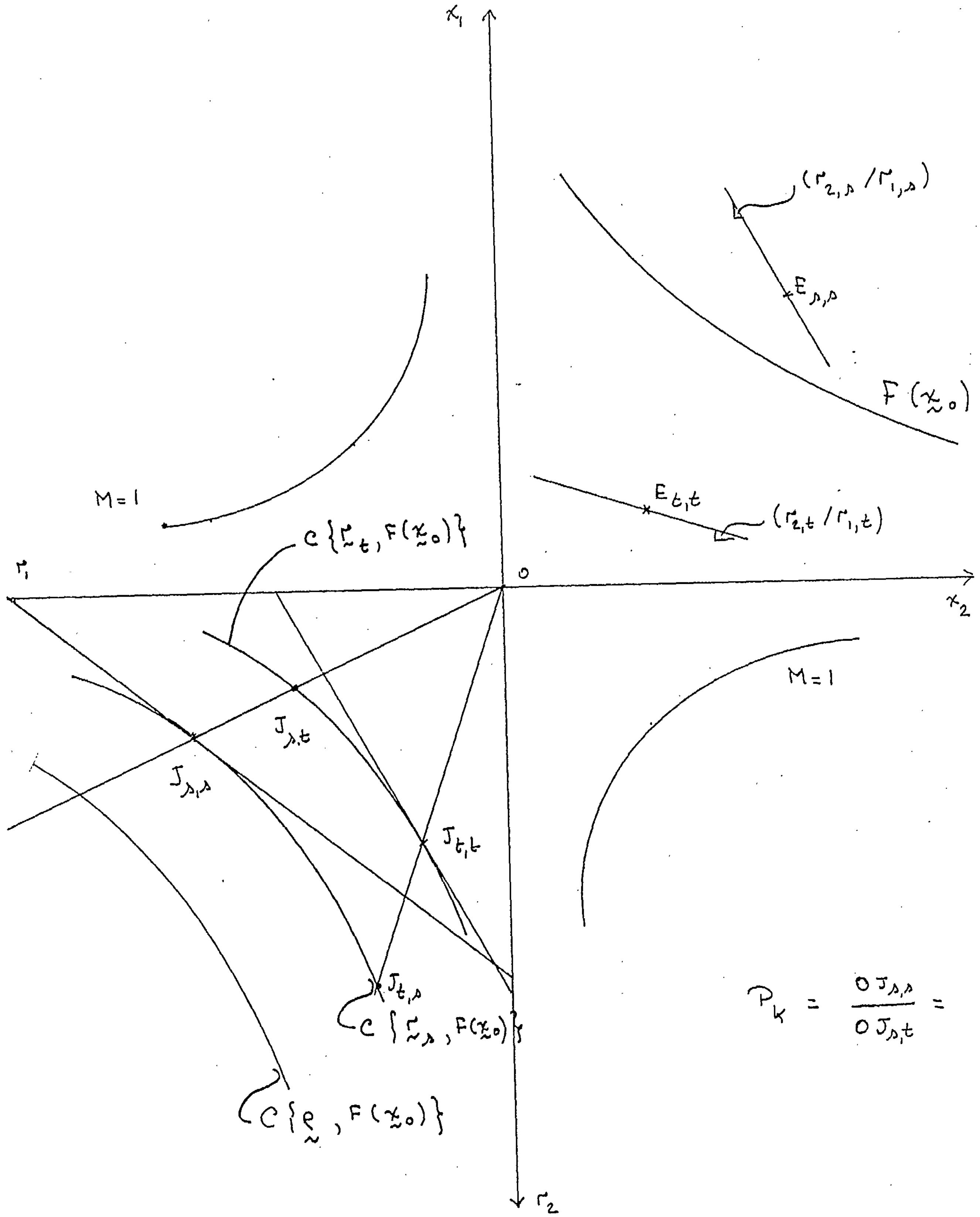

Figure 5.4 Konüs Price Index

$$P = \frac{C\{\underset{\sim}{r}_s, F(.)\}}{C\{\underset{\sim}{r}_t, F(.)\}}$$

$$Q = \frac{F(\underset{\sim}{x}_s)}{F(\underset{\sim}{x}_t)}$$

It follows that we must substitute to the non-homogeneous function $F(\underset{\sim}{x})$ a mapping which is linear homogeneous. From the section on the Malmquist indexes, an obvious candidate is the transformation function $D\{\underset{\sim}{x}, F(\underset{\sim}{x}_o)\}$. At first, it may seem that we would only need $D\{\underset{\sim}{x}_s, F(\underset{\sim}{x}_o)\}$ and $D\{\underset{\sim}{x}_t, F(\underset{\sim}{x}_o)\}$, however, this is not the case. The absence of positive linear homogeneity implies that the slopes of $D\{\underset{\sim}{x}_s, F(\underset{\sim}{x}_o)\}$ and $D\{\underset{\sim}{x}_t, F(\underset{\sim}{x}_o)\}$ at $\underset{\sim}{x}_s$ and $\underset{\sim}{x}_t$ are inconsistent with r_s and r_t. In fact, to derive the quantity vectors needed, quantity vectors such that the budget line be tangent to the transformation function, a new function must be defined as

$$H\{\underset{\sim}{x}^o, F(\underset{\sim}{x}_o)\} = \min_{\underset{\sim}{x}^o}(D\{\underset{\sim}{x}^o, F(\underset{\sim}{x}_o)\}; \underset{\sim}{r}^1.\underset{\sim}{x}^o < M)$$

This is illustrated in Fig. 5.5 where D_1 and D_2 are two transformation isoquants which intersect the budget constraint while D_3 is just tangent to H.

The Konus quantity index is now derived as

$$Q_k(\underset{\sim}{x}_s, \underset{\sim}{x}_t, \underset{\sim}{r}_s, \underset{\sim}{r}_t, \underset{\sim}{x}_o) = \frac{H\{\underset{\sim}{x}_s^o, F(\underset{\sim}{x}_o)\}}{H\{\underset{\sim}{x}_t^o, F(\underset{\sim}{x}_o)\}}$$

where $\underset{\sim}{x}^o$ is the pseudo-quantity vector corresponding to the tangency of the transformation function to the expenditure constraint.

An alternative approach to defining a quantity index is the Allen quantity index which uses simultaneously the expenditure constraint and the cost function:

$$Q_A(\underset{\sim}{x}_s, \underset{\sim}{x}_t, \underset{\sim}{r}) = \frac{C\{\underset{\sim}{r}, F(\underset{\sim}{x}_s)\}}{C\{\underset{\sim}{r}, F(\underset{\sim}{x}_t)\}}$$

An implicit Allen price index is derived through the factor reversal property.

Of particular interest are two situations in which the Konus and the Allen quantity indexes are equal:

$$P_k \cdot Q_k \equiv \frac{C\{\underset{\sim}{r}_s, F(\underset{\sim}{x}_s)\}}{C\{\underset{\sim}{r}_t, F(\underset{\sim}{x}_t)\}}$$

$Q_A = Q_k$ which implies $P_k Q_A = P_k Q_k$ will hold if

$$P_k Q_A = \frac{C\{\underset{\sim}{r}_s, F(\underset{\sim}{x}_o)\}}{C\{\underset{\sim}{r}_t, F(\underset{\sim}{x}_o)\}} \cdot \frac{C\{\underset{\sim}{r}, F(\underset{\sim}{x}_s)\}}{C\{\underset{\sim}{r}, F(\underset{\sim}{x}_t)\}} = \frac{C\{\underset{\sim}{r}_s, F(\underset{\sim}{x}_s)\}}{C\{\underset{\sim}{r}_t, F(\underset{\sim}{x}_t)\}}$$

if and only if either $\underset{\sim}{r} = \underset{\sim}{r}_t$ and $\underset{\sim}{x}_o = \underset{\sim}{x}_s$ or $\underset{\sim}{r} = \underset{\sim}{r}_t$ and $\underset{\sim}{x}_o = \underset{\sim}{x}_t$.

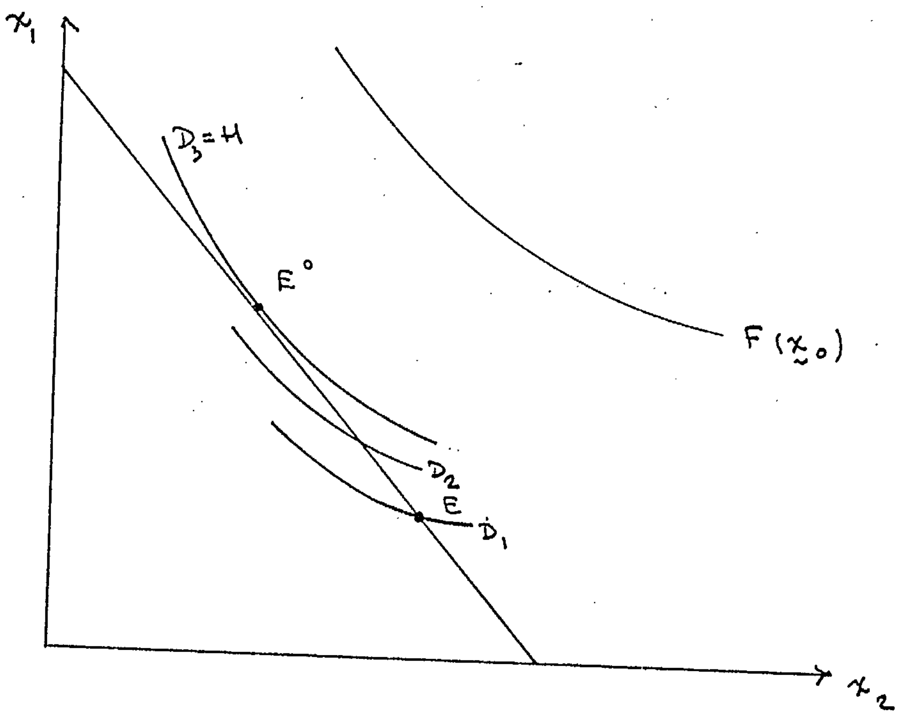

Figure 5.5 $H\{\underset{\sim}{x}^\circ, F(\underset{\sim}{x}_0)\} = \min_{\underset{\sim}{x}^\circ} \{D[\underset{\sim}{x}^\circ, F(\underset{\sim}{x}_0)] ; \underset{\sim}{r}' \cdot \underset{\sim}{x}^\circ \leq M\}$

Malmquist, Konus and Allen Indexes

We have seen that Konus and Allen indexes could unambiguously be compared in two distinct situations. A Laspeyeres (Paasche)-Konus price index is the index based on the reference output level $\underset{\sim}{x}_t(\underset{\sim}{x}_s)$ and a Laspeyres (Paasche)-Allen quantity index is the index defined with respect to the reference price vector $\underset{\sim}{r}_t(\underset{\sim}{r}_s)$. Given these definitions, it was shown that the Laspeyres-Konus index and the Paasche-Allen index could be compared, as the Paasche-Konus and Laspeyres-Allen indexes. However, in both these situations, the Allen-Konus indexes can also be compared to the Malmquist one. Diewert (1979) has established the theorem in terms of the quantity indexes; it can also be shown in terms of price indexes with the present analysis. We illustrate the theorem in terms of the Laspeyres-Malmquist price indexes, where the Laspeyres-Malmquist price index is defined as $P_M(\underset{\sim}{r}_s,\underset{\sim}{r}_t,\underset{\sim}{x}_s,\underset{\sim}{x}_t,\underset{\sim}{x}_t)$.

Let the output level correspond to the original equilibrium, i.e. $F(\underset{\sim}{x}_o) = F(\underset{\sim}{x}_t)$. In Fig. 5.6, this is represented by the original equilibrium point, $E_{t,t}$. The Malmquist index implies, in the price space, the cost function $C(\underset{\sim}{r}^o,D\{\underset{\sim}{x}_t,F(\underset{\sim}{x}_t)\})$ where $\underset{\sim}{r}^o$ is the vector of "pseudo-prices" implied by the expenditure level M_t and by the slope of the transformation function $D\{\underset{\sim}{x}_t,F(\underset{\sim}{x}_t)\}$. The Konus price index implies the cost function $C\{r,F(\underset{\sim}{x}_t)\}$. By definition $D\{\underset{\sim}{x}_t,F(\underset{\sim}{x}_t)\} = 1$ which is a rescaling by $\{1/F(\underset{\sim}{x}_t)\}$ of the production function. Since the expenditure is the same for both the Malmquist and the Konus index, and since the cost functions are both based on the same isoquant in the quantity space, then as $\underset{\sim}{r}_t^o = \underset{\sim}{r}_t$,

$$C\{\underset{\sim}{r}_t,F(\underset{\sim}{x}_t)\} = C(\underset{\sim}{r}_t^o,D\{\underset{\sim}{x}_t,F(\underset{\sim}{x}_t)\})$$

$E_{s,s}$, the new equilibrium point corresponding to the new quantity vector $\underset{\sim}{x}_s$, is both on the production function's isoquant $F(\underset{\sim}{x}_s)$ and on the transformation function isoquant $D\{\underset{\sim}{x}_s,F(\underset{\sim}{x}_t)\}$. Given that the ray from the origin which passes through $E_{s,s}$ intersects $F(\underset{\sim}{x}_t)$ in $E_{s,t}$, the slope of the transformation function in $E_{s,s}$ will be the same as that of $F(\underset{\sim}{x}_t)$ in $E_{s,t}$. If it were the case that the slope of the production function $F(\underset{\sim}{x}_s)$ were the same in $E_{s,s}$ as that of $F(\underset{\sim}{x}_t)$ in $E_{s,t}$, then the isoquant $F(x_s)$ would just be a linear homogeneous expansion of $F(\underset{\sim}{x}_t)$ which would contradict the original assumption. Hence, in general, in $E_{s,s}$, the slope of $F(\underset{\sim}{x}_s)$ differs from that of $D\{\underset{\sim}{x}_s,F(\underset{\sim}{x}_t)\}$. The slopes of those two functions together with the common expenditure constraint M_s implies a difference between the price vector corresponding to $F(\underset{\sim}{x}_s)$ and the pseudo-price vector corresponding to $D\{\underset{\sim}{x}_s,F(\underset{\sim}{x}_t)\}$. Then

$$\underset{\sim}{r}^o \neq \underset{\sim}{r}_s$$

and $E_{s,s}$ is mapped to two different points in the price space, J_s^M

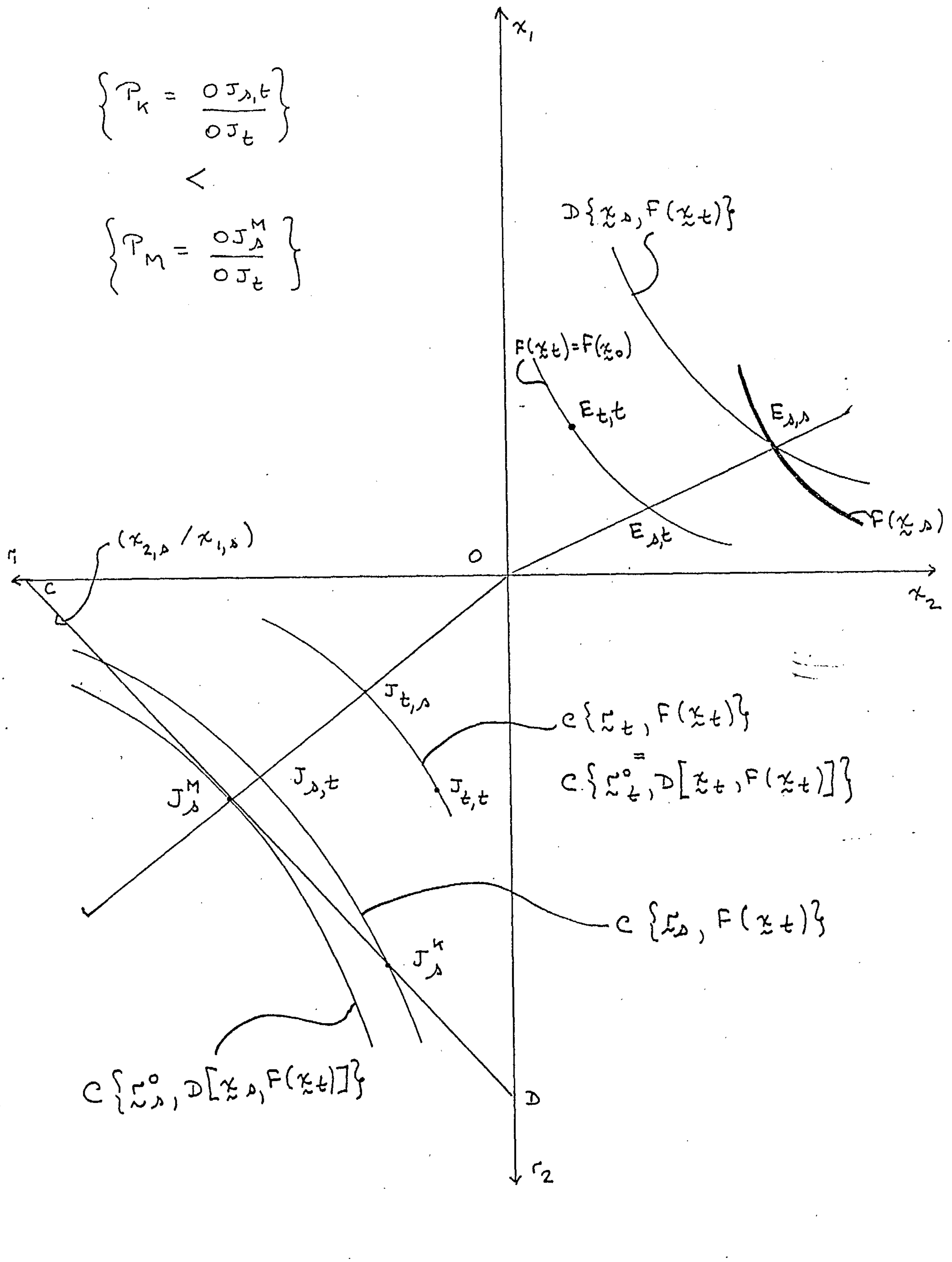

Figure 5.6

Comparison of Laspeyres-Malmquist and Konüs Price Indexes and J_s^k, depending whether the index is a Malmquist index or a Konus index. J_s^M is the price vector which corresponds to the pseudo-cost function $C(\underset{\sim}{r}_s^o, D\{\underset{\sim}{x}_t, F(\underset{\sim}{x}_t)\})$. J_s^k is the price vector given by $C\{\underset{\sim}{r}_s, F(\underset{\sim}{x}_t)\}$.

Since $D\{\underset{\sim}{x}_t, F(\underset{\sim}{x}_t)\}$ is positive linear homogeneous, the slope of the cost function in J_s^M is given by the relative quantities associated with $E_{s,s}$, i.e. $(x_{2;s}/x_{1;s})$. Since J_s^k corresponds to the same quantities, it must be on the line CD with slope $(x_{2;s}/x_{1;s})$ with the third quadrant. Since the cost function of $C(\underset{\sim}{r}_s^o, D\{\underset{\sim}{x}_t, F(\underset{\sim}{x}_t)\})$ is tangent to CD in J_s^M, and since it is positive linear homogeneous, $C\{\underset{\sim}{r}_s, F(\underset{\sim}{x}_t)\}$ must intersect the line CD in J_s^k. The isoquant $C\{\underset{\sim}{r}_s^o, F(\underset{\sim}{x}_t)\}$ is contained in $C\{\underset{\sim}{r}_s, F(\underset{\sim}{x}_t)\}$. Whenever $P_M > 1$,

$$P_M = \frac{C(\underset{\sim}{r}_s^o, D\{\underset{\sim}{x}_t, F(\underset{\sim}{x}_t)\})}{C(\underset{\sim}{r}_t^o, D\{\underset{\sim}{x}_t, F(\underset{\sim}{x}_t)\})} > \underline{P}_k = \frac{C\{\underset{\sim}{r}_s, F(\underset{\sim}{x}_t)\}}{C\{\underset{\sim}{r}_t, F(\underset{\sim}{x}_t)\}}$$

Finally, as we have shown that the Laspeyres or Paasche-Allen index was equal to the Laspeyres or Paasche-Konus index,

$$P_M > P_k = P_A$$

Had the total expenditure decreased or $F(\underset{\sim}{x}_s)$ be used as a reference isoquant and the expenditure increased, the inequality would have been reversed.

The result is illustrated in Fig. 5.7 in terms of the quantity indexes. The Laspeyres-Konus price index indicates by definition that the price vector $\underset{\sim}{r}_s$ with the expenditure level M_s implies that $H\{\underset{\sim}{x}_s^o, F(\underset{\sim}{x}_t)\}$ has the slope $(r_{2;s}/r_{1;s})$ in $\underset{\sim}{x}_s^o$. This is the slope of the isoquant $F(\underset{\sim}{x}_s)$ in $E_{s,s}$. However, as $H\{\underset{\sim}{x}^o, F(\underset{\sim}{x}_t)\}$ is positive linear homogeneous, in general $\underset{\sim}{x}_s \neq \underset{\sim}{x}_s^o$. The expenditure constraint is given by AB and $H\{\underset{\sim}{x}_s^o, F(\underset{\sim}{x}_t)\}$ is tangent to AB by definition. The tangency point is denoted by E_s^k. The Malmquist quantity index is defined in terms of the transformation function $D\{\underset{\sim}{x}_s, F(\underset{\sim}{x}_t)\}$ which passes through $E_{s,s}$. As $H\{\underset{\sim}{x}, F(\underset{\sim}{x}_t)\} - D\{\underset{\sim}{x}, F(\underset{\sim}{x}_t)\}$ i.e. the same family of isoquants is used for both mappings and since $H\{\underset{\sim}{x}_s^o, F(\underset{\sim}{x}_t)\} = D\{\underset{\sim}{x}_s^o, F(\underset{\sim}{x}_t)\}$ is tangent to AB, $D\{\underset{\sim}{x}_s, F(\underset{\sim}{x}_t)\}$ must be on an isoquant which contains $D\{\underset{\sim}{x}_s^o, F(\underset{\sim}{x}_t)\}$. In other words,

$$Q_M = \frac{D\{\underset{\sim}{x}_s, F(\underset{\sim}{x}_t)\}}{D\{\underset{\sim}{x}_t, F(\underset{\sim}{x}_t)\}} < Q_k = \frac{H\{\underset{\sim}{x}_s^o, F(\underset{\sim}{x}_t)\}}{H\{\underset{\sim}{x}_t, F(\underset{\sim}{x}_t)\}}$$

and $Q_M < Q_k = Q_A$.

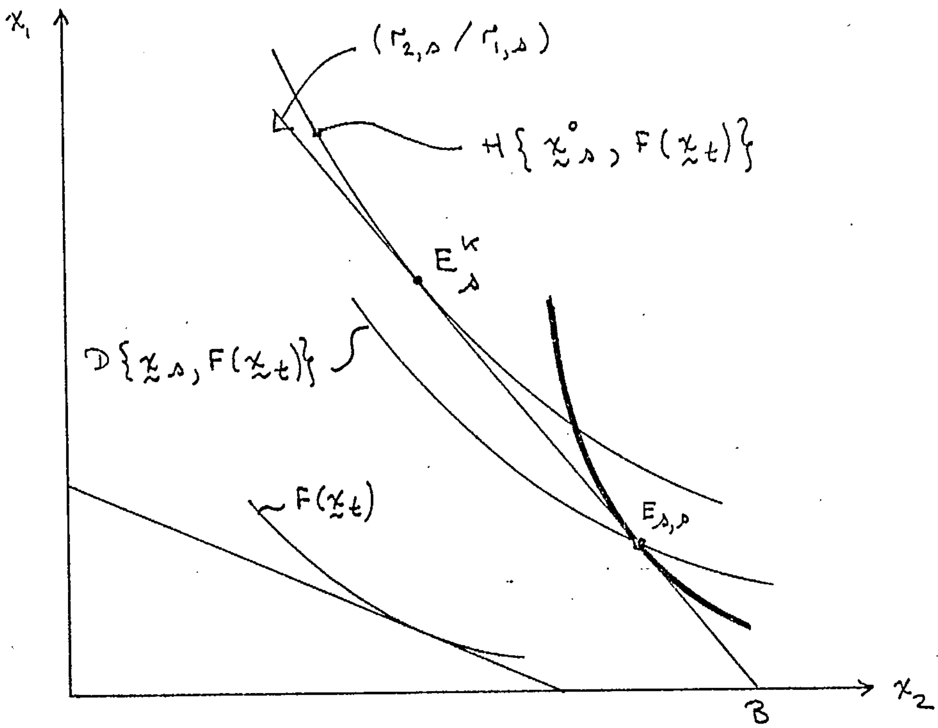

Figure 5.7

Comparison of Laspeyres-Malmquist and -Konüs Quantity Indexes

CONCLUSION: AN APOLOGY OF SUPERLATIVE INDEX NUMBERS

It has been noted that recent econometric studies test and reject the hypothesis of positive linear homogeneity. In fact, they also reject the separability hypothesis. It would, therefore, appear that eonometrics is superior to index methodology except for the number of variables it can handle, and that the latter is nested within modern econometrics. It is the object of this conclusion to show that index number theory is not nested within econometrics, hence that, just as econometrics has clear advantages over index numbers, index numbers in turn have their advantages. We shall restrict the analysis to flexible functional forms in econometrics and superlative index numbers; superlative index numbers are the index numbers which are exact with respect to flexible functional forms (Diewert, 1976).

Let's begin with the benefits of econometric analysis. Econometrics makes it possible to estimate the structural parameters of the flexible function. This flexible function form can be conceived as second order Taylor expansion to approximate the unspecified underlying production function in the neighbourhood of a point such as (x*,z*) (Blackorby, Primont and Russell, 1978). Knowledge of the structure through a formal stochastic estimation is sufficient to test hypotheses regarding questions such as separability, homogeneity... Those results are not obtained freely, however. It is true that functional forms such as the translog, the generalized Leontief, the quadratic mean of order r aggregator function,...are justified in terms of a local approximation in the neighbourhood of a point. However, in all practical situations, that neighbourhood is normally stretched over no less than twenty years of observations. In other words, econometrics stretches to its limit the concept of "local approximation in the neighbourhood of a reference point".

The index number approach, on the other hand, requires no more than two observations in its neighbourhood. If F is the true but unspecified function, over a small neighbourhood, i.e. given the vector $(\underset{\sim}{x}_t, z_t)$, in the neighbourhood of $(\underset{\sim}{x}_t, z_t)$ which contains only that other observation $(\underset{\sim}{x}_{t-\tau}, \underset{\sim}{z}_{t-\tau})$ which is closest (τ will normally be 1), the second order approximation of F( ) is denoted by F_t. F_t is expected to be neither separable nor positive linear homogeneous. If the econometric approach is adopted, the best we can do is to approximate over more than twenty years F_t by another function which needs be neither separable nor positive linear homogeneous, $\hat{F}_t$. In fact, a function $\hat{F}$ about a mean point in the observation space is generally what the econometrician will estimate as approximation to F_t. In the index number approach, in terms of superlative indexes, only two observations are needed. Index numbers can be calculated within the same neighbourhood as F_t.

The separability problem has been treated by Diewert (1979) who shows that it is not a problem provided the approximation is about the geometric mean of the two output levels, i.e. where $t-\tau = s = t-1$, provided the approximation is taken about $(z_{t-1} \cdot z_t)^{\frac{1}{2}}$. This point is in the neighbourhood of $(\underset{\sim}{x}_t, z_t)$. Vice versa, we can design an even smaller neighbourhood about $(z_{t-1} \cdot z_t)^{\frac{1}{2}}$ with both $(\underset{\sim}{x}_{t-1}, z_{t-1})$ and $(\underset{\sim}{x}_t, z_t)$ in it. Hence, provided the flexible form is positive linear homogeneous, Diewert shows in this theorem that the Tornqvist and Fisher index formulae are respectively superlative with respect to the translog and the generalized Leontief functions, even excluding separability.

However, a non-linear homogeneous second order approximation can be approximated in the neighbourhood of a point by a positive linear homogeneous second order approximation. Hence F_t can be approximated, in the neighbourhood with both $(\underset{\sim}{x}_{t-1}, z_{t-1})$ and $(\underset{\sim}{x}_t, z_t)$ in it, by $\tilde{F}_t$. $\tilde{F}_t$ is positive linear homogeneous and needs not be separable. The neighbourhood of approximation is generally very small since it covers generally only one year. It follows that the consistancy of superlative index numbers with flexible functional forms F_t more restrictive than those used in econometrics, $\hat{F}_t$, is not sufficient to specify which of $\tilde{F}_t$ and $\hat{F}_t$ is the best approximation of F_t, hence of F.

At best we may suggest one evidence in favour of index numbers above econometric estimation in terms of the better approximation of F. It has been known since Fisher (1922) that a substantial number of index formulae almost always yield almost the same answer. Two formulae have been shown by him to be particularly desirable, namely his "ideal" index and formula 124 which is now better known as the Tornqvist index. Both of those have been shown to be superlative in terms of the generalized Leontief and the translog respectively (Diewert, 1976, 1979), i.e. that whenever the index is Fisher's ideal

(Tornqvist), $\tilde{F}_t$ is a generalized Leontief (translog) function. If each index indeed gives an approximation of the exact index corresponding to F, it means that each must correspond to generalized Leontief and translog functions respectively both of which approximate well F in the neighbourhood $\{(\underset{\sim}{x}_{t-1}, z_{t-1}), (\underset{\sim}{x}_t, z_t)\}$. If they are close to each other, i.e. if they approximate each other, then they must indeed both approximate well F. In econometrics also, one should obtain different flexible functional forms estimated close to one another, however, this is almost never the case; this would appear to infirm the quality of the econometric estimation as an approximation. In other words, it is suggested that in general, superlative index numbers follow more faithfully the true function, i.e.

$$|\tilde{F}_t(\underset{\sim}{x}_t) - \hat{F}_t(\underset{\sim}{x}_t)| \leq |\hat{F}_t(\underset{\sim}{x}_t) - F(\underset{\sim}{x}_t)|$$

and $$|\tilde{F}_t(\underset{\sim}{x}_{t-1}) - F_t(\underset{\sim}{x}_{t-1})| \leq |\hat{F}_t(\underset{\sim}{x}_{t-1}) - F(\underset{\sim}{x}_{t-1})|$$

Evidently it is likely that for sufficiently big, it will almost always be true that

$$|\tilde{F}_t(\underset{\sim}{x}_{t-\tau}) - F_t(\underset{\sim}{x}_{t-\tau})| \geq |\hat{F}_t(\underset{\sim}{x}_{t-\tau}) - F(\underset{\sim}{x}_{t-\tau})|$$

however, this needs not concern index number theory since, provided a chain index is used - as implied by the previous analysis - the comparison would be based on an approximation of $F_{t-\tau}$, and we would expect

$$|\tilde{F}_{t-\tau}(\underset{\sim}{x}_{t-\tau}) - F_{t-\tau}(\underset{\sim}{x}_{t-\tau})| \leq |\hat{F}_{t-\tau}(\underset{\sim}{x}_{t-\tau}) - F(\underset{\sim}{x}_{t-\tau})|$$

It is felt that the above argumentation does provide both the proper justification for chain index numbers and the framework to evaluate quasi-chained indexes (Star, 1974). By chaining the index, we approximate over very small intervals the mapping by homogeneous flexible forms, different structure being implicity associated with those functions in different periods of time.

The Divisia index has been criticized for its sensitivity to the path of integration (Usher, 1974). Even though it has been indicated earlier that the path problem is much broader and that it is basically a question of homotheticity, it is useful to look at the problem here since existing superlative indexes can be shown to be approximations of the Divisia index. Given a series of a set of observations $(\underset{\sim}{x}_1, \underset{\sim}{r}_1, \underset{\sim}{z}_1)$, $(\underset{\sim}{x}_2, \underset{\sim}{r}_2, \underset{\sim}{z}_2), \ldots, (\underset{\sim}{x}_\tau, \underset{\sim}{r}_\tau, \underset{\sim}{z}_\tau)$, and assuming that, as is often the case, for $\tau > 0$, it is for $\tau = 1$ that $(\underset{\sim}{x}_{t-\tau}, \underset{\sim}{r}_{t-\tau}, \underset{\sim}{z}_{t-\tau})$ is closest to $(\underset{\sim}{x}_t, \underset{\sim}{r}_t, \underset{\sim}{z}_t)$, then, as long as the series does not change too brutally, the path from $(\underset{\sim}{x}_{t-1}, \underset{\sim}{r}_{t-1}, \underset{\sim}{z}_{t-1})$ to $(\underset{\sim}{x}_t, \underset{\sim}{r}_t, \underset{\sim}{z}_t)$ will approximately not matter, $t = 2, 3, \ldots, \tau$. The path will be based upon a homogeneous approximation of the mapping between z_t and $\underset{\sim}{x}_t$. But now the path from $(\underset{\sim}{x}_{t-\tau}, \underset{\sim}{r}_{t-\tau}, \underset{\sim}{z}_{t-\tau})$ to $(\underset{\sim}{x}_t, \underset{\sim}{r}_t, \underset{\sim}{z}_t)$ is made of small independent local approximations of the real path, from $(t-\tau)$ to $(t-\tau+1)$, from $(t-\tau+1)$ to $(t-\tau+2), \ldots$, all the way to t. Since $(\underset{\sim}{x}_{t-s}, \underset{\sim}{r}_{t-s}, \underset{\sim}{z}_{t-s})$ where $s = \tau, \tau+1, \ldots, t$, are by definition on the correct path, a chain superlative index should yield approximately the correct answer.

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