DEPARTMENT OF MATHEMATICS

An Interim Report to the Ministry of State
for Science and Technology

on

**THE QUEEN'S MATHEMATICS DEPARTMENT
POPULATION MODEL**

by

J. Davis and J. Verner
Queen's University, Kingston, Ontario
Queen's Mathematical Preprints No. 1974-8

Queen's University

AT KINGSTON
ONTARIO, CANADA

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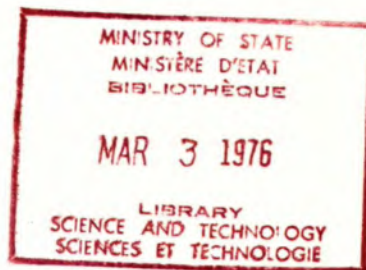
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April 15, 1974

<u>Letter of Transmittal</u>

Dr. A. R. Demirdache,
Director,
Technological Forecasting
   and Technology Assessment Division,
Ministry of State for Science
   and Technology,
Ottawa, Canada.

Dear Dr. Demirdache:

I have pleasure in forwarding to you herewith an Interim Report on the work undertaken by this Department in response to your request for input towards the Canadian contribution to the work of the International Institute for Applied Systems Analysis.

You will **recall** that you asked us to attempt to model the growth of population in Canada, by means of a model involving coupled systems and non-linear differential equations. The importance of the Model derives from the fact that population studies are an indispensable building block in any attempt to study national socio-economic problems with the accuracy necessary to form a basis for policy decisions.

The work on this Model has been carried out principally by Professors Jon Davis and James H. Verner of this Department. As you will see, they have broken new ground in applying relatively sophisticated methods which have not hitherto been deployed on population studies, and they have devised computer programmes to display the distribution of population in age and time in a very graphic manner.

I remind you that in the agreement between us, the right of publishing the details of the procedures contained in this Report in scientific journals has been reserved to Professors Davis and Verner. They would wish me to emphasize strongly their evaluation of the present work as being highly preliminary, and they would caution against any attempt to make exaggerated claims for their model in its present state.

Even so, I trust that you will agree with my view that they have demonstrated that there are extremely worthwhile possibilities in their method, which should be pursued vigorously.

Yours faithfully,

A. J. Coleman, Head,
Department of Mathematics.

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Abstract

The Queen's Mathematics Department population model is a dynamic model for simulating the evolution of a population distribution as a function of age and income level.

The basic structure of the model is such that birth-rates may be generated within the model as <u>endo-genous</u> variables. This allows the inclusion of feedback effects from the population distribution to birth and immigration rates, and so provides a capability for simulations valid over longer time intervals than are possible with <u>exogenous</u> birth and immigration rate variables.

The model has been constructed with two main issues in view. The first is that of compatability of this model with other models with which it might be combined. This requirement dictates a modular structure described in this report. The second issue is the problem of parameter estimation in the model. The model has been formulated in such a way that estimation is made possible.

Effective numerical algorithms for these estimations based on available data formats are also reported.

A description of work that remains to be done in
order to complete development of the model is also
included.

I.   Introduction

Models of population growth form an essential part of any attempt at large scale socio-economic modelling. The age and economic level structure of the population has a direct bearing on various government service requirements ranging from elementary schools to pension plans, as well as the economic base available to support such programs. For this reason, it is essential that population models capable of simulating behaviour over a reasonable length of time be investigated.

Traditional demographic methods project population estimates forward in time by means of an aggregation procedure, followed by a linear extrapolation procedure based essentially on a Markov-chain type of model. Such methods are reasonably accurate over the relatively short term; however, the model structure is such that the fertility curve (the age-specific distribution of the birth-rates) is treated as an "exogenous variable" which must be specified for each run. Some attempts (the so-called "cohort method") have been made to include in the model the observed fact that birth-rates do vary over time, but the problem of extrapolation birth-rates forward in time in order to increase the length of time that model

results are valid remains.

It is clear that <u>many</u> factors affect birth-rates:
economic conditions, perceptions of future economic
conditions, ecological concerns, a host of other
factors affect birth-rates to a greater or lesser
effect. It is also clear that present population
structure affects in turn the economic climate, and
the general environment. The present population is in
turn the result of past birth-rates (and immigration).

The conclusion of the above observations is that
it is impossible to decouple the dynamics of the birth-
rates from those of the population structure without
compromising the long term validity of the model sim-
ulation. In effect, there exists a feedback path from
population structure to birth-rates which may not be
ignored over the long term. (This does not imply that
such decoupling, based on assumptions that certain
factors "vary slowly with time", detracts from the
usefulness of models intended for use over relatively
short time periods).

The model discussed in this report represents an
approach to the problem of including the dynamic feed-
back effect mentioned above in a simulation model.
More specifically, this report contains the results of
some work on what we regard as the <u>basic</u> structural

elements and problems associated with models of this
sort.

The structure and "philosophy" of the model is
discussed more fully in Section II.

It was determined early in our investigation that
partial differential equations were an appropriate
component of the model - in fact, it is hard to con-
sider the effect of the "baby boom" without coming to
the conclusion that a wave equation occupies a central
position in a model of population distribution. In
work on any dynamical model it is necessary to deter-
mine numerical values for parameters occurring in the
model equations before any simulation may be carried
out. At worst, these parameters may have to be
guessed; obviously it is much more desirable that the
parameters be estimated from historical records of the
phenomenon being modelled, if possible. The latter
procedure provides an indirect means of assessing the
validity of the model.

In the case of models governed by partial dif-
ferential equations this estimation problem is even
more severe, as it is often necessary to estimate not
just a finite set of parameters, but a function of one
or more independent variables. Aspects of this prob-
lem are reported in Section III and Appendix C.

Once parameters and functions have been estimated
from the available data, it is possible to simulate the
system on a digital computer. This, of course, involves
the solution of coupled systems of ordinary and partial
differential equations by numerical methods. It is
necessary to investigate the effects of the numerical
methods used on the accuracy of the results obtained,
in order to ensure that the behaviour of the model is
a result of the actual "dynamics" of the model itself,
and not the result of instability caused by inaccurate
numerical methods. The difficulty of this problem is
again increased by the fact that partial differential
equations are involved. The work in this area has been
checked by use of certain exact solutions to the gov-
erning equations (Appendix A) and is described in
Section V and Appendix B.

II.   Structure of the Model

It is helpful in describing the structure of the
model presented here to explain briefly the general
philosophy of "modelling" that the authors of this
report hold, and which has had a strong effect on the
structure adopted for the model discussed here.

In the first place, we feel that a main product
of any modelling and simulation effort should be

insight into the behaviour of the phenomena being model-
led. Perhaps the worst fate that can befall any model
is that it be used to generate one set of trajectories
which are then canonized as "the predictions" of the
model (or worse yet, of the computer used to generate
the output). Rather, the use of a model should itself
be a dynamic process. It is certain that better data
regarding the variables involved in a model will become
available in the future, and it is only prudent that this
data, if possible, be used to improve any "forecasts"
made using the model.

It is also rather likely that there are alter-
native opinions regarding the actual structure of some
sections of any given model. In this situation, it is
essential that simulations be run incorporating these
alternative opinions, rather than selecting one arbit-
rarily and incorporating it permanently into the model.
It is only by simulating each of the reasonable alter-
natives (a matter of judgement is involved here) that
any true insight into the behaviour of the system as a
whole can be gained; this includes an appreciation of
the range of results which might be expected under
reasonable alternative models.

These considerations suggest at least that a use-
ful model must have sufficient flexibility of structure

to accommodate changes of the sort mentioned above. In order to build flexibility of this sort into a model, it is <u>necessary</u> to identify a basic dynamical core around which the model may be constructed.

The basic core of the model in this paper consists of equations for the evolution of the population distribution, and for the evolution of the fertility curve over time.

As was mentioned in the previous section, it is clear that economic conditions interact with the current population distribution and other factors to produce the current instantaneous birth-rate. It is also clear that the exact nature of these interactions is complicated and probably poorly understood in total, although some progress in this direction may be made by various methods. On the other hand, the evolution of the population distribution may be described (see the following section) by a partial differential equation of the conservation law type. Also, by looking at birth-rate records, it is possible to argue that the evolution of the fertility curve is also adequately modelled by a relatively simple partial differential equation. Furhter, the <u>structure</u> of these sections of the model is independent of the details of economic and other interactions which combine to affect birth-rates.

These considerations have led us to the decision
to base the framework of the model on the dynamics of
the population distribution and of the fertility curve.
This leads to an overall model structure which may be
represented in the "block diagram" form illustrated in
Figure 1.
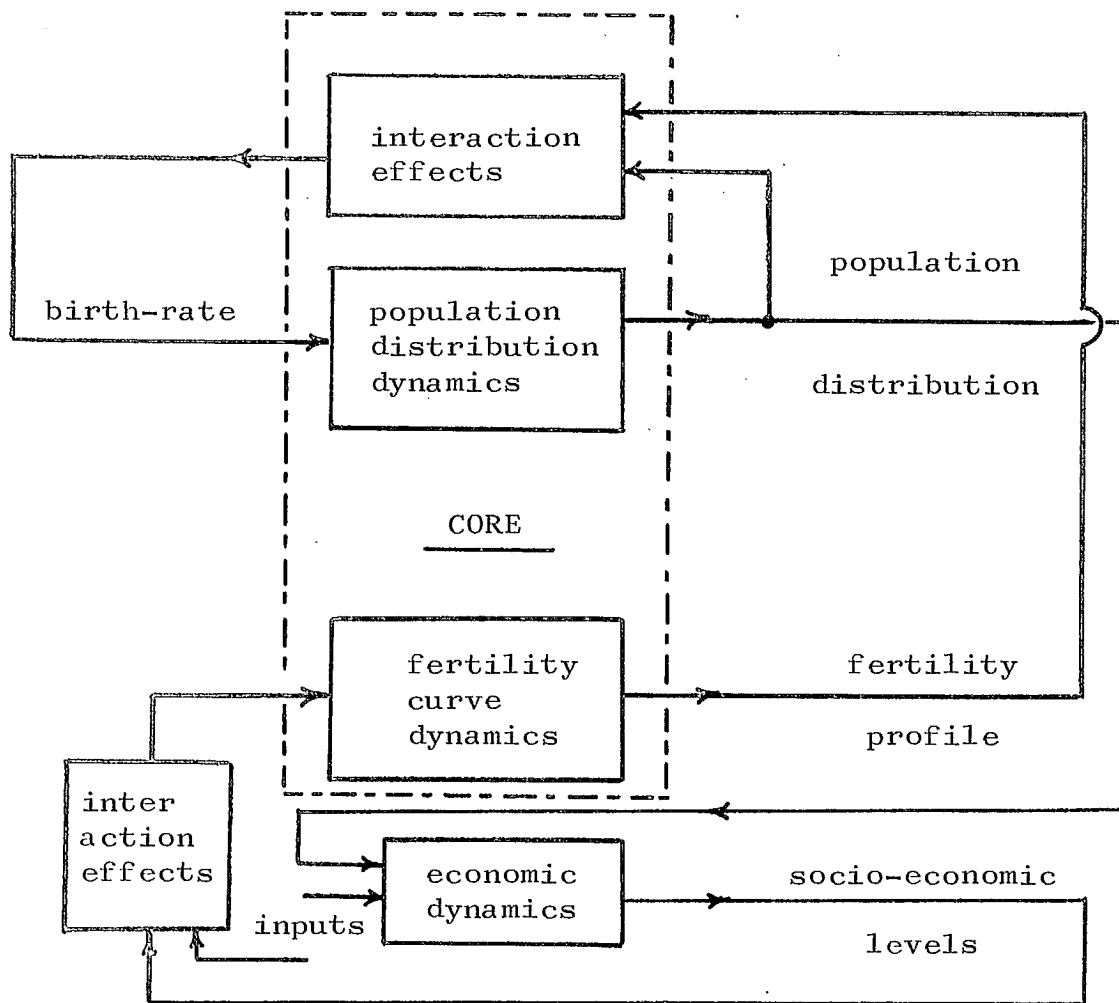
Figure 1.

This block diagram includes a section labelled economic dynamics. It is judged that the exact form of this section will be the subject of some debate, and that repeated simulations with varying socio-economic models will be required. It is expected that models with a relatively long time horizon and moderately high level of aggregation will be found most appropriate. In particular, models of the "Candide" type with high levels of detail and relatively short (eg. ten year) time spans are not felt to be appropriate. As work up to the time of the writing of this report has been concentrated on problems associated with core section of the model, problems in this particular area require further study.

III. Derivation of Equations of Dynamic Core

The core of the model consists of two partial differential equations: one for the evolution of the population density as a function of time, age, and income level, and one for the evolution of the fertility curve (i.e. the curve of age and income specific birth-rates). These two equations are coupled in a non-linear fashion, although the non-linearity appears only in the boundary conditions for the population equation. This fact is of considerable use in connection with the estimation problems discussed in the following section, and makes

the derivations presented below simpler than might other-
wise be the case.

A.   Population Distribution Evolution

The model presented below is formulated as a
basically deterministic model, and processes are model-
led as occurring continuously in time on a macroscopic
level, even though on a microscopic level the events
may occur at discrete intervals of time. In this con-
nection, a first step is to recognize that an averaging
process is taking place whenever what are essentially
discrete events are "smeared out" and modelled cont-
inuously in time. This process is illustrated by the
use of death-rates in population models, decay rates in
radioactive decay problems, and, in the derivation
below, of an economic mobility μ . In these cases,
the use of such rates essentially distinguishes between
deterministic and stochastic modelling approaches.

The equation governing the population distribution
may be derived readily from what are essentially count-
ing or bookkeeping methods. This is most easily demon-
strated by the derivation of a simple model of pop-
ulation as a function of age x , neglecting
death-rates, immigration and any other variables. In
this case, the appropriate counting argument is

essentially that the number of people at age x at time t is the same as the number of people at age $x - \Delta t$ at a time Δt units earlier: In terms of population density p, this becomes

$$\int_{x - \frac{\Delta x}{2}}^{x + \frac{\Delta x}{2}} p(x,t)dx = \int_{x - \frac{\Delta x}{2} - \Delta t}^{x + \frac{\Delta x}{2} - \Delta t} p(x,t-\Delta t)dx \quad ,$$

which for smooth densities p is essentially

$$p(x,t) = p(x-\Delta t, t-\Delta t)$$

or $\quad p(x,t) - p(x,t-\Delta t) = p(x-\Delta t, t-\Delta t) - p(x, t-\Delta t)$.

Dividing the above by Δt and letting $\Delta t \to 0$ results in the partial differential equation

$$\frac{\partial p}{\partial t} = - \frac{\partial p}{\partial x} \quad .$$

As is well known, the general form of solution to the above is $f = \varphi(t-x)$, with φ an arbitrary function which must be evaluated from the boundary conditions appropriate to the problem. The appropriate boundary condition is that

$$p(x=0, t) = \beta(t)$$

where $\beta(t)$ is the birth-rate of time t . That this is the appropriate boundary condition may be verified by noting that this gives the solution

$$p(x, t) = \beta(t-x) \quad ,$$

which says essentially that the number of people at age x at time t is the number of people born at time $t - x$, i.e. x years before time t . This of course is entirely evident from the assumptions made above.

The model considered in this paper includes a partial differential equation for the population density $p(x, s, t)$ as a function of the three variables age x , income s , and time t . As will be shown below in Appendix A, it is unnecessary to specify at this point

the units involved in the income scale s , that is, whether s represents net income, disposable income, or some other measure. This is so because the _form_ of the governing equation is invariant under a (non-linear) change of income scale, so that the units involved become an issue only during the processing of data for estimation purposes. This fact is a pleasant surprise which naturally arises out of the structure of the model equations.

To derive an equation for the population density on a realistic basis, it is necessary to account for effects neglected in the simplified model above, in particular to introduce terms

$$i(x,s,t)$$

representing the immigration rate (as a function of age, income level, and time), and the death-rate

$$r(x,s,t) \quad .$$

It is also necessary to introduce a term which accounts for the change of income level of various segments of the population over time. To accomplish this,

we introduce an <u>economic mobility function</u>,

$$u(x,s,t) \quad .$$

Even though income levels of individuals on a microscopic scale undergo changes at discrete instants of time, perhaps modelled by a Poisson process, on the macroscopic scale of its influence on the income distribution we model the effect as one of a continuous flow across income levels. With this effect in mind, a term of the form

$$u(x,s,t) \cdot \Delta t$$

has an interpretation as the fraction of people at income level and age x crossing through level s in the time interval from t to t + Δt .

With the above definition of terms, it is easy to use a "counting argument" entirely similar to the one above to arrive at an equation representing the evolution of population density. The result is

$$\frac{\partial p}{\partial t} = -\frac{\partial p}{\partial x} - \frac{\partial}{\partial s}(u(x,s,t)p) - r(x,s,t)p + i(x,s,t)$$

Just as in the above derivation, it can be seen
that the appropriate boundary condition for this
equation is again

$$p(x=0) = \text{birth-rate.}$$

There is a technical problem associated with this boun-
dary condition, since taken literally it demands the
assignment of an income level to babies at birth.   In
fact, the model as formulated above is capable of prop-
agating income level migration through childhood.   It
is clear that such a procedure makes little sense;
however, the problem may be avoided rather easily by
the following device.   The income distribution at x = 0
may be set equal to that at the age of entry into the
labour market.   If the economic mobility is equal to zero
for values of age x less than the labour market entry
age, then the income distribution will remain constant
for ages less than entry at the values of the entry
distribution.   Income distribution data generated by sim-
ulation runs may then be considered only for ages
greater than an age of entry into the labour market, and
no further modification of the model is required.   The
income distribution at entry age must be generated as
part of the economic section of the model, and this

effect comprises one of the feedback paths from the
economic to population distribution sector shown on
Figure 1.

Other effects of this sort, for example, an effect
of p on the economic mobility μ , are implicitly
contained in the time dependence of u . As will be
seen below in the section dealing with estimation prob-
lems associated with the model, there are substantial
theoretical and practical benefits which follow from
modelling the income migration process as above. In
particular, it is then possible to devise numerical
methods to estimate u from observed population dist-
tribution data.

B. Fertility Curve Dynamics

Although the observation that socio-economic
conditions, social attitudes, and so on, exert an effect
on birth-rates is a common one, there seems to have been
little effort made to quantify these effects in a dyn-
amic model. Undoubtedly, a major reason for this is
that it appears impossible to "derive" such a set of
relations in the sense of the derivation outlined above
for the population density dynamics.

For this reason, we have decided to approach this

problem as one in system identification. That is, we
attempt to formulate the problem in such a way that the
problem is reduced to that of estimating a dynamical
relationship between a relatively small number of
variables. This in _itself_ is a major reduction, since
in principle a fertility curve is an element requiring
an infinite number of numbers for its specification.

This first reduction may be obtained by examining
typical historical records of the behaviour of fertility
curves over time (See Figure 2). A first observation is
that the curves are all of roughly the same shape. An
examination of their differences shows that their peaks
slide from age to age over time, and that the area under
the curve, representing the total birth-rates to be
expected from a uniformly distributed population, varies
over time.

A simple partial differential equation capable of
reproducing this observed behaviour has been adopted as
the basis for the fertility curve dynamics. This is

$$\frac{\partial f}{\partial t} = - a(t) \frac{\partial}{\partial x} (d(x)f) - b(t)f \quad .$$

The first term in this equation produces the
effect of the shifting peak, while integrating the

equation with respect to x shows that $b(t)$ is the percentage change of the area under the fertility curve per unit time. An equivalent interpretation is that it represents the percentage rate of change of average family size.

The justification of the representation of the fertility curve dynamics by the above equation may be carried on in several ways. In the first place, the interpretation given $b(t)$ guarantees the presence of the term $b(t)f$ in virtually any such equation. The appropriateness of the term representing the "shifts" may be supported on the basis of a time scale argument, combined with the fact that the model fits the observed data reasonably well. The "shifts" occur in the data on a time scale considerably faster than that of the dynamics of the population section of the model. In fact, the shifts appear correlated with variations in the economic climate, recessions, rising and falling unemployment, and the like. Since these effects are expected to be introduced into the model most likely on the basis of "standard" econometric and business cycle models, it is anticipated that it will be possible to include the function $a(t)$ and its dynamics in this section of the model. The dynamics of $a(t)$ are to be

identified by means of either the usual econometric

model identification techniques, or more recent work in

the area of control theory.   Since this identification

problem presupposes knowledge of the term d(x) , work

in this area is dependent on solving the problem of est-

imating d(x) , and applying the algorithm for this

purpose is described in the following section.

Comments similar to the above also apply to the

problem of determining the dynamics governing the term

b(t) , although it is suspected that this will be even

more difficult than the above process.   This is so

because b(t) is dominated more by social attitudes,

education, and other effects much less easy to quantify

than economic ones.   It is felt that this area represents

an example of the need for alternative sub-models and

repeated simulations discussed above in Section II in

connection with the overall structure of the model.

While the above discussion has been carried

through as though the fertility curve were independent

of income, an entirely similar derivation is possible on

the basis of an <u>income dependent fertility curve</u>.   If

one also allows the possibility that the economic inter-

actions occur unevenly across income levels, then the

appropriate equation is

$$\frac{\partial t}{\partial t} \;=\; -\,\frac{\partial}{\partial x}\,(a(x,s,t)f) \;-\; b(s,t)f \qquad .$$

Because of the meaning of a fertility curve (or surface, if s is included as an independent variable) as an age (and income) specific birth-rate, the formula for the total birth-rate is simply

$$b(t) \;=\; \int f(x,s,t)\,p(x,s,t)dxds \qquad .$$

No mention has been made in the above derivations of any geographical aspect of the problem. There are, however, some restrictions implicit in the derivations of the model equations. It is clear that certain of the quantitites involved in the above equations vary with geographical locality. From this, it is obvious that the model must be applied separately over geographical areas between which the relevant quantities vary. To obtain an overall model, then, internal migrations must then be included in the immigration rates of the models for each geographical region.

There is also implicit in the model derivation as assumption of a sufficiently large sample population, so that the modelling of the immigration, death, birth, and economic migration processes as continuous is valid.

IV.   Model Estimation Methods

The dynamical equations governing the core section
of the model derived above involve various auxiliary
functions, namely: death and immigration rates, an
economic mobility function $\mu(x,s,t)$, and functions
$a(t)$, $b(t)$ and $d(x)$ determining the evolution of
the fertility curve. Before it is possible to produce
any simulation runs with the model, it is necessary to
determine suitable estimates of these functions.

Generally speaking, this problem of parameter and
function estimation is one of the most difficult ones
involved in the construction of any model. Consideration
of the conventional techniques of econometric modelling
makes obvious the amount of effort which is expended in
this area. In fact, with a certain amount of injustice
one might view much econometric modelling as consisting
of the development of schemes for the recursive estim-
ation of parameters for short term (often linear) extra-
polation models. This view ignores the effort involved
in determining the extrapolation model whose parameters
are to be estimated, but the fact remains that there
continues to be much work on the development of
regression - estimation methods in this area.

At practically the opposite end of this problem
stand models of the sort proposed by Forrester and his

associates. One of the most consistent criticisms levelled at Forrester's World and Urban Dynamics models is that practically no attempt has been made to estimate the parameters and functions involved in the models in any "realistic" fashion.

This apparent gulf between the Forrester models and conventional econometric models is, in our view, a large contributing factor to the hostile reaction Forrester's models have received in some quarters. It is also a gulf that is not easily overcome by philosophic discussions about differences of purpose between the two approaches.

In the case of the present model, it happens that considerable progress can be made in estimating the functions that are involved in the model of the core dynamics. Of course, this is not entirely unexpected, since an effort has been made to formulate the dynamics of the core in terms of variables which may be readily measured. Also, our definition of what constitutes the core dynamics of the model virtually assures that it must be possible to produce useful quantitative estimates of the functions involved.

The functions $r(s,s,t)$ and $i(x,s,t)$ in the population model are just death and immigration rates, so there is no problem in obtaining historical records of

these. Similarly, the function b(t) may be readily determined on the basis of its interpretation in terms of area under the fertility curve.

This leaves just the terms $\mu(x,s,t)$ in the population equation and $a(t)\,d(x)$ in the fertility equation to be determined. It can be seen that each term enters its equation in an analogous way, so that an estimation method can be derived which can be used to estimate <u>both</u> the economic mobility $u(x,s,t)$ and the term $a(t)\,d(x)$ in the fertility equation.

It is shown in Appendix A that the form of the partial differential equations is such that an integrating factor may be introduced to reduce the problem to that of estimating $\mu(x,s,t)$ and $a(t)\,d(x)$ in the equations

$$\frac{\partial \bar{p}}{\partial t} = -\frac{\partial \bar{p}}{\partial x} - \frac{\partial}{\partial s}\;(u(x,s,t)\bar{p})$$

$$\frac{\partial \bar{f}}{\partial t} = -\frac{\partial}{\partial x}\;(a(t)d(x)\bar{f})\qquad .$$

In the modified population equation, integrate between the limits of s and infinity. There results

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \quad \int_s^\infty \bar{p} \, ds \ = \ \mu(x,s,t) \cdot \bar{p}(x,s,t) \quad ;$$

here $\int_s^\infty \bar{p} \, ds$ has the interpretation of the number

of people at age x , income s , and time t

having an income greater than s . Solving this for

μ gives

$$\mu(x,s,t) \ = \ \frac{1}{\bar{p}(x,s,t)} \quad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \ \int_s^\infty \bar{p}(x,s,t) ds \quad .$$

This provides an estimate of μ wherever $\bar{p}(x,s,t) \neq 0$.
Since $\bar{p}(x,s,t) \neq 0$ except on the "tails of the dist-
ribution", the above formula may be used to determine
μ throughout the age and income brackets containing
the great bulk of the population. On the tails of
distribution (e.g., at very high income levels) the
interpretation of μ makes it clear that μ must tend
to zero, so that the fact that the above formula is less
useful there is of little concern.

Carrying out exactly the same procedure with the
modified fertility curve results in

$$a(t)\, d(x) = \frac{1}{\bar{f}(x,t)}\, \frac{\partial}{\partial t} \int_x^\infty \bar{f}(x,t)\,dx \qquad .$$

This determines $a(t)\, d(x)$ over those portions of the age scale which $\bar{f}(x,t) \neq 0$. Again $\bar{f}$ tends to zero only on the tails of the fertility distribution. Recalling that the term $a(t)\, d(x)$ was introduced to account for changes in the age distribution of fertility we see that intuitively $a(t)\, d(x)$ reflects the effects of shifts in "planned births" for the most part. Since births arising in the extremes of the fertility distribution do not fall into that category, it is clear that $d(x)$ must approach zero at these extremes. Hence it is again true that the fact that the formula derived is less useful in regions where $\bar{f}$ is close to zero is of small consequence.

Once $u(x,s,t)$ and $a(t)\, d(x)$ have been estimated, further estimation problems remain. One problem is that of determining numerically the values of $a(t)$ alone for use in identifying the interactions between the economic sector and the fertility curve. A second related problem is that of isolating the time dependence in $u(x,s,t)$ in such a way that a similar interaction analysis may be carried out. These problems are of a

somewhat more technical nature, so our work on them is reported in Appendix C below.

A further technical complication arises in connection with practical use of the estimation formulas above. This arises from the fact that the actual population density data is not available; rather figures are available for, say, the number of persons between ages 25 and 29 with income between eight and ten thousand dollars. This amounts to the data

$$\int_{8,000}^{10,000} \int_{25}^{29} p(x,s,t) \, dx \, ds$$

at a fixed value of t.

We have expended a moderate amount of effort to develop accurate numerical algorithms with which $u(x,s,t)$ and $a(t) \, d(x)$ may be determined from aggregated data of the sort mentioned above. The method devised uses somewhat delicate application of numerical spline techniques. This work is also described in Appendix C.

V.   Simulation Trial Examples

Simulation runs have been made in order to test the algorithm for numerical solution of the coupled system of ordinary and partial differential equations which constitute the model

Since there is a rather large amount of numerical data associated with each simulation run, the results are produced by the simulation program in a visual as well as numerical format. This is accomplished through a plotting routine which constructs perspective drawings of the three dimensional surfaces generated by a simulation run.

The (steady state) age distribution which results from a constant birth-rate and an absence of immigration is illustrated in Figure 2. The age distribution which results in this case is of course determined solely by the death-rate.

The wave-like nature of the solutions of the governing equations may be clearly observed in a simulation which creates a rise in the fertility curce, starting from an initial condition of the steady state illustrated in Figure 3. Since the dynamics governing $a(t)$ and $b(t)$ have not yet been determined, a simulation has been carried out by introducing $b(t)$ as an exogenous variable; $a(t)$ was determined through the
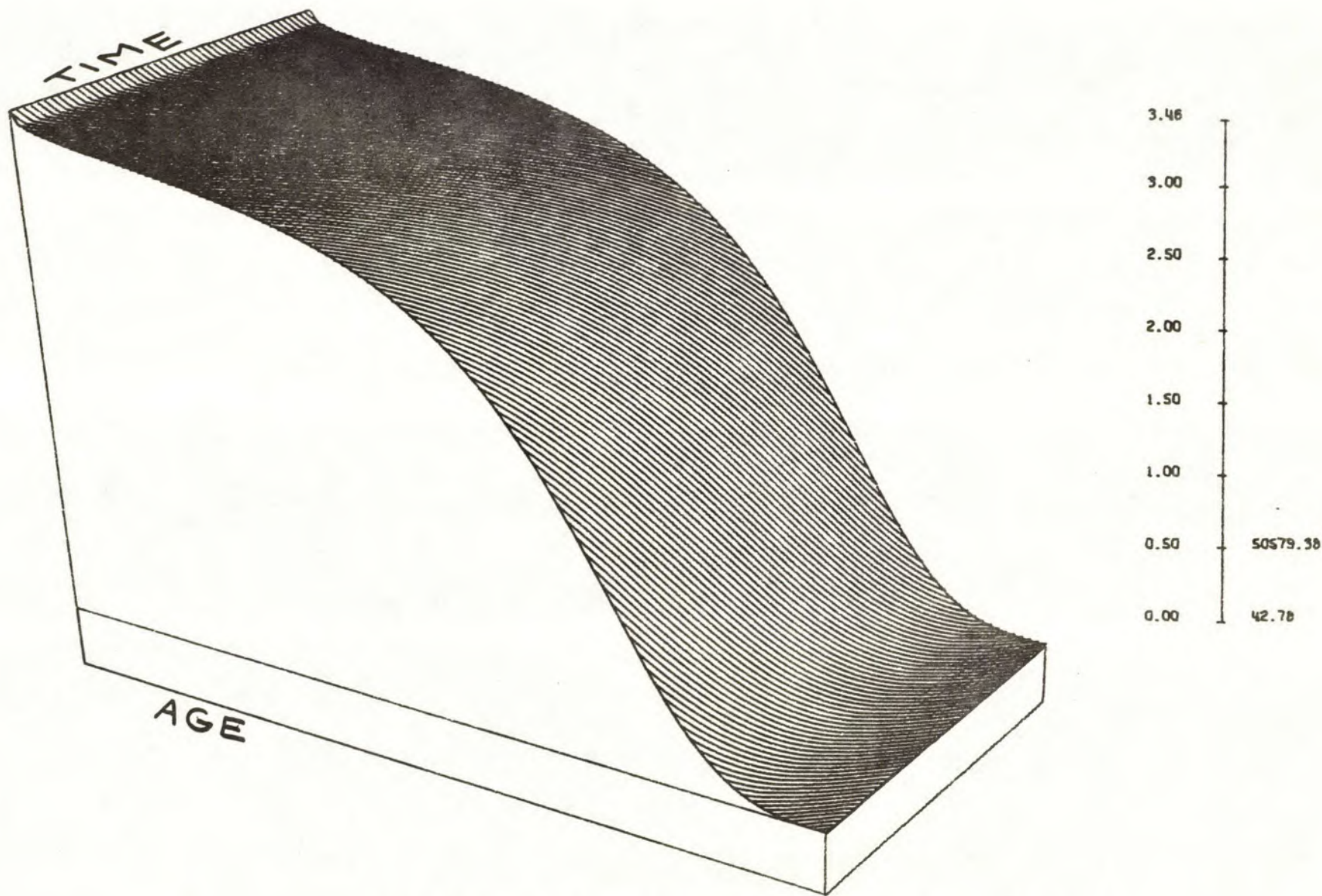
EVOLUTION OF POPULATION OVER 50 YEARS: CONSTANT PROFILE

Figure 2.

dynamic equation

$$\left(\frac{d}{dt} + 1 \right)^2 a(t) = b(t) \quad .$$

The functions b(t) and a(t) have been determined so
that $\int_0^T a(t) \, dt = \int_0^T b(t) \, dt = 0$, so that the fert-
ility curve returns to its original value. This
produces the response to Figure 4 in the fertility
curve, which corresponds to a "baby boom" of duration
approximately five years. The effect of this rise in
the fertility curve on the age distribution is il-
lustrated in Figure .

The varying total birth-rate may be clearly seen
at the back edge of the figure; the secondary rise in
the birth-rate which occurs as the original "offspring"
of the boom pass through the childbearing ages is
plainly visible. It is also easy to see the original
boom passing as a wave through the age structure.

Both the wave nature of the solutions and the
birth-rate variations which occur due to a non-
uniform age distribution are illustrated in Figure 5 .
This output results from an initial age distribution
which is significantly different from the steady-state
distribution. Such a distribution might be viewed as
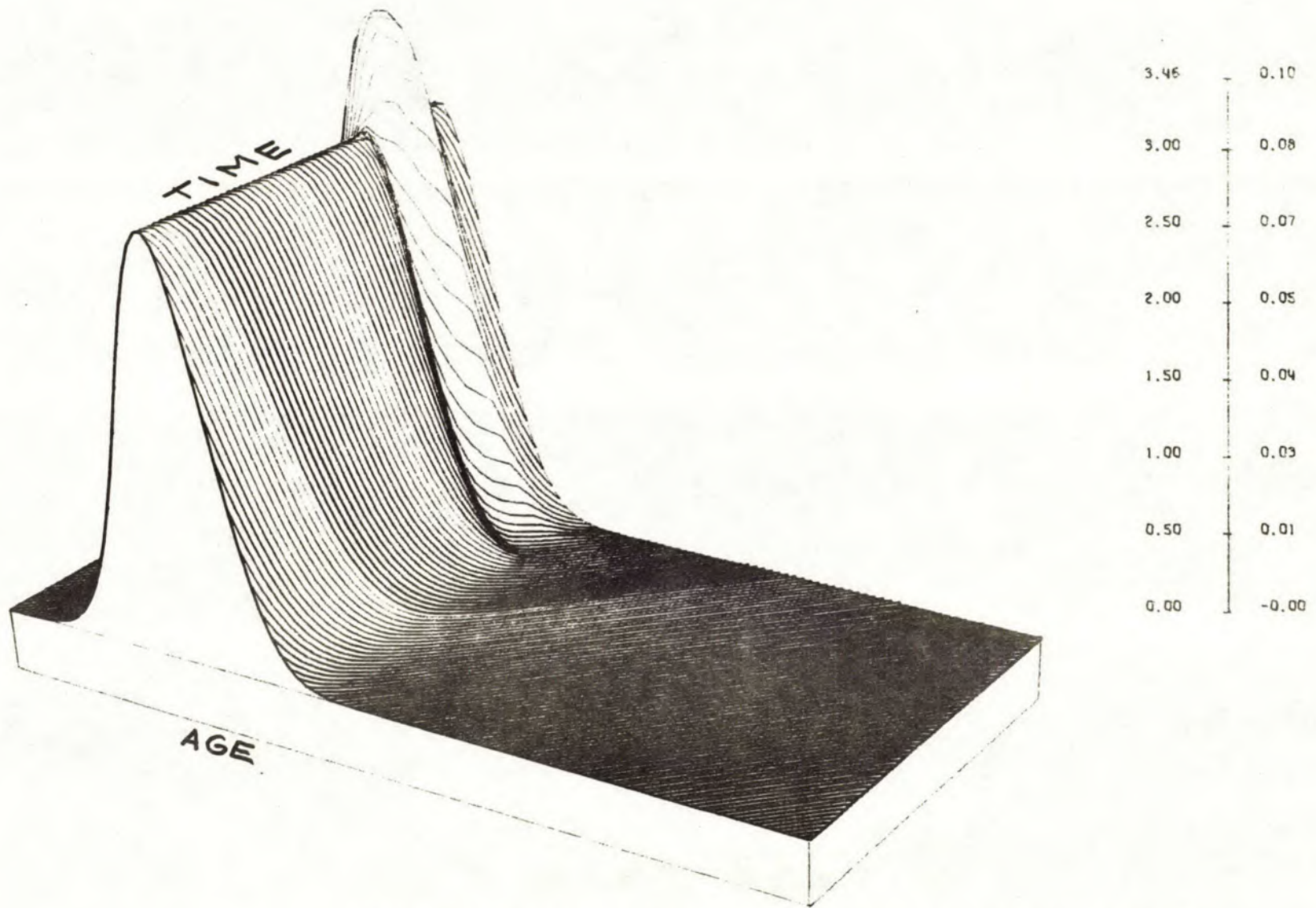
EVOLUTION OF FERTILITY OVER 50 YEARS: WITH FERTILITY PROGRESSION

Figure 3.
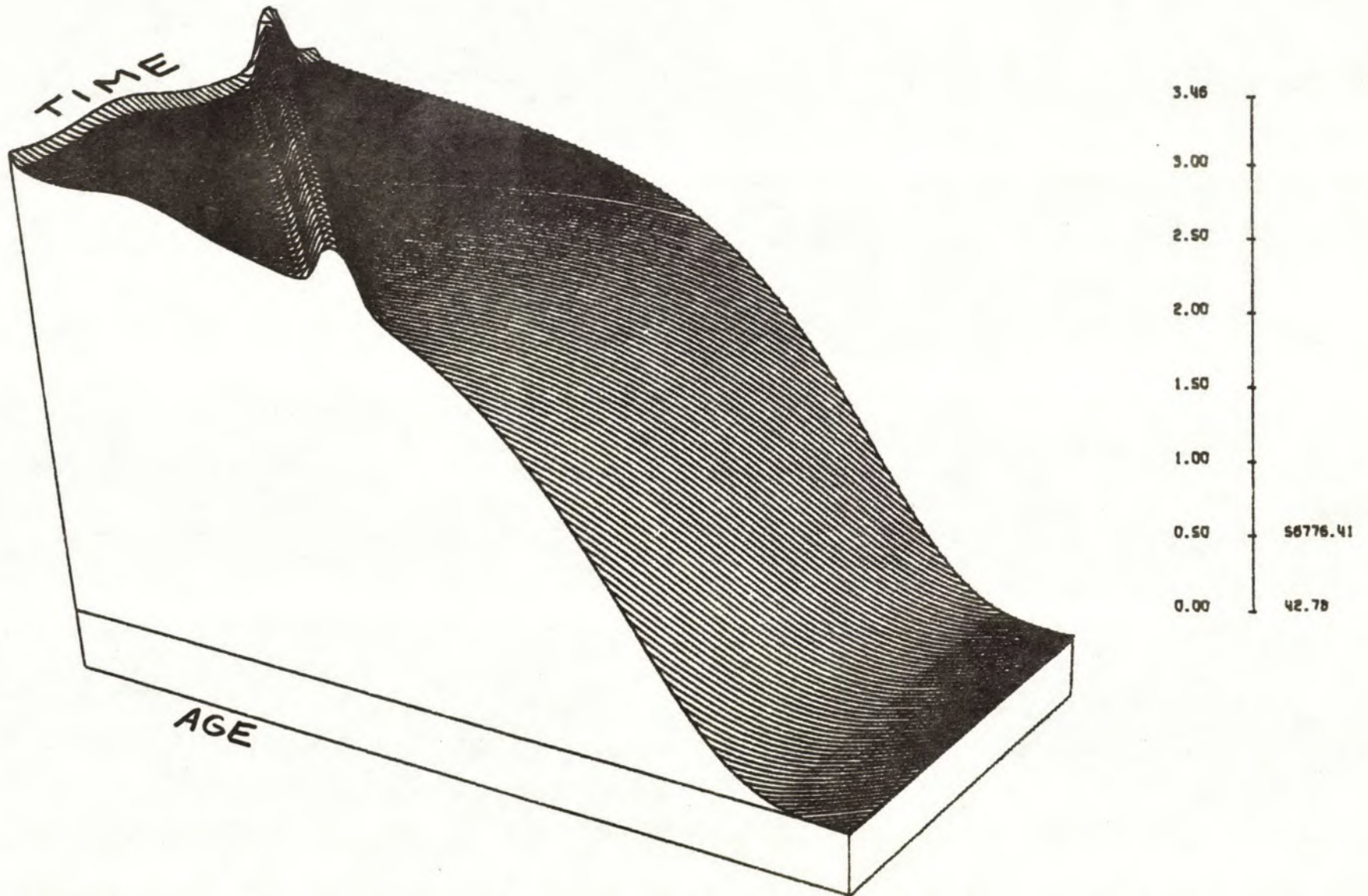
3.46

3.00

2.50

2.00

1.50

1.00

0.50   S6776.41

0.00   42.78

EVOLUTION OF POPULATION OVER 50 YEARS: WITH FERTILITY PROGRESSION

Figure 4.

TIME

AGE

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3.46	
3.00	
2.50	
2.00	
1.50	
1.00	
0.50	56149.99
0.00	38.64
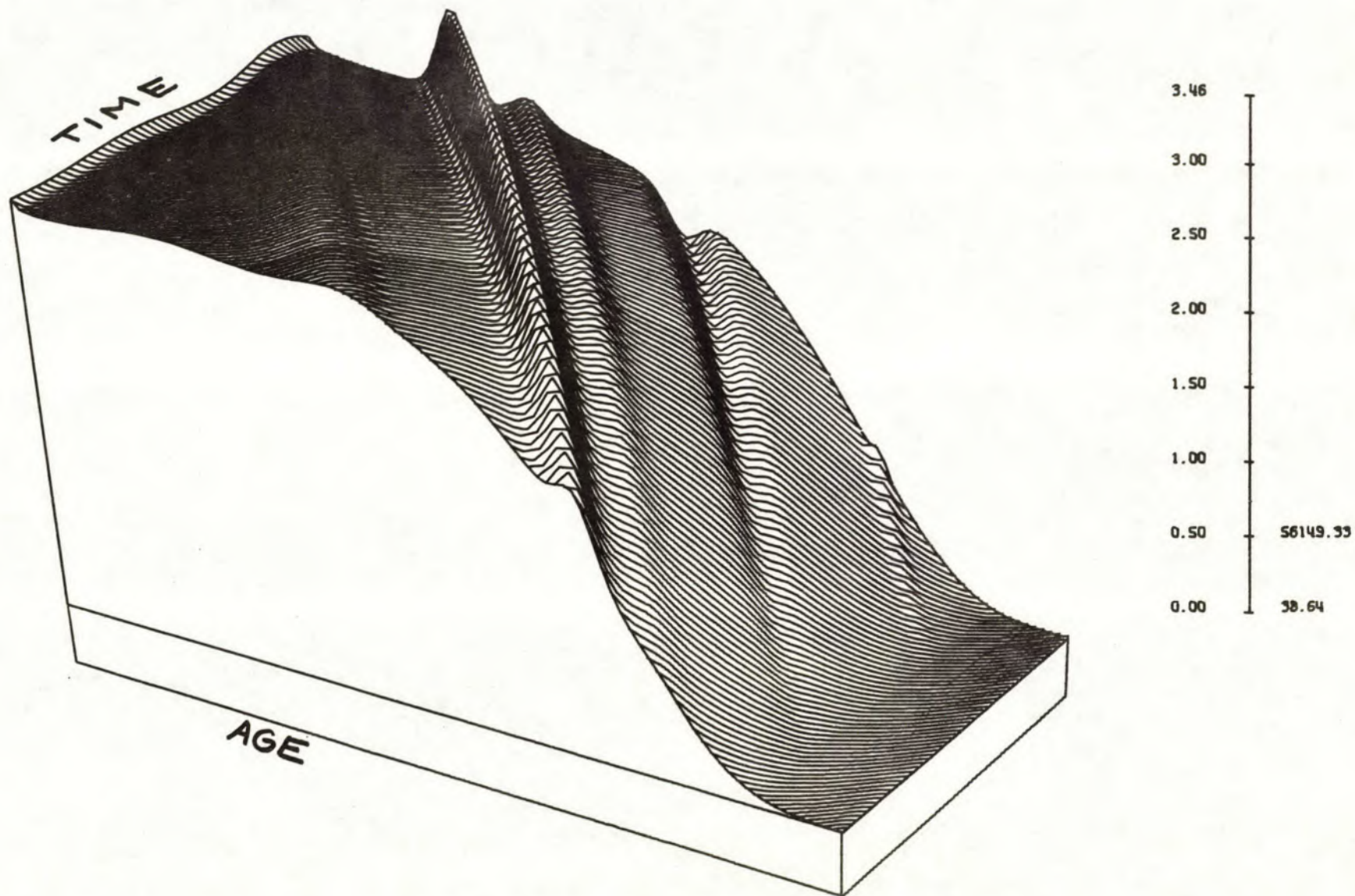
EVOLUTION OF POPULATION OVER 50 YEARS: WITH UNCHANGING FERTILITY

Figure 5.

the result of variations in fertility and immigration which have occurred previous to the time interval covered by the simulation. In this simulation the fertility curve has been held constant, so that the birth-rate variations which occur are due to the varying number of people in the child-bearing age brackets.

VI. Future Work

As of the time of writing of this report, the model has progressed to the point that the basic structure has been established, and the crucial numerical problems associated with the model are well in hand. In particular, numerical methods have been devised for the simultaneous integration of the partial differential equations involved in the model core dynamics and the accuracy of the method has been tested by means of comparison with explicit special case solutions of the model equations obtained by analytical means. Also, a considerable amount of effort has been expended on the problem of devising efficient numerical procedures for extracting estimates of the functional coefficients of the partial differential equations from date available in the form of a histogram. This algorithm has been tested again by use of explicit solutions of the governing equations,

and has produced accurate results in the tests.

With these two obstacles removed, the next step
in the development of the model is to begin the process
of modelling the dynamics of the interaction between the
parameters occurring in the core section of the model
and various economic variables. The first step in this
procedure is to apply the estimation algorithms to the
actual historical data in order to determine the time
history of $a(t)$, $b(t)$ and the variations associated
with the economic mobility μ . Once these functions
have been extracted from the data, various approaches to
establishing the interaction can be started.

The problem of determining the interaction will
first be treated by conventional time-series techniques,
that is, correlation analysis based on the assumption of
a linear dynamical system of finite dimension as the
dynamical intermediary between the economic variables
and those in the model core. More recent methods as-
sociated with input-output analysis of control systems
and identification from operating records will also be
tried if the time-series methods are found unsuitable.
The relative effectiveness of these two techniques will
probably depend on the "actual" location of the stoch-
astic noise element in the real system, length of the
operating records, and other factors which are difficult

to predict in advance. Non-linear regression methods and techniques of non-linear system identification are held in reserve in case the above methods prove incapable of modelling the interactions.

After a suitable dynamic model of the interaction effects has been determined, full scale model simulations may begin. This requires models to generate economic variables as mentioned above in Section II, and it is currently planned to adapt standard economic models to this purpose. It is also expected that in this context stochastic as well as deterministic simulations will be carried out. This is desirable for two reasons: first, it is a means of assessing the sensitivity of the overall model; second, it is clearly more realistic to model economic behaviour to include random fluctuations if possible.

Finally, we mention that there is a considerable amount of additional work which should be carried out in connection with an investigation of this model. In this area, we mention here only two possibilities which might be considered. The first is described here only because of its possible relevance to the problem of interaction identification discussed above.

The model as it currently exists has been form-
ulated on a "macroscopic" level, that is, the processes
which transfer people from one level to another as well
as birth and death processes have been modelled as
occurring continuously in time. On the microscopic
level of an individual, these processes obviously occur
at discrete instants of time and are most suitably
modelled as a stochastic process. Such modelling will
involve the determination of the probability of the
occurances of the various "elementary events" which
occur on the microscopic level. In such a model, the
various interaction effects which are to be estimated in
the continuous model appear in the form of dependence of
transition probabilities on the current state of the
other variables involved in the model. On this level,
there is then the possibility of estimating these trans-
ition probabilities and their dependence on the other
model variables, and thus modelling the interaction
effects directly. Close examination of the resulting
stochastic process model should then shed light on the
form of the interaction in the continuous model form-
ulation of the problem.

A second area where very useful work may be done
is in the area of the construction of highly effecient
numerical methods for the solution of the governing

equations. In our work so far relatively standard numerical techniques have been used for numerical integration of the evolution equations. It may be quite possible to make use of the special forms of the equations to construct more efficient methods. Some preliminary investigation indicates that methods based on Lie algebraic techniques hold promise in this regard. It will be especially important to the usefulness of the final simulation programs that program execution time be kept as low as possible in order that the required number of repeated simulations may be carried out at a reasonable cost.

Appendix A: Analytical Properties of the
                Governing Equations

In this Appendix we report some of the analytical properties of the partial differential equations governing the dynamics of the population distribution and the fertility curve. The study of these analytical properties in itself provides considerable insight into the problems of population dynamics, as well as providing material essential for the testing of the accuracy of numerical methods developed for use in the model.

It was mentioned in Section III that the population distribution equation enjoys an invariance property

which makes it unnecessary to specify in the model formulation the exact measure of income represented by the variable s .

This can be readily demonstrated mathematically as follows. Suppose that instead of considering the distribution function $p(x,s,t)$ as a function of the income scale s , governed by

$$\frac{\partial p}{\partial t}(x,s,t) = -\frac{\partial p}{\partial x}(x,s,t) - \frac{\partial}{\partial s}(\mu(x,s,t)p(x,s,t))$$

$$+ i(x,s,t) - r(x,s,t)\,p(x,s,t)$$

we ask for the evolution of the distribution expressed as a function of the income measure σ . Here the new scale σ is related to the scale s according to

$$\sigma = \varphi(s)$$

where φ is a monotone, smooth (non-linear) function otherwise arbitrary.

By the Chain Rule,

$$\frac{\partial}{\partial s} = \frac{\partial \sigma}{\partial s} \cdot \frac{\partial}{\partial \sigma} = \varphi'(s)\frac{\partial}{\partial \sigma}$$

so that the evolution equation for p becomes

$$\frac{\partial}{\partial t} p(x, \varphi^{-1}(\sigma), t) = - \frac{\partial}{\partial x} p(x, \varphi^{-1}(\sigma), t)$$

$$- \varphi'(\varphi^{-1}(\sigma)) \frac{\partial}{\partial \sigma} (\mu(x, \varphi'(\varphi^{-1}(\sigma)), t) \cdot$$

$$p(x, \varphi^{-1}(\sigma), t)) + i - r \ p(x, \varphi^{-1}(\sigma), t) \qquad .$$

The Jacobian rule shows that the population density in terms of x , σ and t is given by

$$\tilde{p}(x, \sigma, t) = \frac{1}{\varphi'(\varphi^{-1}(\sigma))} \ p(x, \varphi^{-1}(\sigma), t) \qquad .$$

Rearranging the previous equation to introduce $\tilde{p}$ gives

$$\frac{\partial \tilde{p}}{\partial t} = - \frac{\partial \tilde{p}}{\partial x} - \frac{\partial}{\partial \sigma} (\tilde{\mu}(x, \sigma, t)\tilde{p}) + \tilde{\imath} - \tilde{r} \ \tilde{p}$$

with $\tilde{\mu}(x, \sigma, t) = \mu(x, \varphi^{-1}(\sigma), t) \cdot \varphi'(\varphi^{-1}(\sigma)) \qquad .$

$$\tilde{\imath}(x,\sigma,t) \;=\; \frac{i(x,\varphi^{-1}(\sigma),t)}{\varphi'(\varphi^{-1}(\sigma))}$$

$$\tilde{r}(x,\sigma,t) \;=\; r(x,\varphi^{-1}(\sigma),t) \qquad .$$

This identifies the transformation law of the economic mobility, and shows the invariance of the governing equation under such a change of scale.

While the coupled system consisting of the population anf fertility evolution equation is a non-linear one, the non-linear interaction occurs only in the calculation of the instantaneous birth-rate (so long as μ , $a(t)$ and $b(t)$ are treated as exogoneous variables). Since the birth-rate enters only as a boundary condition, it is possible to get useful results from explicit solutions of the equations.

Both the population and fertility equations fall into the class of evolution equations governed by first order partial differential equations. While the equations in general have variable coefficients, they are linear in the dependent variable; hence, in principle, the method of characteristics is applicable.

This observation does not dispose of the problem, however. A principal reason for carrying out the invest-

igation into the analytical properties of the equations
is to obtain if possible _explicit_ solutions to the
equations. By explicit solutions, we mean solutions
obtained in closed form analytically.

These solutions have been used to test the accuracy
of the numerical methods used both to integrate the evol-
ution equations and to estimate the functional co-
efficients of the equations $\mu(x,s,t)$ and $a(t) \cdot d(x)$.
In the absence of explicit solutions, only lengthy (and
expensive) trial runs with varying step sizes can be
employed to attempt to estimate accuracy; with explicit
solutions available, it is far easier to estimate the
step sizes required for a given level of numerical
accuracy.

The above remarks pertain to evaluation of the
integration scheme; in the case of the estimation prob-
lem, the unavailability of explicit solutions would force
one to the use of the integration routine to generate the
data on which to test the estimation algorithm. In the
case of inaccurate results, it then becomes tedious to
determine whether the inaccuracy arises from the est-
imation scheme, or from the numerically generated data.

It is this need for explicit solutions, at least in
particular cases, that has led to the work reported

below. The method of characteristics in general produces a solution in <u>implicit</u> form; it is essentially impossible to carry out the required function inversions numerically with enough control on accuracy to make such implicit solutions useful for our purposes.

Fertility Equation

An explicit solution to the fertility equation may be obtained by the method of characteristics. For

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(a(t)d(x)f) - b(t)\, f$$

introduction of an integrating factor of $\; e^{\int_0^t b(t)dt}$ reduces the problem to

$$\frac{\partial \bar{f}}{\partial t} = -\frac{\partial}{\partial x}(a(t)d(x)\bar{f})\quad .$$

Solution of the above by the method of characteristics gives

$$\bar{f}(x,t) = \frac{1}{d(x)}\,\emptyset\,(h^{-1}(h(x)-\int_0^t a(s)dx))$$

$$\cdot\; d(h^{-1}(h(x) - \int_0^t a(s)ds))\;,$$

$$\text{with} \quad h(x) - h(x_o) = \int_{x_o}^{x} \frac{1}{d(x)} \, dx \,,$$

$$\text{and} \quad \emptyset(x) = \bar{f}(x, 0) \,.$$

Population Equation

As may be seen from the above example, explicit solutions are generally very involved in form. For this reason, explicit solutions of the population equation will not be exhibited here. We remark that such solutions may be found; the case in which the death-rate varies linearly with age is one example of use in connection with the estimation problem for $\bar{\mu}$. (Results from this example allow the removal of the death-rate term from the governing equations by means of an integrating factor.)

The use of explicit solutions has some potential use beyond evaluation of numerical methods. This is in the area of decreasing the size and cutting down the execution time for the simulation of the model. This may become important in later phases of development of the model, and will have an effect on the frequency of use of the completed model.

The key to such reduction of time expenditure is the observation that an explicit solution reduces the problem of evolution over an <u>arbitrary</u> time interval to a <u>single</u> function evaluation. This is to be contrasted with the repeated evaluations involved in a numerical integration. Of course, the full benefit of this discrepancy is available only if the interaction effects are specified exogenously. In the case of the full model, however, it seems likely that explicit solutions could be used together with extrapolation methods to improve simulation execution time.

This leads naturally to the question of which classes of coefficient functions tive rise to explicit solution formulae. Of particular interest is the prob-lem of explicit solutions to models in which the co-efficient functions appear in "separable form" (see Appendix B below), so that the equation has the form

$$\frac{\partial p}{\partial t} = -\frac{\partial p}{\partial x} - \frac{\partial}{\partial s}\left(\sum_{i=1}^{N}\eta_i(t)\nu_i(x,s)p\right) - \xi(t)\ r(x,s)\ p$$

Progress in the direction of explicit solutions to the above equation may be made by recourse to the theory of Lie Algebras. In particular, if the Lie algebra gen-erated by the partial differential operators on the

right side of the above equation is <u>solvable</u>, then
(global) explicit expressions are possible. Other con-
ditions on the Lie algebra lead to (local) results which
may prove useful.

<u>Appendix B</u>:

<u>Numerical Methods for Partial and Ordinary Differential</u>

<u>Equations</u>

In Section III of the report, the population and
fertility are dynamically modelled by a pair of partial
differential equations:

$$\frac{\partial p}{\partial t} = -\frac{\partial p}{\partial x} - \frac{\partial}{\partial s} (\mu(x,s,t) \cdot p) - r(x,s,t)p + i(x,s,t)$$

$$\frac{\partial f}{\partial t} = -a(t) \frac{\partial}{\partial x} (d(x)f) - b(t)f \quad .$$

In Section IV, techniques for estimating the parameters
$\mu(x,s,t)$, $a(t)$, $d(x)$ are discussed and there, it
is pointed out that full simulation of the overall model
requires dynamic modelling of these functions using
partial and/or ordinary differential equations.

Since this appendix deals with aspects of the actual simulation of the model, it is assumed that ordinary differential equations for $a(t)$ and $b(t)$ have been obtained, estimated values of $d(x)$ and $u(x,s,t)$ have been obtained by use of the estimation procedures described elsewhere in this report, and that initial fertility and population distributions are known: $f(x,s,o)$ and $p(x,s,o)$. Values of $p(x,s,t)$ and $f(x,s,t)$ are required, and these are simulated using numerical techniques. The techniques have been chosen in order to be consistent with the conservation law character of the governing equations, to attain a reasonable accuracy in the simulated values subject to restrictions on the size of data groupings which are expected in currently available data, and to balance these with economy of computation.

In the numerical simulation which has been carried out up to the time of writing of this report, the income level dependence of the population density has been suppressed. As well as yielding computational efficiencies during the development of the model, this procedure has made the analysis of numerical problems arising in the modelling considerably easier. The extension of the numerical methods developed so far to include the income variable s is expected to cause no significant dif-

ficulty, as the problems which arise should parallel those already encountered.

Numerical Integration of the Fertility Equation

The fertility equation has been transformed by an integration factor to

$$\frac{\partial \bar{f}}{\partial x}(x,t) = -a(t)\frac{\partial}{\partial x}(d(x)\bar{f}(x,t))$$

where

$$f(x,t) = e^{-\int_{o}^{t}b(t)dt}\bar{f}(x,t) \quad .$$

An approximation F_{ij} to $\bar{f}(ih,jk)$ is obtained using

$$F_{ij+1} = F_{ij} - \frac{k}{2h}a_j(d_{i+1}F_{i+1j}-d_{i-1}F_{i-1j})$$

$$+ \frac{1}{2}(\frac{ka_j}{h})^2[d_{i+\frac{1}{2}}(d_{i+1}F_{i+1j}-d_iF_{ij})$$

$$- d_{i-\frac{1}{2}}(d_iF_i-d_{i-1}F_{i-1j})]$$

and $f(ih,jk)$ is estimated by F_{ij} using numerical

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integration for the integration factor (see below). This scheme is almost second order, and has two desirable properties: For $d(x) = d$ constant, it is numerically stable provided that the step size ratio is chosen to satisfy

$$\frac{k}{h} \leq \left| \frac{1}{d \cdot a_j} \right| \qquad .$$

Also this scheme has the property that it removes a distortion of the fertility profile when the effect causing the distortion is removed. This property is exhibited in Figure 3.

The dynamical equations governing the variables $a(t)$ and $b(t)$ must of course be integrated simultaneously with the partial differential equations. Numerical approximations are currently calculated using a modified Euler method over time steps of length k .

While this procedure may be easily replaced by a more accurate process, this method was selected in view of the decision to use simple routines initially as an aid to algorithmic development, and later to replace these by more sophisticated routines as dictated by accuracy and economy in large simulations.

Since dynamic modelling of $a(t)$ and $b(t)$ has not yet been carried out, the dynamics

$$(D+1)^2 \, a(t) = b(t)$$

have been assumed in order to verify the integration methods. In this case

$$\underline{a}(t^*) = a(t) + h\underline{f}(\underline{a}(t))$$

$$\underline{a}(t+k) = \underline{a}(t) + \frac{h}{2} \left[f(\underline{a}(t)) + f(\underline{a}(t^*)) \right]$$

where

$$\underline{f} \begin{bmatrix} I \; b(t) \\ I \; a(t) \\ a(t) \\ a^1(t) \end{bmatrix} = \begin{bmatrix} b(t) \\ a(t) \\ a^1(t) \\ b(t) - a(t) - 2a^1(t) \end{bmatrix}$$

describes the modified Euler method.

Numerical Integration of the Population Equation

Initially only the age-time dynamics of population have been considered; hence the equation is

$$\frac{\partial p}{\partial t}(x,t) = -\frac{\partial p}{\partial x}(x,t) - r(x,t) \cdot p(x,t) \quad .$$

To solve this numerically, we approximate $p(x,t)$ by $P(x,t)$ where

$$P(x+h, t+h) = (1-r(x,t)) P(x,t) \quad .$$

The fertility is used to estimate the population birth-rate

$$p(o, t+h) = \int_0^\infty p(x,t) f(x,t) dx \quad ,$$

and this is approximated numerically by

$$P(o, t+h) = \sum_{i=1}^{100} P(x,t) F(x, t+\frac{h}{2})$$

where $F(x, t+\frac{h}{2})$ is obtained from the numerical approximations of the fertility curve.

The low accuracy method for simulating the population is reasonably accurate for that section of the profile where the death-rates are almost constant. It is expected that improvements will be possible after additional work. Improvements in the simple scheme used for estimating $P(o, x+h)$ would lead only to a change in scale of values, but not their dynamics.

In conclusion we point out that certain portions of the model are particularly sensitive to errors - that is small errors may lead to very inaccurate simulations of the dynamics, whereas other portions of the model are not so sensitive. For this reason, it is possible (and economically reasonable) to tailor the accuracy of the methods used to the sensitivity of that part of the model being simulated.

Appendix C: Numerical Determination of Partial Differential Equation Coefficients

In Section IV it is shown that integration of the partial differential equations leads to analytic formulas for the estimation of $\mu(x,s,t)$ from the population equation, and $a(t) \cdot d(x)$ from the fertility equation. To use these formulas, available data must be used to estimate the quantities required. In particular, it is required that histogram data be used

1. to generate (continuous) density functions, that

2. partial derivatives of these density functions be estimated, and that

3. the required integrals be estimated.

The distributions involved appear to be very smooth, and as a result piecewise approximation by polynomials with continuous first derivatives is necessary; additional smoothness is desirable. The algorithm employed is described below for the problem of estimating the economic mobility $u(x, s, t)$. The procedure for estimating the term $a(t) d(x)$ in the fertility equation is entirely similar.

1. A function which might be best described as a fourth-order spline (having three continuous derivatives) is determined so that its integrals over the appropriate intervals are equal to the given values from the histogram data.

2. Differentiation of the fourth-order spline with respect to the x-variable provides an estimate of $\frac{\partial p}{\partial x}$; determination of an additional cubic spline in the t-variable followed by a t-differentiation provides an estimate of $\frac{\partial p}{\partial t}$.

3. Finally, the required integral is estimated by integration of the result of 2. above.

In the use of spline methods in approximation problems, it is necessary to provide additional boundary

conditions beyond the requirement that the spline inter-
polate the appropriate sample points. Unfortunately, in
the present application, use of the so-called "natural
boundary conditions" was found to give particularly poor
estimates near the boundary of the region involved.
Further, it was found that these errors were quite sens-
itive to the values assigned.

After considerable experimentation, it was deter-
mined that adequate results could be obtained through
estimation of the third derivatives near the endpoints
by third order finite differences, and use of this data
to determine the boundary conditions. The scheme for
approximating the distribution function requires the
solution of a system of linear equations including three
different types

(a) $\quad r_i'' - 2r_{i-1}'' + r_{i-2}'' - \dfrac{r_i''''}{2} - \dfrac{r_{i-1}''''}{2} = 0 \qquad i = 2,\ldots,100$

(b) $\quad r_i - 2r_{i-1} + r_{i-2} - \dfrac{r_i''}{6}$

$\qquad -\dfrac{2}{3} r_{i-1}'' - \dfrac{r_{i-2}''}{6} + \dfrac{r''''}{24} + \dfrac{r''''}{24} = 0 \qquad i = 2,\ldots,100$

(c) $\quad r_i + r_{i-1} - \dfrac{r''_i}{12} - \dfrac{r''_{i-1}}{12} + \dfrac{r''''_i}{60} = 2\,P_i \qquad i = 1,\ldots,100$

where P_i is the number of persons between ages i and $i + 1$, and the boundary conditions are

$$r''_0 = r''_{100} = 0 \quad , \quad r'''_0 = 0 \Rightarrow -r''_0 + r''_1 - \frac{r''''_1}{2} = 0$$

$$r''''_0 = 0 \; .$$

Here r_i is interpreted as the population distribution at age i years, and it is assumed that values P_i are available for $i = 0,1,\ldots,99$.

To solve this system, a reduction method for a sparse matrix is used, and the equations are ordered so that coefficients of moderate size are maintained on the diagonal. For a test distribution, the error in re-generating the histogram was less than 1 percent.

Spline Approximation

A standard analysis for cubic spline approximation represents this function in terms of estimates for the

second derivative at nodes. These are obtained as solutions of the system of equations

$$\frac{h_i M_{i-1}}{h_i + h_{i+1}} + 2M_i + \frac{h_{i+1} M_{i+1}}{h_i + h_{i+1}} = 6 \frac{\dfrac{r_{i+1} - r_i}{h_{i+1}} - \dfrac{r_i - r_{i-1}}{h_i}}{h_i + h_{i+1}}$$

to interpolate $\{r_i\}$ with a spacing $\{h_i\}$ (which for our model is either h or k constant), and boundary conditions used are

$$\frac{M_1 - M_o}{h} = \frac{1}{h^3} (-r_o + 3r_1 - 3r_2 + r_3)$$

$$\frac{M_n - M_{n-1}}{h} = \frac{-1}{h^3} (-r_{n-3} + 3r_{n-2} - 3r_{n-1} + r_n)$$

With this approximation, errors in $u(x,s,t)$ and $a(t)$ $d(x)$ obtained using the estimation procedures are less than 1 percent on the interior of the domain. Although errors are large where $P(x,s,t)$ is small, values of $u(x,s,t)$ there are not crucial (see Section IV).

Separability of Coefficient Functions

It was mentioned in Section III that one aspect of the structure of the model was that it was formulated in such a way as to make it possible to model the feedback effects on the core section of the model in terms of a finite number (even a <u>small</u> finite number) of functions of time. This was illustrated in Section III in the hypothesis that the effect of the rest of the world on the fertility curve could be adequately modelled by

$$\frac{\partial f}{\partial t} = -a(t) \frac{\partial}{\partial x} (d(x)f) - b(t)f \quad .$$

In this formulation, the world affects f only through $a(t)$ and $b(t)$. However, it was also mentioned that

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x} (a(x,s,t)f) - b(t,s)f$$

might well represent a more realistic model, and it is easy to verify that the estimation procedure described in Section IV and Appendix C above will equally well produce an estimate of the coefficient function $a(x,s,t)$. A problem that arises, then, is that of distinguishing between $a(x,s,t)$ and $a(t) \cdot d(x)$ at the

stage of the output of the estimation algorithm. From

the point of view of subsequent modelling effort, it

might well be hoped that the result had the form

$a(t) \, d(x,s)$, or perhaps even $a(t) \cdot d(x) \cdot c(s)$.

A similar problem occurs in the case of estimation

of $\mu(x,s,t)$, where the validity of a representation

of the form

$$\mu(x,s,t) \;=\; \eta_1(t) \, \nu_1(x,s) + \ldots + \eta_N(t) \, \nu_N(x,s) \;,$$

with N a "reasonably small" integer is at least a

practical requirement for the success of any attempt to

model the interaction affects.

Given the implicit smoothness assumptions on the

coefficient functions of the model and the fact that the

ranges of the x , s , and t variables involved

are finite, there is no problem in applying standard

approximation theorems to deduce that μ may be closely

approximated by a function of the above form. (A

similar remark obviously applies to $a(x,s,t)$). For

convenience, we refer to the above form as a "separable

representation for μ ".

Since we have shown above that separable repres-

entations exist, the only problems which remain are those

of the number N of terms to be expected in the representation, and the numerical determination of N and $\eta_i(t)$ from the available data.

Consideration of the effects that the t variation in μ (and a) is intended to model, and the probable variability of these effects across age and income brackets suggests strongly that N is small. It would be surprising if N were greater than 3 in the case of the estimation of $\mu(x,s,t)$, and it appears entirely possible that a single term will suffice in the case of the estimation of $a(x,s,t)$.

It remains to show the feasibility of determining separability of the representation numerically. To distinguish a separable μ from a non-separable one we proceed as follows:

A smooth function $\mu(x,s,t)$, defined for $s \in S$, $t \in T$, $x \in X$, with S, T, X compact subsets of R^1 defines the kernel of a compact linear operator L mapping from $L^2(X \times S) \to L^2(T)$ according to the formula

$$Lf(t) = \int\int_{X \times S} \mu(s,s,t)\, f(x,s)\, dx ds \quad .$$

Now a _separable_ μ is distinguished by the fact
that the associated L is an operator of finite dimen-
sional range, and this observation essentially solves the
problem.

When $\mu(s,s,t)$ has been estimated numerically, μ
is not obtained as a continuous function. What is ob-
tained in fact is a set of sample values
$\{\mu(x_i,s_j,t_k)\}$, with $x_i \in \tilde{X}$, $s_j \in \tilde{S}$, $t_k \in \tilde{T}$.
Here $\tilde{X}$, $\tilde{S}$, $\tilde{T}$ are each Euclidean space of dimen-
sion equal to the number of sample points in each of the
independent variables. The discrete analog of the def-
inition of L is to use the above three dimensional
array to define a linear mapping (matrix)

$$\tilde{L} : R^{\dim(\tilde{X} \times \tilde{S})} \to R^{\dim \tilde{T}} .$$

The problem of finding the $\eta_i(t)$ is now equiv-
alent to determining the range space of $\tilde{L}$, and N
is simply the rank of $\tilde{L}$.

The problem is simplified still further by invoking
the fact that

$$\text{Range } \tilde{L} = \text{Range } \tilde{L} \, \tilde{L}^* ,$$

where $\tilde{L}^*$ is the adjoint (actually transpose in this
case) of the matrix $\tilde{L}$. This reduces the problem to

the entirely standard one of an eigenvector/eigenvalue

analysis of a <u>symmetric</u> matrix, and hence effectively

solves it.

Appendix D   Computer Program Listing

In this appendix we list the computer programs

developed up to the time of this report for use with

the model. Included below are both the programs used

for numerical integration of the governing evolution

equations in simulation runs, and the programs designed

to estimate model coefficients from the available data.

The programs listed here are written in FORTRAN.

Given the relatively large arrays of data which must be

handled in connection with this model, it is clear that

FORTRAN is not the most convenient language in which to

program the numerical algorithms required. With a view

to future uses of the model, however, such factors as

the wide availability of FORTRAN compilers, the exist-

ence of the I.B.M. CSMP (Continuous System Modelling

Package) which is FORTRAN compatible, and of FORTRAN

packages for the Calcomp plotter used to produce output

data plots make FORTRAN a reasonable language choice.

Program I

Simulation of population and fertility propagation over time. Initial age-specific profiles of population, fertility and mortality, and dynamics for $a(t)$ and $d(x)$ are required. Here, the values for population are taken as the number of live individuals at age x in the population as given by the Commissioner's 1941 Standard Ordinary Mortality Table. Values for mortality are also taken from this table.

Values for the fertility are given by the artificial distribution:

$$f(x,o) = C e^{-\frac{(x-24.5)^2}{8}}$$

where the constant C is chosen so that

$$\int_0^{100} f(x,o) \, p(x,o) \, dx = p(o) \quad .$$

Values of $d(x)$ are assumed from (the artificial distribution)

$$d(x) = \frac{(1-e^{-\frac{(x+1)^2}{600}})}{1+e^{\frac{(x-24.5)^2}{4}}}$$

The parameter $a(t)$ is determined by numerical integration of the differential equation

$$(D+1)^2 \, a(t) \;=\; b(t)$$

where

$$b(t) \;=\; -\,\frac{\sin(e^{-|t-5|}-e^{-|t-10|})}{15} \qquad 0 \le t < 15$$

$$=\; 0 \qquad\qquad\qquad t > 15$$

```
          $JOB    ACCT-NUM,'VERNER',TIME=60
     1            REAL X(100)
     2            IC=100
     3            IF=0
     4            DO 9 IR=1,10
     5            CALL DATA(X,IR,IC,IF)
     6      9          CONTINUE
     7            STOP
     8            END


     9            SUBROUTINE DATA(X,IR,IC,IF)
    10            REAL D(100),F(100),P(100),X(100),R(100),US(100),D5(100),F1(100)
    11            REAL HH,KK,SUM,IV(10),DLY(10)
    12            REAL PLTPER,PRTPER,I1
    13            INTEGER YRS,AGES,YOUNG,NN
          C       **************************************************************
          C       SET IFORM=0 IF YOU WANT PLOT,    =1 IF YOU WANT PRINT
          C       **************************************************************
    14            IFORM=1
    15            PL=1E-5
    16            PR=PI
    17            TINT=1E-4
    18            ICM1=IC-1
    19            IF (IR.GT.1) GO TO 31
          C       RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
          C       HH IS AGE STEP   -   KK IS TIME STEP
    20            READ(5,101) HH,KK
    21      101   FORMAT(2F10.6)
          C       PLTPER IS PLOT INTERVAL   -   PRTPER IS PRINT INTERVAL
    22            READ(5,101) PLTPER,PRTPER
          C       NDE IS THE NUMBER OF FIRST ORDER ODE'S AFTER TRANSFORMATION
          C       IF NDE > 10 CHANGE REAL DIMENSION IN DATA AND $STEP
          C       NN IS THE NUMBER OF INTEGRATION STEPS IN KK FOR SOLVING THE ODES
    23            READ(5,103) NDE, NN
    24      103   FORMAT(2I4)
          C       PUT INITIAL VALUES OF ODES ONE TO A CARD
    25            DO 97 I=1,NDE
    26                 READ(5,107) DLY(I)
    27      107        FORMAT(F10.4)
    28      97         CONTINUE
          C       RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
    29            PR=PR+KK
    30            T=0.0
          C       **************************************************************
          C                    READ DEATH RATES
          C                    READ POPULATION DENSITY
          C       **************************************************************
          C       RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
    31            READ(5,100) (R(I),I=1,IC)
    32      100   FORMAT(5F10.6,30X)
    33            READ(5,104) (P(I),I=1,IC)
    34      104   FORMAT(10F8.0)
          C       RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
    35            SUM=0.0
    36            DO 99 I=1,IC
    37                 SUM=SUM+P(I)
    38      99    CONTINUE
    39            DO 98 I=1,IC
    40                 P(I)=P(I)*22000000./SUM
    41            X(I)=P(I)
```

```
42    98        CONTINUE
      C         PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
43              IF(IFORM.NE.1) GOTO 61
44              IPG=1
45              WRITE(6,109)
46    109       FORMAT('1','POPULATION FOR AGES  1   TO 20   IS')
47              WRITE (6,102) (P(I),I=1,20)
48    102       FORMAT(5F10.0/)
49              WRITE(6,110)
50    110       FORMAT(' ','AT BEGINNING OF YEAR            0, AND')
51              WRITE(6,111)
52    111       FORMAT(' ','------------------------------------')
53    61        CONTINUE
      C         PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
54              DO 70 I=1,IC
55                  I1=FLOAT(I)
56                  D(I)=(1-EXP(-11*I1/600.))/(1+((I1-25.5)**2)/4.)
57    70          CONTINUE
58              YOUNG=0
59              F1(1)=EXP(-(.125*HH*(YOUNG-24.5))**2)
60                  SUM=F1(1)*P(1)
61              DO 14 I=2,IC
62                  F1(I)=EXP(-(.125*HH*(I+YOUNG-25.5))**2)
63                  SUM=SUM+F1(I)*P(I)
64                  D5(I)=.5*(D(I)+D(I-1))
65    14          CONTINUE
66              DQ5=.5*(3.*D(1)-D(2))
67              DT5=.5*(3.*D(IC)-D(IC-1))
68              SUM=P(1)/SUM
69              DO 15 I=1,IC
70                  F1(I)=F1(I)*SUM
71    15          CONTINUE
72              PA=KK/HH
73              PA1=0.0
74              P2=0.0
75              FA=1
76              GO TO 33
77    31        CONTINUE
78              PR=PR+KK
      C         ****************************************************************
      C                           UPDATE COEFFICIENTS OF FERTILITY EQUATION
      C         ****************************************************************
79              PA1=DLY(2)
80              P2=DLY(2)*DLY(2)*PA
81              FA=EXP(-DLY(4))
      C         PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
82              IF (PRTPER.GT.PR.OR.IFORM.NE.1) GOTO 62
83              WRITE(6,106) T
84    106       FORMAT(' AT TIME',F6.2)
85              WRITE(6,112) DLY(1)
86    112       FORMAT('0','DELAY IN PEAK OF FERTILITY CURVE IS  ',F9.6)
87    62        CONTINUE
      C         PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
      C         ****************************************************************
      C                           MULTIPLY FERTILITY BY INTEGRATING FACTOR
      C         ****************************************************************
88    33        CONTINUE
89              SUM=0.0
90              DO 42 I=1,IC
91                  SUM=SUM+F1(I)
```

- 63 -

```
 92                 F(I)=FA*F1(I)
 93      42         CONTINUE
       C     PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
 94              IF (PRTPER.GT.PR.OR.IFORM.NE.1) GOTO 63
 95              WRITE(6,113) SUM
 96      113    FORMAT(' ','    INTEGRAL OF NORMALIZED FERTILITY IS        ',F9.6)
 97              WRITE(6,114) FA
 98      114    FORMAT(' ','      FRACTION OF INITIAL FERTILITY IS                    '
              2,F9.6)
 99              WRITE(6,115)
100      115    FORMAT('0','VALUES OF FERTILITY CURVE FOR AGES  21 TO 30   ARE')
101              WRITE(6,108)  (F(I), I=21,30)
102      108    FORMAT(5F10.6/)
103      63     CONTINUE
       C     PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
       C     ********************************************************************
       C                              INCREMENT POPULATION BY ONE YEAR
       C     ********************************************************************
104              TINT=TINT+KK*2
105              IF (HH.GT.TINT) GOTO 45
106              SUM=0.
107              SUM1=0.
108              DO 44 I=1,ICM1
109                  J=IC-I
110                  SUM=SUM+P(J+1)*F(J+1)
111                  P(J+1)=P(J)*(1-R(J))
112                  SUM1=SUM1+P(J+1)
113                  X(J+1)=P(J+1)
114      44         CONTINUE
115              SUM=SUM+P(1)*F(1)
116              P(1)=SUM
117              X(1)=P(1)
118              SUM1=SUM1+P(1)
119              TINT=-1E-4
120      45     CONTINUE
       C     PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
121              IF (PRTPER.GT.PR.OR.IFORM.NE.1) GOTO 64
122              WRITE(6,116) SUM1
123      116    FORMAT('0','TOTAL POPULATION IS  ',F10.0)
124              PR=+1E-5
125              WRITE(6,117)
126      117    FORMAT('0','*****************************************************
              I******************** ')
127      64     CONTINUE
       C     PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
       C     ********************************************************************
       C                              INCREMENT FERTILITY BY ONE YEAR
       C     ********************************************************************
128              DO 34 I=1,IC
129                  IF (1E-20.GT.F1(I)) F1(I)=0.0
130                  US(I)=U(I)*F1(I)
131      34         CONTINUE
132              UQ=2.*US(1)-US(2)
133              UT=2.*US(IC)-US(IC-1)
134              F1(IC)=F1(IC)-.5*PA*((PA1*(UT-US(IC-1)))-P2*(DT5*UT+D5(IC)*US
              2(IC-1)-(DT5+D5(IC))*US(IC)))
135              F1(1)=F1(1)-.5*PA*((PA1*(US(2)-UQ))-P2*(D5(2)*US(2)+DQ5*UQ-(D5
              2(2)+DQ5)*US(1)))
136              DO 35 I=2,ICM1
137                  F1(I)=F1(I)-.5*PA*((PA1*(US(I+1)-US(I-1)))-P2*(D5(I+1)*US
```

- 64 -

```
                2(I+1)+D5(I)*US(I-1)-(D5(I+1)+D5(I))*US(I)))
138       35        CONTINUE
      C
      C         **********************************************************
      C                   SOLVE ODES FOR DELAY PARAMETERS AND INCREMENT TIME
      C         **********************************************************
139             DO 36 I=1,NDE
140                 IV(I)=DLY(I)
141       36        CONTINUE
142             CALL $RUN(IV,DLY,T,KK,NDE,NN)
143             PL=PL+KK
144             IF (PLTPER.GT.PL) GOTO 31
      C         PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
145             IF (TFORM.NE.1) GOTO 65
146             IPG=IPG+1
147             IF (IPG.NE.2) GOTO 79
148             IPG=0
149             WRITE(6,121)
150      121    FORMAT('1')
151       79    CONTINUE
152             WRITE(6,118)
153      118    FORMAT(' ','POPULATION FOR AGES  1 TO 20  IS')
154                 WRITE(6,102) (P(I), I=1,20)
155             WRITE(6,119) IR
156      119    FORMAT(' ','AT BEGINNING OF YEAR',I8,',AND')
157             WRITE(6,120)
158      120    FORMAT(' ','-------------------------------')
159       65    CONTINUE
      C         PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
160             RETURN
161             END

162             SUBROUTINE $RUN(IV,Z,T,KK,NDE,NN)
      C         **********************************************************
      C         THIS SUBROUTINE GIVES A SIMULTANEOUS SIMULATION OF VALUES
      C         Z=(IA(T),A(T),A'(T),IB(T),B(T)) AT  T+H  FROM INITIAL VALUES  IV
      C         OF THE SAME VARIABLES AT T.  FOR EACH DIFFERENT SIMULATION THE
      C         SUBROUTINE  $DE  MUST BE REWRITTEN TO EVALUATE THE DERIVATIVES
      C         D(I)  AS A FUNCTION OF TIME  T  AND SOLUTION VALUES  Y(I)
      C         **********************************************************
163             REAL IV(NDE),Z(NDE),T,T2,KK,K1
164             K1=KK/FLOAT(NN)
165             T2=T
166             T=T+KK
167       28    CONTINUE
168             CALL $STEP(IV,Z,T2,K1,NDE)
169             T2=T2+K1
170             DO 29 I=1,NDE
171                 IV(I)=Z(I)
172       29        CONTINUE
173             IF (T.GT.T2+K1/2) GOTO 28
174             RETURN
175             END

176             SUBROUTINE $STEP(IV,Z,T,KK,NDE)
177             REAL IV(NDE),Z(NDE),Z1(10),D(10),D1(10),KK
178             CALL $DE(T,IV,NDE,D)
179             DO 11 I=1,NDE
180                 Z1(I)=IV(I)+KK*D(I)
181       11        CONTINUE
182             TT=T+KK
```

```fortran
      183          CALL $DE(TT,Z1,NDE,D1)
      184          DO 12 I=1,NDE
      185              Z(I)=IV(I)+.5*KK*(D(I)+D1(I))
      186     12      CONTINUE
      187          RETURN
      188          END

      189          SUBROUTINE $DE(T,Y,NDE,D)
      190          REAL Y(NDE),D(NDE),KK
      191          D(1)=Y(2)
      192          D(2)=Y(3)
      193          T1=0.
      194          IF (T.GT.15.) GOTO 78
      195          TT1=ABS(T-5.)
      196          TT2=ABS(T-10.)
      197          T1=(-SIN(EXP(-TT1)-EXP(-TT2)))/15.
      198     78   CONTINUE
      199          D(3)=15.*T1-Y(2)-2.*Y(3)
      200          D(4)=T1
      201          D(5)=0.
      202          RETURN
      203          END

          $ENTRY
```

POPULATION FOR AGES   1 TO 20   IS
    350170.   342263.   340289.   338880.   337734.

    336725.   335795.   334919.   334091.   333320.

    332613.   331958.   331324.   330688.   330033.

    329350.   328642.   327922.   327184.   326431.

AT BEGINNING OF YEAR   0, AND

AT TIME   0.75

DELAY IN PEAK OF FERTILITY CURVE IS   -0.000373
    INTEGRAL OF NORMALIZED FERTILITY IS   1.089507
       FRACTION OF INITIAL FERTILITY IS   1.000501

VALUES OF FERTILITY CURVE FOR AGES 21 TO 30   ARE
  0.056025 0.063485 0.069726 0.074224 0.076578

  0.076574 0.074216 0.069721 0.063482 0.056023

TOTAL POPULATION IS   22000110.

##
POPULATION FOR AGES   1 TO 20   IS
    350215.   342263.   340289.   338880.   337734.

    336724.   335795.   334919.   334091.   333320.

    332613.   331958.   331324.   330688.   330033.

    329350.   328642.   327922.   327184.   326431.

AT BEGINNING OF YEAR   1, AND

AT TIME   1.75

DELAY IN PEAK OF FERTILITY CURVE IS   -0.004762
    INTEGRAL OF NORMALIZED FERTILITY IS   1.089505
       FRACTION OF INITIAL FERTILITY IS   1.002134

VALUES OF FERTILITY CURVE FOR AGES 21 TO 30   ARE
  0.056127 0.063609 0.069877 0.074401 0.076738

  0.076675 0.074285 0.069795 0.063563 0.056101

TOTAL POPULATION IS   22000510.

##

```
POPULATION FOR AGES   1 TO 20   IS
  350562.    342307.    340289.    338880.    337734.

  336725.    335795.    334919.    334092.    333320.

  332613.    331958.    331324.    330688.    330033.

  329350.    328642.    327922.    327184.    326431.

AT BEGINNING OF YEAR        2,AND
------------------------------------
AT TIME   2.75

DELAY IN PEAK OF FERTILITY CURVE IS   -0.020237
     INTEGRAL OF NORMALIZED FERTILITY IS        1.089506
          FRACTION OF INITIAL FERTILITY IS                 1.006584

VALUES OF FERTILITY CURVE FOR AGES   21 TO 30   ARE
  0.056417   0.063967   0.070328   0.074940   0.077210

  0.076924   0.074418   0.069958   0.063759   0.056301


TOTAL POPULATION IS   22001840.

#######################################################################################
POPULATION FOR AGES   1 TO 20   IS
  351507.    342646.    340332.    338880.    337734.

  336724.    335795.    334919.    334091.    333320.

  332613.    331958.    331324.    330687.    330033.

  329350.    328642.    327922.    327184.    326432.

AT BEGINNING OF YEAR        3,AND
------------------------------------
AT TIME   3.75

DELAY IN PEAK OF FERTILITY CURVE IS   -0.064413
     INTEGRAL OF NORMALIZED FERTILITY IS        1.089506
          FRACTION OF INITIAL FERTILITY IS                 1.018705

VALUES OF FERTILITY CURVE FOR AGES   21 TO 30   ARE
  0.057215   0.064960   0.071590   0.076457   0.078518

  0.077577   0.074741   0.070374   0.064276   0.056834


TOTAL POPULATION IS   22005710.

#######################################################################################
```

POPULATION FOR AGES   1 TO 20   IS
   354085.   343570.   340669.   338923.   337734.

   336724.   335795.   334919.   334091.   333320.

   332613.   331958.   331324.   330688.   330033.

   329350.   328642.   327922.   327184.   326432.

AT BEGINNING OF YEAR   4,AND

AT TIME   4.75

DELAY IN PEAK OF FERTILITY CURVE IS   -0.184397
     INTEGRAL OF NORMALIZED FERTILITY IS   1.089502
          FRACTION OF INITIAL FERTILITY IS   1.050882

VALUES OF FERTILITY CURVE FOR AGES 21 TO 30   ARE
  0.059358 0.067649 0.075042 0.080616 0.082016

  0.079207 0.075498 0.071418 0.065613 0.058225

TOTAL POPULATION IS   22016510.

###
POPULATION FOR AGES   1 TO 20   IS
   361071.   346090.   341588.   339259.   337778.

   336725.   335795.   334919.   334091.   333320.

   332613.   331958.   331324.   330688.   330033.

   329349.   328642.   327922.   327184.   326432.

AT BEGINNING OF YEAR   5,AND

AT TIME   5.75

DELAY IN PEAK OF FERTILITY CURVE IS   -0.480408
     INTEGRAL OF NORMALIZED FERTILITY IS   1.089503
          FRACTION OF INITIAL FERTILITY IS   1.098618

VALUES OF FERTILITY CURVE FOR AGES 21 TO 30   ARE
  0.062964 0.072470 0.081761 0.088994 0.088098

  0.080292 0.074753 0.071688 0.066822 0.059826

TOTAL POPULATION IS   22043640.

###

29282

POPULATION FOR AGES   1 TO 20   IS
     377589.   352918.   344093.   340174.   338112.

     336768.   335795.   334919.   334091.   333320.

     332613.   331958.   `331324.   330688.   330033.

     329350.   328641.   327922.   327184.   326432.

AT BEGINNING OF YEAR   6,AND

AT TIME   6.75

DELAY IN PEAK OF FERTILITY CURVE IS   -0.909306
     INTEGRAL OF NORMALIZED FERTILITY IS   1.089499
         FRACTION OF INITIAL FERTILITY IS   1.118620

VALUES OF FERTILITY CURVE FOR AGES   21 TO 30   ARE
  0.065618   0.076765   0.088974   0.098260   0.092387

  0.076925   0.069761   0.068678   0.065408   0.059325

TOTAL POPULATION IS   22082090.

##
POPULATION FOR AGES   1 TO 20   IS
     389333.   369063.   350881.   342668.   339024.

     337101.   335838.   334919.   334091.   333320.

     332613.   331958.   331324.   330688.   330033.

     329350.   328642.   327922.   327184.   326432.

AT BEGINNING OF YEAR   7,AND

AT TIME   7.75

DELAY IN PEAK OF FERTILITY CURVE IS   -1.277342
     INTEGRAL OF NORMALIZED FERTILITY IS   1.089500
         FRACTION OF INITIAL FERTILITY IS   1.121859

VALUES OF FERTILITY CURVE FOR AGES   21 TO 30   ARE
  0.067241   0.079908   0.094899   0.105441   0.093917

  0.072075   0.064586   0.065381   0.063390   0.058125

TOTAL POPULATION IS   22123450.

##

```
  POPULATION FOR AGES   1 TO 20   IS
     393105.      380542.      366934.      349429.      341510.

     338010.      336170.      334962.      334091.      333320.

     332613.      331958.      331324.      330688.      330033.

     329350.      328642.      327922.      327184.      326431.
```

AT BEGINNING OF YEAR   8, AND

AT TIME   8.75

DELAY IN PEAK OF FERTILITY CURVE IS   -1.496449
     INTEGRAL OF NORMALIZED FERTILITY IS   1.089500
          FRACTION OF INITIAL FERTILITY IS   1.111391

```
VALUES OF FERTILITY CURVE FOR AGES   21 TO 30   ARE
  0.067541   0.081096   0.097754   0.108647   0.093259

  0.067985   0.060862   0.062787   0.061503   0.056761
```

TOTAL POPULATION IS   22164480.

\#
```
  POPULATION FOR AGES   1 TO 20   IS
     392049.      384229.      378346.      365415.      348248.

     340489.      337077.      335293.      334134.      333320.

     332613.      331958.      331324.      330687.      330033.

     329350.      328642.      327922.      327184.      326431.
```

AT BEGINNING OF YEAR   9, AND

AT TIME   9.75

DELAY IN PEAK OF FERTILITY CURVE IS   -1.532525
     INTEGRAL OF NORMALIZED FERTILITY IS   1.089499
          FRACTION OF INITIAL FERTILITY IS   1.078118

```
VALUES OF FERTILITY CURVE FOR AGES   21 TO 30   ARE
  0.065753   0.079162   0.095780   0.106416   0.090456

  0.065073   0.058294   0.060436   0.059348   0.054862
```

TOTAL POPULATION IS   22197900.

\#

```
POPULATION FOR AGES   1 TO 20   IS
   385079.     383196.     382012.     376780.     364179.

   347206.     339549.     336197.     334465.     333362.

   332613.     331958.     331324.     330688.     330033.

   329350.     328642.     327922.     327184.     326431.
AT BEGINNING OF YEAR        10,AND
---------------------------------------------

CORE USAGE        OBJECT CODE=    9856 BYTES,ARRAY AREA=     3400 BYTES,TOTAL AREA

DIAGNOSTICS       NUMBER OF ERRORS=       0, NUMBER OF WARNINGS=        0, NUMBER

COMPILE TIME=    3.98 SEC,EXECUTION TIME=    14.47 SEC,   QUEEN'S WATFOR VERSION

 COST FOR THIS PROGRAM IS  $     1.42            RUN IN HSC     MAR  19,  1974
```

The above output simulates population and fert-
ility over ten years with "Baby boom" dynamics – wide
peak of intermediate height.   See Figures 3 and 4 in
Section V.

Program Ib

This is a copy of the program which was used to
plot the profile in Figure 6.

```
  ~ $JOB   STF55952,'VERNER'
    // EXEC SYMVU,TIME=5
    //COMP.SYSIN DD *
            SUBROUTINE DATA(X,IR,IC,IF)
            REAL D(100),F(100),P(100),X(100),R(100),US(100),D5(100),F1(100)
            REAL HH,KK,SUM,IV(10),DLY(10)
  ~         REAL PLTPER,PRTPER,I1
            INTEGER YRS,AGES,YOUNG,NN
  •
  C         ##################################################################
  ~ C       SET IFORM=0 IF YOU WANT PLOT,    =1 IF YOU WANT PRINT
  • C       ##################################################################
            IFORM=1
            IFORM=0
            PL=1E-5
            PR=PL
  ~         TINT=1E-4
            ICM1=IC-1
            IF (IR.GT.1) GO TO 31
  ~ C       RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
    C       HH IS AGE STEP  -  KK IS TIME STEP
            READ(5,101) HH,KK
  ~   101   FORMAT(2F10.6)
    C       PLTPER IS PLOT INTERVAL  -  PRTPER IS PRINT INTERVAL
            READ(5,101) PLTPER,PRTPER
  ~ C       NDE IS THE NUMBER OF FIRST ORDER ODE'S AFTER TRANSFORMATION
    C       IF NDE > 10 CHANGE REAL DIMENSION IN DATA AND $STEP
    C       NN IS THE NUMBER OF INTEGRATION STEPS IN KK FOR SOLVING THE ODES
  ~         READ(5,103) NDE, NN
      103   FORMAT(2I4)
    C       PUT INITIAL VALUES OF ODES ONE TO A CARD
  ~         DO 97 I=1,NDE
                READ(5,107) DLY(I)
  •   107       FORMAT(F10.4)
  ~    97       CONTINUE
  • C         RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
            PR=PR+KK
  ~         T=0.0
    C         ##################################################################
    C                         READ DEATH RATES
  ~ C                         READ POPULATION DENSITY - OBTAINED FROM SPLINE
    C         ##################################################################
    C         RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
  ~         READ(5,100) (P(I),I=1,IC)
      100   FORMAT(5F10.6,30X)
            READ(5,104) (P(I),I=1,IC)
  ~   104   FORMAT(10F8.0)
    C       RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
            SUM=0.0
  ~         DO 99 I=1,IC
                SUM=SUM+P(I)
     99     CONTINUE
  ~         DO 98 I=1,IC
                P(I)=P(I)*22000000./SUM
            X(I)=P(I)
  ~    98   CONTINUE
    C       PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
            IF(IFORM.NE.1) GOTO 61
  ~.        IPG=0
            WRITE(6,109)
      109   FORMAT('1','POPULATION FOR AGES  1   TO 20   IS')
  ~         WRITE (6,102) (P(I),I=1,20)
  •   102   FORMAT(5F10.0/)
            WRITE(6,110)
  ~   110   FORMAT(' ','AT BEGINNING OF YEAR            0, AND')
```

```
  111     FORMAT(                                              )
   61     CONTINUE
  C       PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
          DO 70 I=1,IC
               I1=FLOAT(I)
               D(I)=(1-EXP(-I1*I1/600.))/(1+((I1-25.5)**2)/4.)
   70          CONTINUE
          YOUNG=0
          F1(1)=EXP(-(.125*HH*(YOUNG-24.5))**2)
               SUM=F1(1)*P(1)
          DO 14 I=2,IC
               F1(I)=EXP(-(.125*HH*(I+YOUNG-25.5))**2)
               SUM=SUM+F1(I)*P(I)
               D5(I)=.5*(D(I)+D(I-1))
   14          CONTINUE
          DQ5=.5*(3.*D(1)-D(2))
          DT5=.5*(3.*D(IC)-D(IC-1))
          SUM=P(1)/SUM
          DO 15 I=1,IC
               F1(I)=F1(I)*SUM
   15          CONTINUE
          PA=KK/HH
          PA1=0.0
          P2=0.0
          FA=1
          GO TO 33
   31     CONTINUE
          PR=PR+KK
  C       ****************************************************************
  C                             UPDATE COEFFICIENTS OF FERTILITY EQUATION
  C       ****************************************************************
          PA1=DLY(2)
          P2=DLY(2)*DLY(2)*PA
          FA=EXP(-DLY(4))
  C       PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
          IF (PRTPER.GT.PR.OR.IFORM.NE.1) GOTO 62
          WRITE(6,106) T
  106     FORMAT(' AT TIME',F6.2)
          WRITE(6,112) DLY(1)
  112     FORMAT('0',,'DELAY IN PEAK OF FERTILITY CURVE IS  ',F9.6)
   62     CONTINUE
  C       PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
  C       ****************************************************************
  C                             MULTIPLY FERTILITY BY INTEGRATING FACTOR
  C       ****************************************************************
   33     CONTINUE
          SUM=0.0
          DO 42 I=1,IC
               SUM=SUM+F1(I)
               F(I)=FA*F1(I)
   42          CONTINUE
  C       PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
          IF (PRTPER.GT.PR.OR.IFORM.NE.1) GOTO 63
          WRITE(6,113) SUM
  113     FORMAT(' ',' INTEGRAL OF NORMALIZED FERTILITY IS         ',F9.6)
          WRITE(6,114) FA
  114     FORMAT(' ','       FRACTION OF INITIAL FERTILITY IS
         2,F9.6)
          WRITE(6,115)
  115     FORMAT('0','VALUES OF FERTILITY CURVE FOR AGES  21 TO 30   ARE')
          WRITE(6,108) (F(I), I=21,30)
  108     FORMAT(5F10.6/)
   63     CONTINUE
  C       PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
  C       ****************************************************************
```

```
              TINT=TINT+KK*5
              IF (HH.GT.TINT) GOTO 45
              SUM=0.
              SUM1=0.
              DO 44 I=1,ICM1
                    J=IC-I
                    SUM=SUM+P(J+1)*F(J+1)
                    P(J+1)=P(J)*(1-R(J))
                    SUM1=SUM1+P(J+1)
                    X(J+1)=P(J+1)
     44           CONTINUE
              SUM=SUM+P(1)*F(1)
              P(1)=SUM
              X(1)=P(1)
              SUM1=SUM1+P(1)
              TINT=-1E-4
     45       CONTINUE
     C        PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
              IF (PRTPER.GT.PR.OR.IFORM.NE.1) GOTO 64
              WRITE(6,116) SUM1
     116      FORMAT('0','TOTAL POPULATION IS   ',F10.0)
              PR=+1E-5
              WRITE(6,117)
     117      FORMAT('0','*************************************************
         I*********************')
     64       CONTINUE
     C        PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
     C        *****************************************************************
     C                             INCREMENT FERTILITY BY ONE YEAR
     C        *****************************************************************
              DO 34 I=1,IC
                    IF (1E-20.GT.F1(I)) F1(I)=0.0
                    US(I)=Q(I)*F1(I)
     34           CONTINUE
              UQ=2.*US(1)-US(2)
              UT=2.*US(IC)-US(IC-1)
              F1(IC)=F1(IC)-.5*PA*((PA1*(UT-US(IC-1)))-P2*(DT5*UT+D5(IC)*US
         2(IC-1)-(DT5+D5(IC))*US(IC)))
              F1(1)=F1(1)-.5*PA*((PA1*(US(2)-UQ))-P2*(D5(2)*US(2)+DQ5*UQ-(D5
         2(2)+DQ5)*US(1)))
              DO 35 I=2,ICM1
                    F1(I)=F1(I)-.5*PA*((PA1*(US(I+1)-US(I-1)))-P2*(D5(I+1)*US
         2(I+1)+D5(I)*US(I-1)-(D5(I+1)+D5(I))*US(I)))
     35           CONTINUE
     C        *****************************************************************
     C                      SOLVE ODES FOR DELAY PARAMETERS AND INCREMENT TIME
     C        *****************************************************************
              DO 36 I=1,NDE
                    IV(I)=DLY(I)
     36           CONTINUE
              CALL $RUN(IV,DLY,T,KK,NDE,NN)
              PL=PL+KK
              IF (PLTPER.GT.PL) GOTO 31
     C        PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
              IF (IFORM.NE.1) GOTO 65
              IPG=IPG+1
              IF (IPG.NE.2) GOTO 79
              IPG=0
              WRITE(6,121)
     121      FORMAT('1')
     79       CONTINUE
              WRITE(6,118)
     118      FORMAT(' ','POPULATION FOR AGES  1 TO 20   IS')
                    WRITE(6,102) (P(I), I=1,20)
```

```
117    FORMAT('  AT BEGINNING OF YEAR',I5,'AND',
       WRITE(6,120)
  120  FORMAT(' ','------------------------------------')
   65  CONTINUE
C      PPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPPP
       RETURN
       END
       SUBROUTINE $RUN(IV,Z,T,KK,NDE,NN)
C      ****************************************************************
C      THIS SUBROUTINE GIVES A SIMULTANEOUS SIMULATION OF VALUES
C      Z=(IA(T),A(T),A'(T),IB(T),B(T)) AT  T+H  FROM INITIAL VALUES  IV
C      OF THE SAME VARIABLES AT T.  FOR EACH DIFFERENT SIMULATION THE
C      SUBROUTINE  $DE  MUST BE REWRITTEN TO EVALUATE THE DERIVATIVES
C      D(I)  AS A FUNCTION OF TIME  T  AND SOLUTION VALUES  Y(I)
C      ****************************************************************
       REAL IV(NDE),Z(NDE),T,T2,KK,K1
       K1=KK/FLOAT(NN)
       T2=T
       T=T+KK
   28  CONTINUE
       CALL $STEP(IV,Z,T2,K1,NDE)
       T2=T2+K1
       DO 29 I=1,NDE
            IV(I)=Z(I)
   29       CONTINUE
       IF (T.GT.T2+K1/2) GOTO 28
       RETURN
       END
       SUBROUTINE $STEP(IV,Z,T,KK,NDE)
       REAL IV(NDE),Z(NDE),Z1(10),D(10),D1(10),KK
       CALL $DE(T,IV,NDE,D)
       DO 11 I=1,NDE
            Z1(I)=IV(I)+KK*D(I)
   11       CONTINUE
       TT=T+KK
       CALL $DE(TT,Z1,NDE,D1)
       DO 12 I=1,NDE
            Z(I)=IV(I)+.5*KK*(D(I)+D1(I))
   12       CONTINUE
       RETURN
       END
       SUBROUTINE $DE(T,Y,NDE,D)
       REAL Y(NDE),D(NDE),KK
       D(1)=Y(2)
       D(2)=Y(3)
       T1=0.
       IF (T.GT.15.) GOTO 78
       TT1=ABS(T-5.)
       TT2=ABS(T-10.)
       T1=(-SIN(EXP(-TT1)-EXP(-TT2)))/15.
   78  CONTINUE
       D(3)=15.*T1-Y(2)-2.*Y(3)
       D(4)=T1
       D(5)=0.
       RETURN
       END
//GO.SYSIN DD *
EVOLUTION OF POPULATION OVER 50 YEARS:   WITH FERTILITY PROGRESSION
   50 100    2   4           0                    1         3
   30.0330.0  7.0        4.0                  20.0
 1.0         .25
 1.0         1.0
     5     1
  0.0
  0.0
```

.02258	.00577	.00414	.00338	.00299
.00276	.00261	.00247	.00231	.00212
.00197	.00191	.00192	.00198	.00207
.00215	.00219	.00225	.00230	.00237
.00243	.00251	.00259	.00268	.00277
.00288	.00299	.00311	.00325	.0034
.00356	.00373	.00392	.00412	.00435
.00459	.00486	.00515	.00546	.00581
.00618	.00659	.00703	.00751	.00804
.00861	.00923	.00991	.01064	.01145
.01232	.01327	.01430	.01543	.01665
.01798	.01943	.021	.02271	.02457
.02659	.02878	.03118	.03376	.03658
.03964	.04296	.04656	.05046	.0547
.0593	.06427	.06966	.0755	.08181
.08864	.09602	.10399	.11259	.12186
.13185	.1426	.15416	.16657	.17988
.19413	.20937	.22563	.243	.26144
.28099	.30173	.32364	.34666	.371
.39621	.44719	.54826	.72467	1.0

1023102.	1000000.	994230.	990114.	986767.	983817.	981102.	978541.	976124.	973869.
971804.	969890.	968038.	966179.	964266.	962270.	960201.	958098.	955942.	953743.
951483.	949171.	946789.	944337.	941806.	939197.	936492.	933692.	930788.	927763.
924609.	921317.	917880.	914282.	910515.	906554.	902393.	898007.	893382.	888504.
883342.	877883.	872098.	865967.	859464.	852554.	845214.	837413.	829114.	820292.
810900.	800910.	790282.	778981.	766961.	754191.	740631.	726241.	710990.	694843.
677771.	659749.	640761.	620782.	599824.	577882.	554975.	531133.	506403.	480850.
454548.	427593.	400112.	372240.	344136.	315982.	287973.	260322.	233251.	206989.
181765.	157799.	135297.	114440.	95378.	78221.	63036.	49838.	38593.	29215.
21577.	15514.	10833.	7327.	4787.	3011.	1818.	1005.	454.	125.

```
/*
//
```

HASP-II JOB STATISTICS -- 296 CARDS READ -- 0 LINES PRINTED -- 0

Estimation of $\mu(x, s, t)$: this program uses values of the population density to estimate $\mu(x, s, t)$ through differentiation of two cubic spline approximations and integration of a subsequent spline approximation.

As a test problem, a separable economic mobility was chosen

$$\mu(x, s, t) = \alpha(x)\, \beta(s) \quad,$$

$$\alpha(x) = \frac{(x+20)(80-x)}{10000} \quad, \quad \beta(s) = \frac{s+10}{200} \quad.$$

With an initial population density

$$p(x, s, o) = p_0(x, s) \equiv e^{-\frac{x}{200}} \frac{(2s^3 - 405s^2 + 21000s)}{540}$$

the population density without deaths

$$\bar{p}(x, s, t) = [\, e^{\frac{w}{2000000}} p_0(x-t, (s+10)e^{\frac{w}{2000000}} - 10)] \quad,$$

$$w = -\frac{t^3}{3} + t^2(30-x) + t(x-80)(x+20) \quad,$$

evolves, and with a death-rate of

$$r(x,s,t) = (1-.01t)(.0003x+.0006)$$

the density with deaths is

$$p(x,s,t) = \bar{p}(x,s,t)e^{-R} \quad,$$

$$R = (t - \frac{.01t^2}{2})(.0003x+.0006) - .0003(\frac{t^2}{2} - \frac{.01t^3}{6}).$$

```
$JOB    ACCT-NUM,'VERNER',TIME=60,PAGES=20
C
C
C       ****************************************************************
C       THIS PROGRAM IS USED TO ESTIMATE THE ECONOMIC MOBILIITY
C       PARAMETER MU.  AGE- TIME- AND INCOME-SPECIFIC VALUES OF THE
C       POPULATION ARE REQUIRED.
C       FOR A TEST RUN WE ASSUME THAT  4D'7#S,T)=ALPHA(X)#BETA(S).
C       FOR THIS PARTICULAR EXAMPLE THE EIGENVECTOR TECHNIQUE USED
C       TO ESTIMATE  A(T) AND D(X)  IN THE FERTILITY EQUATION COULD
C       HAVE BEEN USED.
C       ****************************************************************
C
1              REAL P(7,11,11),MU(7,11,11),ALPHA(7),BETA(11),X,S,T
2              REAL H(11),A(3,11),COF(4,11)
3              REAL K(2,11),D(11),TOT,W,W1,Y
4              INTEGER DX,DS,DT,XE,SM,TM,XX,SS,TT
5              XE=7
6              SM=11
7              TM=11
8              DX=1
9              DS=1
10             DT=2.
11             E=2.71828182846
C
C
C       ****************************************************************
C       DEFINE FUNCTION ALPHA
C       ****************************************************************
C
12             WRITE (6,112)
13       112   FORMAT('1','   THE ECONOMIC MOBILITY IS  ALPHA(X)#BETA(S) WHERE')
14             WRITE(6,106)
15       106   FORMAT('0')
16             DO 10 IX=1,XE
17             XX=DX*(IX-1)
18             ALPHA(IX)=(XX+20.)*(80.-XX)/10.**4.
C
C       PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRIN
19             WRITE(6,100) XX,ALPHA(IX)
20       100   FORMAT('      ALPHA(',I3,') IS  ',F10.5)
21        10   CONTINUE
22             WRITE(6,106)
C       PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRIN
C
C
C       ****************************************************************
C       DEFINE FUNCTION BETA
C       ****************************************************************
C
23             DO 11 IS=1,SM
24             SS=DS*(IS-1)
25             BETA(IS)=(SS+10.)/200.
C
C       PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRIN
26             WRITE(6,101) SS,BETA(IS)
27       101   FORMAT('      BETA(',I3,') IS  ',F10.5)
28        11   CONTINUE
C       PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRIN
C
C       ****************************************************************
C       FIND ANALYTIC SOLUTION TO THE PDE
```

```
C              DP/DX  +  DP/DT = D/DS(MU*P)  -  R*P
C       ********************************************************************
C
29        DO 12 LT=1,TM
30        T=DT*(LT-1)
31        DO 12 LX=1,XE
32        X=DX*(LX-1)
33        W=(.0003*X +.0006)*(T-.005*T*T)-.0003*(T*T/2-(.005*T**3)/3)
34        W1=E**(-W)
35        W2=T*((X+20.)*(X-80.)+T*((30.-X)+T/3.))*(-1.)
36        DO 12 LS=1,SM
37        SS=DS*(LS-1)
38        S=SS+10.
39        W=S*T**(W2/2000000.)-10.
40        Y=W*(21000.+W*((-405.)+2.*W))
41        W=(W+10.)/S
42        P(LX,LS,LT)=(W*Y/540.)*E**((T-X)/200.)*W1
43   12   CONTINUE
C
C       *********************************************************************
C           SPLINE  P(X,S,T)   AGAINST   X   AND DIFFERENTIATE THE SPLINE
C                MU(X,S,T)   IS   DP/DX
C       *********************************************************************
C
44        DO 20 TX=1,XE
45        X=DX*(TX-1)
46        K(1,TX)=X
47   20   CONTINUE
48        CALL SETUP(XE,A,H,K)
49        DO 21 IS=1,SM
50        DO 21 IT=1,TM
51        DO 22 TX=1,XE
52        K(2,TX)=P(TX,IS,IT)
53   22   CONTINUE
54        CALL SOLVE(A,K,H,XE,D,COF)
55        DO 21 TX=1,XE
56        X=DX*(TX-1)
57        MU(TX,IS,IT)=COF(3,IX)+X*(2.*COF(2,IX)+X*3.*COF(1,IX))
58   21   CONTINUE
C
C       *********************************************************************
C           SPLINE  P(X,S,T)   AGAINST   T   AND DIFFERENTIATE THE SPLINE
C                MU(X,S,T)   IS   DP/DX+DP/DT+R(X,S,T)*P(X,S,T)
C       *********************************************************************
C
59        DO 30 IT=1,TM
60        T=DT*(IT-1)
61        K(1,IT)=T
62   30   CONTINUE
63        CALL SETUP(TM,A,H,K)
64        DO 31 IX=1,XE
65        X=DX*(IX-1)
66        DO 31 IS=1,SM
67        DO 32 IT=1,TM
68        K(2,IT)=P(IX,IS,IT)
69   32   CONTINUE
70        CALL SOLVE(A,K,H,TM,D,COF)
71        DO 31 IT=1,TM
72        T=DT*(IT-1)
73        MU(IX,IS,IT)=MU(IX,IS,IT)+COF(3,IT)+T*(2.*COF(2,IT)+
```

```
                    &T*3.*COF(1,IT))+(1.-.01*T)*(.0003*X+.0006)*P(IX,IS,IT)
  74        31      CONTINUE
            C
            C
            C       ************************************************************
            C            SPLINE  DP/DX+DP/DT+R(X,S,T)*P(X,S,T)    AGAINST  S   AND
            C            INTEGRATE THE SPLINE FROM  S  TO THE MAXIMUM VALUE OF  S.
            C                    MU(X,S,T) IS  END CONDITION PLUS THIS INTEGRAL
            C       ************************************************************
            C
  75                DO 40  IS=1,SM
  76                S=DS*(IS-1)
  77                K(1,IS)=S
  78        40      CONTINUE
  79                CALL SETUP(SM,A,H,K)
  80                DO 41 IX=1,XE
  81                DO 41 IT=1,TM
  82                DO 42 IS=1,SM
  83                K(2,IS)=MU(IX,IS,IT)
  84        42      CONTINUE
  85                CALL SOLVE(A,K,H,SM,0,COF)
  86                MU(IX,SM,IT)=0.
  87                S2=DS*SM
  88                S1=S2-DS
  89                DO 41 IS=2,SM
  90                S1=S1-DS
  91                S2=S2-DS
  92                I=SM+1-IS
  93                    MU(IX,I,IT)=MU(IX,I+1,IT)+(((S2*COF(1,I)/4.+COF(2,I)/3.)*
            &           S2+COF(3,I)/2.)*S2+COF(4,I))*S2-(((S1*COF(1,I)/4.
            &           +COF(2,I)/3.)*S1+COF(3,I)/2.)*S1+COF(4,I))*S1
  94        41      CONTINUE
            C
            C       PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRIN
  95                WRITE(6,110)
  96       110      FORMAT('1',' VALUES ARE % ERROR IN MU, S AND T GIVEN, X ACROSS')
  97                WRITE(6,114)
  98       114      FORMAT('0','      S     T   X=     0          10        20        30
            X    40        50        60')
            C       PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRIN
            C
  99                DO 55 KS=1,SM
 100                WRITE(6,106)
 101                SS=DS*(KS-1)
 102                DO 55 KT=1,TM
 103                TT=DT*(KT-1)
 104                DO 50 KX=1,XE
 105                IF (P(KX,KS,KT).EQ.0.) GOTO 51
 106                MU(KX,KS,KT)=(BETA(SM)*ALPHA(KX)*P(KX,SM,KT)-MU(KX,KS,KT))/
            &P(KX,KS,KT)
 107                GOTO 52
 108        51          MU(KX,KS,KT)=ALPHA(KX)*BETA(KS)
 109        52      CONTINUE
 110                MU(KX,KS,KT)=(MU(KX,KS,KT)/(BETA(KS)*ALPHA(KX))-1)*100.
 111        50      CONTINUE
            C
            C       PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRIN
 112                WRITE(6,113) SS,TT,(MU(KX,KS,KT), KX=1,XE)
 113       113      FORMAT(' ',2I5,'   ',7F9.2)
 114        55      CONTINUE
            C       PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRIN
```

```
115              WRITE(6,105)
116        105   FORMAT('1')
117              STOP
118              END
       C

119              SUBROUTINE SETUP(N,A,H,K)
       C
       C
       C     ************************************************************************
       C     THIS ROUTINE SETS UP A TRIDIAGONAL MATRIX OF THE SPLINE EQUATION
       C     IN A   3 BY N   ARRAY   -   FIRST ROW IS SUPERDIAGONAL
       C                                 SECOND ROW IS DIAGONAL
       C                                 THIRD ROW IS SUBDIAGONAL
       C     AND THEN DECOMPOSES A TO   LU   SO THAT
       C         FIRST ROW IS SUPERDIAGONAL OF U, SECOND ROW IS DIAGONAL OF U
       C       THIRD ROW IS SUBDIAGONAL OF L, DIAGONAL OF L IS UNITY
       C     ************************************************************************
       C
120              REAL A(3,N),H(N),K(2,N)
121              NM1=N-1
122              H(2)=K(1,2)-K(1,1)
123              DO 10 I=2,NM1
124                  H(I+1)=K(1,I+1)-K(1,I)
125                  A(1,I)=H(I+1)/(H(I+1)+H(I))
126                  A(2,I)=2.0
127                  A(3,I)=1-A(1,I)
128        10       CONTINUE
129              A(1,1)=-2.
130              A(2,1)=2.
131              A(3,1)=0.
132              A(1, )=0.
133              A(2, )=2.
134              A(3, )=-2.
135              DO 11 I=2,N
136                  A(3,I)=A(3,I)/A(2,I-1)
137                  A(2,I)=A(2,I)-A(3,I)*A(1,I-1)
138        11       CONTINUE
139              RETURN
140              END
       C

141              SUBROUTINE SOLVE(A,K,H,N,D,COF)
       C
       C
       C     ************************************************************************
       C     THIS ROUTINE CALCULATES THE SECOND ORDER FINITE DIFFERENCES OF
       C     THE SPLINE, AND THEN SOLVES   AM=D,   (BY FORWARD AND BACKWARD
       C     SUBSTITUTION), PLACING M(VECTOR OF SECOND DERIVATIVES) IN D
       C     END POINT CONDITIONS NOW USE THIRD ORDER FINITE DIFFERENCES
       C     TO ESTIMATE THE THIRD ORDER DERIVATIVES AT   XO+H/2   AND  XN-H/2.
       C     ************************************************************************
       C
142              REAL H(N),D(N),A(3,N),K(2,N),COF(4,N)
143              D(2)=(K(2,2)-K(2,1))/H(2)
144              NM1=N-1
145              DO 12 I=2,NM1
146                  D(I+1)=(K(2,I+1)-K(2,I))/H(I+1)
147                  D(I)=6*(D(I+1)-D(I))/(H(I+1)+H(I))
148        12       CONTINUE
149              D(1)  =-2.*(-K(2,1)+K(2,4)+3*(K(2,2)-K(2,3)))/(H(2)*H(2))
150              D(N)=-2.*(-K(2,N)+K(2,N-3)+3.*(K(2,N-1)-K(2,N-2)))/(H(N-1)**2)
```

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```
151          DO 13 I=2,N
152             D(I)=D(I)-A(3,I)*D(I-1)
153    13       CONTINUE
154          D(N)=D(N)/A(2,N)
155          DO 14 I=2,N
156             J=N+1-I
157             D(J)=(D(J)-A(1,J)*D(J+1))/A(2,J)
158    14       CONTINUE
159          CALL POLLY(N,D,K,H,COF)
160          RETURN
161          END
C


162          SUBROUTINE POLLY(N,M,K,H,COF)
C
C
C            ***************************************************************
C            THIS ROUTINE COMPUTES THE COEFFICIENTS OF THE SPLINE POLYNOMIAL
C               ON EACH SUBINTERVAL
C            K  IS THE ARRAY OF DATA POINTS
C            H  IS THE VECTOR OF SUBINTERVAL LENGTHS
C            M  IS THE SOLUTION VECTOR TO THE EQUATION AM=D
C            ***************************************************************
C
163          REAL M(N),K(2,N),H(N),COF(4,N)
164          NM1=N-1
165          DO 11 I=1,NM1
166          COF(1,I)=(M(I+1)-M(I))/(6.*H(I+1))
167          COF(2,I)=(K(1,I+1)*M(I)-K(1,I)*M(I+1))/(2.*H(I+1))
168          DD=M(I+1)*K(1,I)*K(1,I)-M(I)*K(1,I+1)*K(1,I+1)
            &+2.*K(2,I+1)-2.*K(2,I)
169          COF(3,I)=(DD/(2.*H(I+1)))+H(I+1)*(M(I)-M(I+1))/6.
170          DD=M(I)*(K(1,I+1)**3)-M(I+1)*(K(1,I)**3)+6.*K(1,I+1)*K(2,I)
            &-6.*K(1,I)*K(2,I+1)+K(1,I)*M(I+1)*(H(I+1)**2)-K(1,I+1)*M(I)
            &*(H(I+1)**2)
171          COF(4,I)=DD/(H(I+1)*6.)
172    11    CONTINUE
173          DO 16 J=1,4
174    16    COF(J,N)=COF(J,NM1)
175          RETURN
176          END

     $ENTRY
```

THE ECONOMIC MOBILITY IS   ALPHA(X)*BETA(S) WHERE

```
ALPHA(  0) IS      0.16000
ALPHA( 10) IS      0.21000
ALPHA( 20) IS      0.24000
ALPHA( 30) IS      0.25000
ALPHA( 40) IS      0.24000
ALPHA( 50) IS      0.21000
ALPHA( 60) IS      0.16000


BETA(  0) IS       0.05000
BETA( 10) IS       0.10000
BETA( 20) IS       0.15000
BETA( 30) IS       0.20000
BETA( 40) IS       0.25000
BETA( 50) IS       0.30000
BETA( 60) IS       0.35000
BETA( 70) IS       0.40000
BETA( 80) IS       0.45000
BETA( 90) IS       0.50000
BETA(100) IS       0.55000
```

VALUES ARE % ERROR IN MU, S AND T GIVEN, X ACROSS

S	T	X= 0	10	20	30	40	50	60
0	0	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0	2	403.75	359.65	75.85	115.51	170.63	160.18	803.37
0	4	-41.60	18.10	-20.49	27.85	-27.13	-15.82	83.30
0	6	-203.99	-7.42	2.34	18.17	13.93	33.79	99.21
0	8	-133.41	-1.23	16.22	-3.06	6.51	-2.07	100.96
0	10	-131.42	25.24	10.08	-3.89	-11.09	-1.89	107.53
0	12	-125.68	66.76	-4.97	0.09	17.15	-8.16	95.22
0	14	-86.40	18.42	-1.60	5.88	8.73	-15.13	105.93
0	16	-95.43	13.09	6.82	7.75	10.38	-11.17	103.78
0	18	-237.34	27.41	-10.80	-3.91	10.89	-40.15	86.25
0	20	-343.99	18.83	-38.13	17.28	1.92	16.66	283.49
10	0	-0.70	-0.86	-0.28	-0.44	-0.51	-1.00	-1.71
10	2	0.36	0.43	0.08	0.17	0.22	0.15	0.75
10	4	-0.07	0.07	-0.03	0.07	-0.07	-0.05	0.23
10	6	-0.59	-0.02	-0.01	0.05	0.06	0.11	0.30
10	8	-0.44	0.00	0.06	0.01	0.05	-0.03	0.49
10	10	-0.53	0.15	0.06	0.00	-0.06	-0.04	0.74
10	12	-0.65	0.43	-0.03	-0.03	0.13	-0.06	0.75
10	14	-0.51	0.16	-0.01	0.07	0.07	-0.15	0.95
10	16	-0.61	0.18	0.04	0.11	0.13	-0.18	1.10
10	18	-1.37	0.28	-0.13	-0.06	0.17	-0.53	1.15
10	20	-1.89	0.24	-0.37	0.30	-0.02	0.29	3.78
20	0	-0.39	-0.37	-0.15	-0.22	-0.25	-0.45	-0.71
20	2	0.13	0.15	0.02	0.05	0.07	0.04	0.27
20	4	-0.03	0.04	-0.01	0.03	-0.02	-0.00	0.13
20	6	-0.26	-0.01	-0.01	0.01	0.03	0.03	0.13
20	8	-0.19	0.01	0.02	0.01	0.01	-0.02	0.24
20	10	-0.22	0.07	0.02	0.01	-0.02	-0.02	0.34
20	12	-0.33	0.15	-0.02	-0.02	0.05	-0.02	0.33
20	14	-0.28	0.09	-0.01	0.02	0.02	-0.06	0.41
20	16	-0.32	0.11	-0.00	0.05	0.07	-0.11	0.51
20	18	-0.64	0.13	-0.06	-0.02	0.08	-0.23	0.62
20	20	-0.79	0.14	-0.08	0.14	-0.06	0.12	1.60
30	0	-0.24	-0.25	-0.09	-0.14	-0.18	-0.29	-0.43
30	2	0.05	0.07	-0.01	0.02	0.04	0.01	0.15
30	4	-0.02	0.03	-0.01	0.02	-0.00	0.01	0.11
30	6	-0.17	0.00	-0.01	0.00	0.01	0.01	0.08
30	8	-0.12	0.01	0.01	0.02	0.00	-0.02	0.16
30	10	-0.12	0.05	0.02	0.01	-0.01	-0.02	0.24
30	12	-0.25	0.09	-0.02	-0.02	0.03	-0.01	0.22
30	14	-0.22	0.06	-0.00	0.01	0.00	-0.04	0.26
30	16	-0.24	0.09	-0.01	0.03	0.05	-0.09	0.36
30	18	-0.45	0.10	-0.04	-0.01	0.07	-0.14	0.46
30	20	-0.54	0.08	0.01	0.09	-0.09	0.06	1.03
40	0	-0.16	-0.20	-0.06	-0.11	-0.14	-0.22	-0.30
40	2	0.00	0.05	-0.01	0.00	0.03	-0.00	0.10
40	4	-0.01	0.02	-0.01	0.02	0.01	0.00	0.11

40	6	-0.13	0.01	-0.01	0.00	0.00	-0.00	0.06
40	8	-0.03	0.01	0.00	0.01	-0.00	-0.02	0.11
40	10	-0.03	0.03	0.01	0.02	-0.01	-0.02	0.21
40	12	-0.22	0.07	-0.02	-0.02	0.02	-0.00	0.16
40	14	-0.19	0.04	0.00	0.01	-0.01	-0.03	0.20
40	16	-0.22	0.08	-0.01	0.03	0.04	-0.08	0.30
40	18	-0.37	0.09	-0.04	0.00	0.07	-0.11	0.39
40	20	-0.46	0.06	0.06	0.07	-0.11	0.03	0.79
50	0	-0.14	-0.17	-0.05	-0.11	-0.09	-0.19	-0.23
50	2	-0.02	0.04	-0.02	0.01	0.01	-0.01	0.07
50	4	0.01	0.02	-0.01	0.01	0.00	0.01	0.10
50	6	-0.11	0.00	-0.01	-0.01	0.00	-0.01	0.05
50	8	-0.06	0.01	-0.00	0.01	-0.01	-0.03	0.10
50	10	-0.06	0.02	0.01	0.02	-0.00	-0.01	0.18
50	12	-0.22	0.05	-0.01	-0.02	0.02	0.01	0.13
50	14	-0.18	0.05	-0.00	0.00	-0.01	-0.02	0.16
50	16	-0.22	0.08	-0.02	0.02	0.04	-0.08	0.26
50	18	-0.36	0.08	-0.04	0.01	0.05	-0.09	0.37
50	20	-0.39	0.05	0.07	0.06	-0.11	0.03	0.62
60	0	-0.12	-0.11	-0.03	-0.11	-0.05	-0.16	-0.17
60	2	-0.02	0.02	-0.03	0.01	-0.01	-0.02	0.05
60	4	0.01	0.02	-0.01	0.01	-0.00	0.03	0.08
60	6	-0.09	0.00	-0.01	-0.01	0.01	-0.01	0.03
60	8	-0.05	0.01	-0.01	0.02	-0.01	-0.04	0.09
60	10	-0.05	0.02	0.02	0.03	-0.00	-0.02	0.16
60	12	-0.23	0.04	0.00	-0.01	0.01	0.01	0.12
60	14	-0.19	0.05	-0.02	-0.00	-0.01	0.00	0.14
60	16	-0.23	0.08	-0.02	0.02	0.04	-0.07	0.23
60	18	-0.37	0.08	-0.02	0.01	0.04	-0.09	0.36
60	20	-0.35	0.05	0.07	0.08	-0.10	0.05	0.51
70	0	-0.10	-0.09	-0.03	-0.07	-0.02	-0.14	-0.12
70	2	-0.01	0.02	-0.01	0.01	-0.01	-0.00	0.04
70	4	-0.01	0.01	-0.01	-0.00	-0.00	0.02	0.05
70	6	-0.07	0.00	-0.01	-0.00	0.01	-0.02	0.04
70	8	-0.05	0.01	-0.01	0.01	-0.02	-0.03	0.07
70	10	-0.09	0.03	-0.00	0.02	0.00	-0.02	0.12
70	12	-0.21	0.04	0.00	0.00	0.01	0.00	0.13
70	14	-0.21	0.04	-0.01	-0.01	-0.00	-0.01	0.15
70	16	-0.27	0.08	-0.03	0.02	0.03	-0.07	0.22
70	18	-0.37	0.10	-0.02	0.01	0.03	-0.08	0.33
70	20	-0.34	0.08	0.05	0.05	-0.06	0.03	0.48
80	0	-0.10	-0.09	-0.01	-0.04	-0.00	-0.12	-0.08
80	2	-0.00	0.03	0.00	0.01	-0.02	0.01	0.04
80	4	-0.02	0.01	-0.01	-0.00	0.01	0.01	0.02
80	6	-0.04	0.00	-0.01	-0.00	0.00	-0.03	0.05
80	8	-0.06	0.01	0.00	-0.00	-0.02	-0.01	0.06
80	10	-0.13	0.02	-0.02	0.01	0.01	-0.01	0.09
80	12	-0.19	0.05	-0.00	0.00	0.01	-0.01	0.14
80	14	-0.22	0.04	-0.00	-0.01	0.01	-0.02	0.15
80	16	-0.31	0.08	-0.03	0.03	0.02	-0.05	0.19
80	18	-0.38	0.11	-0.02	0.01	0.00	-0.07	0.27
80	20	-0.37	0.09	0.01	-0.02	-0.01	-0.01	0.44

90	0	-0.06	-0.05	0.00	-0.02	-0.01	-0.07	-0.06
90	2	-0.00	0.02	-0.00	-0.00	-0.01	-0.01	0.01
90	4	-0.01	0.01	-0.01	-0.00	0.00	0.02	0.02
90	6	-0.03	0.01	-0.01	0.00	0.00	-0.02	0.04
90	8	-0.03	0.01	-0.01	-0.01	-0.02	-0.02	0.02
90	10	-0.10	0.01	0.00	0.01	0.01	0.00	0.03
90	12	-0.17	0.05	0.00	0.00	-0.00	-0.01	0.09
90	14	-0.19	0.04	-0.01	-0.01	0.00	-0.01	0.10
90	16	-0.25	0.06	-0.02	0.02	0.02	-0.03	0.12
90	18	-0.31	0.09	-0.02	0.01	-0.01	-0.05	0.18
90	20	-0.35	0.03	-0.01	-0.01	-0.01	-0.01	0.23
100	0	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
100	2	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
100	4	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
100	6	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
100	8	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
100	10	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
100	12	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
100	14	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
100	16	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
100	18	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00
100	20	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00

CORE USAGE   OBJECT CODE=   10216 BYTES,ARRAY AREA=   7332 BYTES,TOTAL AREA

DIAGNOSTICS   NUMBER OF ERRORS=   0, NUMBER OF WARNINGS=   0, NUMBE

COMPILE TIME=   4.78 SEC,EXECUTION TIME=   41.80 SEC,   QUEEN'S WATFOR VERSIO

COST FOR THIS PROGRAM IS $ 3.21   RUN IN HSC   MAR   26, 1974

Program III

Estimation of the parameters $a(t)$ and $d(x)$ from values of the normalized fertility. The normalized fertility might be estimated from the fertility curve and the parameter $b(t)$ which represents the family size.

Here we chose as a test case:

$$d(x) = \frac{x}{100} \qquad x = 0, 10, \ldots, 50$$

$$a(t) = 1 + .4\sin\left(\frac{t}{6}\right) \qquad t = 0, 2, \ldots, 20$$

```
$JOB    ACCT-NUM,'VERNER',TIME=60
        C
        C
        C       ***********************************************************
        C       THIS PROGRAM ESTIMATES THE PARAMETERS  A(T)  AND  D(X) FOR THE
        C       FERTILITY DIFFERENTIAL EQUATION.  DATA IN THE FORM OF VALUES
        C       OF THE NORMALIZED FERTILITY ARE REQUIRED, AND THESE MAY BE
        C       ESTIMATED FROM VALUES OF THE FERTILITY CURVE AND THE PARAMETER
        C       B(T) - THE FAMILY SIZE.
        C       TO TEST THE PROGRAM A CLOSED FORM SOLUTION FOR A SPECIAL CASE
        C       IS USED.
        C       ***********************************************************
        C
   1            COMMON K,A,H,COF,D
   2            REAL FBAR(51,11),FBARST(51,11),K(2,51),G(51,11),GT(11,51),ID(51)
   3            REAL AT(11),D(51),PROD(11,11)
   4            REAL A(3,51),H(51),COF(4,51)
   5            REAL NU(51)
        C       ***********************************************************
        C       OBSERVE THAT DYNAMIC PARAMETERS ARE BEING USED WITH COMMON AND
        C       AN ERROR MAY OCCUR AS A RESULT OF MIXED INDEXING - INSURE THAT
        C       COLUMNS ARE COMPLETELY FILLED ON USE OF A DYNAMIC INDEX
        C       ***********************************************************
   6            NT=11
   7            NX=51
   8            CALL ANAL(NX,NT,FBAR,AT,D)
   9            D2=D(2)
  10            AT2=AT(2)
  11            CALL OBS(NX,NT,FBAR,FBARST,G)
  12            CALL TRANS(NX,NT,G,GT)
  13            CALL MULT(11,51,GT,G,PROD)
  14            CALL EIGEN(PROD,11,AT,ID)
        C       ***********************************************************
        C       TO EXHIBIT ERRORS IN THIS APPROACH WE CALCULATE A(1) EXACTLY
        C       AND MULTIPLY THE OTHER COMPONENTS BY THE APPROPRIATE FACTOR.
        C       ***********************************************************
  15            AT2=AT2/AT(2)
  16            DO 82 IT=1,NT
  17                AT(IT)=AT(IT)*AT2
  18     82         CONTINUE
  19            DO 85 IX=1,NX
  20                SUM=AT(1)*G(IX,1)
  21                SUM1=AT(1)*AT(1)
  22                DO 84 IT=2,NT
  23                    SUM=SUM+AT(IT)*G(IX,IT)
  24                    SUM1=SUM1+AT(IT)*AT(IT)
  25     84             CONTINUE
  26                D(IX)=SUM/SUM1
  27     85         CONTINUE
        C
        C       PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRIN
  28            WRITE(6,114)
  29    114     FORMAT('0',' A(T)   OBTAINED FROM  11 BY 11   MATRIX IS')
  30            WRITE(6,104) (AT(I),I=1,11)
  31    104     FORMAT('0',5F12.6)
  32            WRITE(6,115)
  33    115     FORMAT('0',' D(X)   OBTAINED AS  (G(X,T),A(T))/(A(T),A(T))')
  34            WRITE(6,104) (D(I),I=1,51)
        C       PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRIN
        C
  35            STOP
```

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```
36          END
    C
    C

37          SUBROUTINE ANAL(NX,NT,FBAR,AT,D)
38          REAL FBAR(NX,NT),AT(NT),D(NX)
    C
    C
    C       ***********************************************************************
    C       SUPPOSE   D(X)=X/100
    C       SUPPOSE   A(T)=1.+.4SIN(T/6.)
    C       SUPPOSE FOBAR(X)=(1-COS(X/8))*X/100
    C       THEN THIS SUBROUTINE COMPUTES THE CORRESPONDING ANALYTIC SOLUTION
    C       TO THE FERTILITY PDE
    C            D(FBAR)/DT = -A(T)*D(D(X)*FBAR)/DX
    C       IF   A(T) < 0   THERE IS AN ADVANCE IN THE FERTILITY
    C       IF   A(T) > 0   THERE IS A DELAY IN THE FERTILITY
    C       ***********************************************************************
    C
39          DT=2.
40          DO 8 IT=1,NT
41              T=DT*(IT-1)
42              AT(IT)=1+.4*SIN(T/6)
43              W=(2.4+T-2.4*COS(T/6.))/400.
44              W=EXP(-W)
45              DO 8 IX=1,NX
46                  X=FLOAT(IX-1)
47                  D(IX)=X/400.
48                  FBAR(IX,IT)=.01*X*W*W*(1-COS(X*W/8.))
49      8           CONTINUE
    C
    C       PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRIN
50          WRITE(6,116)
51      116 FORMAT('1',' A(T)  FROM THE CLOSED FORM IS')
52          WRITE(6,104) (AT(I),I=1,11)
53      104 FORMAT('0',5F12.6)
54          WRITE(6,117)
55      117 FORMAT('0','  D(X)   FROM THE CLOSED FORM IS')
56          WRITE(6,104) (D(I),I=1,51)
57          WRITE(6,105)
58      105 FORMAT('0')
    C       PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRIN
    C
59          RETURN
60          END
    C

61          SUBROUTINE OBS(NX,NT,FBAR,FBARST,G)
    C       ***********************************************************************
    C       THIS ROUTINE COMPUTES THE 'OBSERVATIONS' OF THE FORM
    C       INTEGRAL FROM 0 TO X OF -D/DT(FBAR(X,T)) ALL DIVIDED BY FBAR(X,T)
    C        THE ARRAY G STORES THE OBSERVATIONS
    C       ***********************************************************************
62          COMMON K,A,H,COF,D
63          REAL K(2,51),A(3,51),H(51),COF(4,51),D(51)
64          REAL FBAR(NX,NT),FBARST(NX,NT),G(NX,NT)
65          DT=2.
66          DO 5 I=1,NT
67      5       K(1,I)=DT*(I-1.)
68          CALL SETUP(NT,A,H,K)
69          DO 25 IX=1,NX
```

- 93 -

```
70              DO 15 IT=1,NT
71      15          K(2,IT)=FBAR(IX,IT)
72              CALL SOLVE(A,K,H,NT,D,COF)
73              DO 24 J=1,NT
74                  T=DT*(J-1.)
75                  FBARST(IX,J)=3.*COF(1,J)*T*T+2.*COF(2,J)*T+COF(3,J)
76      24          CONTINUE
77      25      CONTINUE
78          DO 35 I=1,NX
79              K(1,I)=I-1.
80      35          CONTINUE
81          CALL SETUP(NX,A,H,K)
82          DO 55 IT=1,NT
83              DO 45 IX=1,NX
84      45          K(2,IX)=FBARST(IX,IT)
85              CALL SOLVE(A,K,H,NX,D,COF)
86              G(1,IT)=0.
87              X=0.
88              NXM1=NX-1
89              DD=0.
90              DO 55 J=1,NXM1
91                  DD=DD-X*(COF(4,J)+X*(COF(3,J)/2.+X*(COF(2,J)/3.+
        2           X*COF(1,J)/4.)))
92                  X=X+1.
93                  DD=DD+X*(COF(4,J)+X*(COF(3,J)/2.+X*(COF(2,J)/3.+
        2           X*COF(1,J)/4.)))
94                  G(J+1,IT)=-DD/FBAR(J+1,IT)
95      55          CONTINUE
96          RETURN
97          END
        C

98          SUBROUTINE SETUP(N,A,H,K)
        C
        C
        C      ***************************************************************
        C      THIS ROUTINE SETS UP A TRIDIAGONAL MATRIX A OF THE SPLINE EQUATION
        C      IN A  3 BY N  ARRAY  -   FIRST ROW IS SUPERDIAGONAL
        C                                SECOND ROW IS DIAGONAL
        C                                THIRD ROW IS SUBDIAGONAL
        C      AND THEN DECOMPOSES A TO  LU  SO THAT
        C         FIRST ROW IS SUPERDIAGONAL OF U, SECOND ROW IS DIAGONAL OF U
        C         THIRD ROW IS SUBDIAGONAL OF L, DIAGONAL OF L IS UNITY
        C      ***************************************************************
        C
99          REAL A(3,N),H(N),K(2,N)
100         NM1=N-1
101         H(2)=K(1,2)-K(1,1)
102         DO 10 I=2,NM1
103             H(I+1)=K(1,I+1)-K(1,I)
104             A(1,I)=H(I+1)/(H(I+1)+H(I))
105             A(2,I)=2.0
106             A(3,I)=1-A(1,I)
107     10          CONTINUE
108         A(1,1)=-2.
109         A(2,1)=2.
110         A(3,1)=0.
111         A(1,N)=0.
112         A(2,N)=2.
113         A(3,N)=-2.
114         DO 11 I=2,N
```

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```
115              A(3,I)=A(3,I)/A(2,I-1)
116              A(2,I)=A(2,I)-A(3,I)*A(1,I-1)
117     11    CONTINUE
118           RETURN
119           END
      C

120           SUBROUTINE SOLVE(A,K,H,N,D,COF)
      C
      C
      C      ***************************************************************
      C      THIS ROUTINE CALCULATES THE SECOND ORDER FINITE DIFFERENCES OF THE
      C      SPLINE, AND THEN SOLVES   AM=D  , (BY FORWARD AND BACKWARD
      C      SUBSTITUTION), PLACING M(VECTOR OF SECOND DERIVATIVES) IN D
      C      END POINT CONDITIONS NOW USE THIRD ORDER FINITE DIFFERENCES
      C      TO ESTIMATE THIRD ORDER DERIVATIVES AT X0+H/2 AND XN-H/2
      C      ***************************************************************
      C
121           REAL H(N),D(N),A(3,N),K(2,N),COF(4,N)
122           D(2)=(K(2,2)-K(2,1))/H(2)
123           NM1=N-1
124           DO 12 I=2,NM1
125              D(I+1)=(K(2,I+1)-K(2,I))/H(I+1)
126              D(I)=6*(D(I+1)-D(I))/(H(I+1)+H(I))
127     12    CONTINUE
128           D(1)  =-2.*(-K(2,1)+K(2,4)+3*(K(2,2)-K(2,3)))/(H(2)*H(2))
129           D(N)=-2.*(-K(2,N)+K(2,N-3)+3.*(K(2,N-1)-K(2,N-2)))/(H(N-1)**2)
130           DO 13 I=2,N
131              D(I)=D(I)-A(3,I)*D(I-1)
132     13    CONTINUE
133           D(N)=D(N)/A(2,N)
134           DO 14 I=2,N
135              J=N+1-I
136              D(J)=(D(J)-A(1,J)*D(J+1))/A(2,J)
137     14    CONTINUE
138           CALL POLLY(N,D,K,H,COF)
139           RETURN
140           END
      C

141           SUBROUTINE POLLY(N,M,K,H,COF)
      C
      C      ***************************************************************
      C      THIS ROUTINE COMPUTES THE COEFFICIENTS OF THE SPLINE POLYNOMIAL
      C        ON EACH SUBINTERVAL
      C      K IS THE ARRAY OF DATA POINTS
      C      H IS THE VECTOR OF SUBINTERVAL LENGTHS
      C      M IS THE SOLUTION VECTOR TO THE EQUATION AM=D
      C      ***************************************************************
      C
142           REAL M(N),K(2,N),H(N),COF(4,N)
143           NM1=N-1
144           DO 11 I=1,NM1
145           COF(1,I)=(M(I+1)-M(I))/(6.*H(I+1))
146           COF(2,I)=(K(1,I+1)*M(I)-K(1,I)*M(I+1))/(2.*H(I+1))
147           DD=M(I+1)*K(1,I)*K(1,I)-M(I)*K(1,I+1)*K(1,I+1)
             &+2.*K(2,I+1)-2.*K(2,I)
148           COF(3,I)=(DD/(2.*H(I+1)))+H(I+1)*(M(I)-M(I+1))/6.
149           DD=M(I)*(K(1,I+1)**3)-M(I+1)*(K(1,I)**3)+6.*K(1,I+1)*K(2,I)
             &-6.*K(1,I)*K(2,I+1)+K(1,I)*M(I+1)*(H(I+1)**2)-K(1,I+1)*M(I)*
             &(H(I+1)**2)
```

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```
150         COF(4,I)=DD/(H(I+1)*6.)
151    11   CONTINUE
152         DO 16 J=1,4
153    16   COF(J,N)=COF(J,NM1)
154         RETURN
155            END
     C

156         SUBROUTINE TRANS(NX,NT,G,GT)
     C
     C     **********************************************************************
     C     THIS ROUTINE TRANSPOSES A MATRIX
     C     **********************************************************************
     C
157         REAL G(NX,NT),GT(NT,NX)
158         DO 65 IX=1,NX
159             DO 65 IT=1,NT
160                 GT(IT,IX)=G(IX,IT)
161    65          CONTINUE
162         RETURN
163         END
     C

164         SUBROUTINE MULT(NA,NB,A,B,PROD)
     C
     C     **********************************************************************
     C     THIS ROUTINE MULTIPLIES TWO MATRICES
     C     **********************************************************************
     C
165         REAL        A(NA,NB),B(NB,NA),PROD(NA,NA)
166         DO 67 IX=1,NA
167             DO 67 IT=1,NA
168                 SUM=A(IX,1)*B(1,IT)
169                 DO 66 K=2,NB
170                 SUM=SUM+A(IX,K)*B(K,IT)
171    66              CONTINUE
172                 PROD(IX,IT)=SUM
173    67          CONTINUE
174         RETURN
175         END
     C

176         SUBROUTINE EIGEN(PROD,NA,NU,EV)
     C
     C     **********************************************************************
     C     THIS ROUTINE FINDS THE DOMINANT EIGENVECTOR OF THE   NA BY NA
     C     MATRIX   PROD   AND PLACES IT IN   EV   WHICH IS NORMALIZED BY THE
     C     1-NORM.   ITERATION   RUNS UNTIL ROUNDING ERROR IS DOMINANT.
     C     **********************************************************************
     C
177         REAL PROD(NA,NA),EV(NA),NU(NA)
178         SUM1=0.
179         DO 77 IX=1,NA
180             SUM=PROD(IX,1)
181             DO 76 IT=2,NA
182                 SUM=SUM+PROD(IX,IT)
183    76          CONTINUE
184             NU(IX)=SUM
185             SUM1=SUM1+SUM
186    77      CONTINUE
```

```
187            ERROR=1.
188     89     CONTINUE
189            SUM2=0.
190            DO 87 IX=1,NA
191                 SUM=PROD(IX,1)*NU(1)
192                 DO 86 IT=2,NA
193                      SUM=SUM+PROD(IX,IT)*NU(IT)
194     86           CONTINUE
195                 EV(IX)=SUM/SUM1
196                 SUM2=SUM2+EV(IX)
197     87      CONTINUE
198            SUM1=0.
199            BOUND=.5*ERROR
200            ERROR=0.
201            DO 97 IX=1,NA
202                 SUM=PROD(IX,1)*EV(1)
203                 DO 96 IT=2,NA
204                      SUM=SUM+PROD(IX,IT)*EV(IT)
205     96           CONTINUE
206                 NU(IX)=SUM/SUM2
207                 SUM1=SUM1+NU(IX)
208                 ERROR=ERROR+ABS(EV(IX)-NU(IX))
209     97      CONTINUE
210            IF (ERROR.LT.BOUND) GOTO 89
211            RETURN
212            END

        $ENTRY
```

A(T) FROM THE CLOSED FORM IS

1.000000	1.130877	1.247348	1.336588	1.388775
1.398163	1.363718	1.289234	1.182909	1.056447
0.923773				

D(X) FROM THE CLOSED FORM IS

0.000000	0.002500	0.005000	0.007500	0.010000
0.012500	0.015000	0.017500	0.020000	0.022500
0.025000	0.027500	0.030000	0.032500	0.035000
0.037500	0.040000	0.042500	0.045000	0.047500
0.050000	0.052500	0.055000	0.057500	0.060000
0.062500	0.065000	0.067500	0.070000	0.072500
0.075000	0.077500	0.080000	0.082500	0.085000
0.087500	0.090000	0.092500	0.095000	0.097500
0.100000	0.102500	0.105000	0.107500	0.110000
0.112500	0.115000	0.117500	0.120000	0.122500
0.125000				

A(T) OBTAINED FROM 11 BY 11 MATRIX IS

0.997756	1.130877	1.248731	1.339316	1.390486
1.399143	1.365165	1.290611	1.183847	1.058285
0.921801				

D(X) OBTAINED AS (G(X,T),A(T))/(A(T),A(T))

0.000000	0.002437	0.004990	0.007491	0.009990
0.012487	0.014985	0.017483	0.019980	0.022478
0.024975	0.027473	0.029970	0.032468	0.034965
0.037463	0.039960	0.042457	0.044955	0.047452
0.049950	0.052447	0.054945	0.057442	0.059939
0.062437	0.064934	0.067432	0.069929	0.072426
0.074924	0.077421	0.079918	0.082415	0.084913
0.087410	0.089907	0.092403	0.094899	0.097395

```
   0.099892        0.102386        0.104882        0.107379        0.109874

   0.112368        0.114861        0.117357        0.119841        0.122342

   0.124579
```

CORE USAGE   OBJECT CODE=   12528 BYTES,ARRAY AREA=   12156 BYTES,TOTAL AREA

DIAGNOSTICS   NUMBER OF ERRORS=   0, NUMBER OF WARNINGS=   0, NUMBER

COMPILE TIME=   4.75 SEC,EXECUTION TIME=   20.16 SEC,   QUEEN'S WATFOR VERSION

COST FOR THIS PROGRAM IS $ 1.83   RUN IN HSC   MAR 21, 1974

Program IV

It is expected that available data will be in the form of a histogram: that is, for population the number of individuals between the ages of x and x + h years is known. To generate a density function for this histogram, a fourth order spline approximation routine is used. To investigate the accuracy of the scheme, the histogram is regenerated by integrating the spline constructed.

```
$JOB     ACCT-NUM,VERNER,TIME=60
   C     ****************************************************************
   C     THIS ROUTINE USES A POPULATION HISTOGRAM TO GENERATE A POPULATION
   C     DENSITY FUNCTION USING A FOURTH ORDER SPLINE APPROXIMATION
   C     ****************************************************************
 1       REAL A(303,9),P(100),Q(303),PO
 2       INTEGER I,J,K,TN,TO,TT
 3       INTEGER T3
   C     RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
 4       READ(5,100) TN
 5   100 FORMAT(I3)
 6       READ(5,101) (P(I), I=1,TN)
 7   101 FORMAT(10F8.0)
   C     RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRR
   C     PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINT
 8       WRITE(6,106)
 9   106 FORMAT('1')
10       PRINT, 'THE GIVEN POPULATION IS'
11       PRINT, ' '
12       WRITE(6,102) (P(I),I=1,TN)
13   102 FORMAT(' ',10F8.0)
   C     PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINT
14       TO=0
15       T3=3*(TN-TO+1)
16       CALL ISPLIN(A,TO,TN,T3,P,Q)
17       CALL CHECK(P,Q,T3,TN)
18       P(1)=Q(3)
19       P(2)=Q(5)
20           DO 25 I=3,TN
21           P(I)=Q(3*I-2)
22   25      CONTINUE
   C     PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINT
23       PRINT, ' '
24       PRINT, ' '
25       PRINT, 'THE POPULATION DENSITY   AT AGES   0,1,2,...,99   IS'
26       PRINT, ' '
27       WRITE(6,102) (P(I),I=1,TN)
28       WRITE(6,106)
   C     PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINT
   C     PUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCH
   C     WRITE(7,101) (P(I),I=1,TN)
   C     PUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCHPUNCH
29       STOP
30       END

31       SUBROUTINE CHECK(P,Q,T3,TN)
   C     ****************************************************************
   C     THIS SUBROUTINE REGENERATES THE POPULATION HISTOGRAM FROM THE
   C     DENSITY FUNCTION BY EXACT INTEGRATION OF THE FOURTH ORDER SPLINE.
   C     HENCE THE ERRORS ARE THOSE SUFFERED AS A RESULT OF ROUND-OFF ERROR
   C     ****************************************************************
32       INTEGER T3,TN
33       REAL Q(T3),P(TN)
34       P(1)=(Q(3)+Q(5))/2.-(Q(2)+Q(4))/24.+Q(6)/120
35       P(2)=(Q(7)+Q(5))/2.-(Q(8)+Q(4))/24.+Q(9)/120.
36       DO 26 I=3,TN
37       P(I)=(Q(3*I+1)+Q(3*I-2))/2.-(Q(3*I+2)+Q(3*I-1))/24.+Q(3*I+3)/120
38   26      CONTINUE
   C     PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINT
39       PRINT, ' '
```

```
40            PRINT, ' '
41            PRINT,'THE POPULATION REGENERATED FROM THE FOURTH ORDER SPLINE IS'
42            PRINT, ' '
43            WRITE(6,102) (P(I),I=1,TN)
44      102   FORMAT(' ',10F8.0)
     C         PRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTPRINTF
45            RETURN
46            END

47            SUBROUTINE CREATE(A,TO,TN,T3,P,Q)
     C         ***************************************************************
     C         THIS ROUTINE CREATES THE INTSPLINE MATRIX FOR APPROXIMATING
     C         A DENSITY FUNCTION GIVEN SUBINTERVAL INTEGRALS
     C         THE UNKNOWN VALUES Q(I) ARE THE VALUES  R4(0),R2(0),R(0),R2(1),
     C         R(1),R4(1),R(2),R2(2),R4(2),...,R(I),R2(2),R4(I),...
     C         R(TO),R2(TO),R4(TO).
     C         BOUNDARY CONDITIONS ARE R''(0)=R'''(0)=R(100)=R''(100)=0
     C         ***************************************************************
48            INTEGER I,J,K,TN,TO,TT
49            INTEGER T3
50            REAL A(T3,9),P(TN),Q(T3)
51            DO 30 I=1,13
     C         RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRF
52            READ(5,103) D
53      103   FORMAT(F10.4)
54                READ(5,104) (A(I,J),J=1,9)
55      104   FORMAT(9F8.5)
56            DO 30 J=1,9
57            A(I,J)=A(I,J)/D
58      30        CONTINUE
     C         RRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRRF
59            TT=TN-TO-1
60            DO 31 I=4,TT
61                DO 31 J=1,9
62                    A(2+3*I,J)=A(11,J)
63                    A(3+3*I,J)=A(12,J)
64                    A(4+3*I,J)=A(13,J)
65      31            CONTINUE
66            TT=TT+1
67            READ(5,104) (A(3*TT+2,J),J=1,9)
68            READ(5,104) (A(3*TT+3,J),J=1,9)
69            N1=TO+2
70            DO 32 I=N1,TT
71                Q(3*I-1)=0.
72                Q(3*I)=0.
73                Q(3*I+1)=2*P(1)
74      32        CONTINUE
75            Q(1)=0.
76            Q(2)=0.
77            Q(3)=2*P(TO+1)
78            Q(4)=0.
79            Q(3*TT+2)=0.
80            Q(3*TT+3)=0.
81            RETURN
82            END

83            SUBROUTINE ISPLIN(A,TO,TN,T3,P,Q)
     C         ***************************************************************
     C         THIS ROUTINE SOLVES THE INTEGRAL SPLINE PROBLEM FOR A POPULATION
     C         ***************************************************************
```

```
84          INTEGER TO ,TN,I,J,K ,T1
85          INTEGER T3
86          REAL A(T3,9),P(TN),Q(T3)
87          CALL CREATE(A,TO,TN,T3,P,Q)
88          DO 33 I=1,T3
89               Q(I)=Q(I)/A(I,5)
90          DO 34 J=6,9
91                    A(I,J)=A(I,J)/A(I,5)
92     34        CONTINUE
93          A(I,5)=1
94          DO 33 K=1,4
95               IF((I+K).GT.T3) GOTO 33
96               Q(I+K)=Q(I+K)-A(I+K,5-K)*Q(I)
97               DO 35 J=2,5
98                        A(I+K,4-K+J)=A(I+K,4-K+J)-A(I+K,5-K)*A(I,4+J)
99     35            CONTINUE
100                   A(I+K,5-K)=0
101    33        CONTINUE
C          ***********************************************************
C          BACKSUBSTITUTION
C          ***********************************************************
102         N=3*(TN-TO+1)
103         Q(N-1)=Q(N-1)-Q(N)*A(N-1,6)
104         Q(N-2)=Q(N-2)-(Q(N)*A(N-2,7)+Q(N-1)*A(N-2,6))
105         Q(N-3)=Q(N-3)-(Q(N)*A(N-3,8)+Q(N-1)*A(N-3,7)+Q(N-2)*A(N-3,6))
106         DO 36 I=5,N
107              K=N-I+1
108              DO 36 J=1,4
109                   Q(K)=Q(K)-A(K,5+J)*Q(K+J)
110    36       CONTINUE
111         RETURN
112         END

       $ENTRY
```

THE GIVEN POPULATION IS

```
1023102.1000000.  994230.  990114.  986767.  983817.  981102.  978541.  976124.  973869.
 971804.  969890.  968038.  966179.  964266.  962270.  960201.  958098.  955942.  953743.
 951483.  949171.  946789.  944337.  941806.  939197.  936492.  933692.  930788.  927763.
 924609.  921317.  917880.  914282.  910515.  906554.  902393.  898007.  893382.  888504.
 883342.  877883.  872098.  865967.  859464.  852554.  845214.  837413.  829114.  820292.
 810900.  800910.  790282.  778981.  766961.  754191.  740631.  726241.  710990.  694843.
 677771.  659749.  640761.  620782.  599824.  577882.  554975.  531133.  506403.  480850.
 454548.  427593.  400112.  372240.  344136.  315982.  287973.  260322.  233251.  206989.
 181765.  157799.  135297.  114440.   95378.   78221.   63036.   49838.   38593.   29215.
  21577.   15514.   11833.    7327.    4787.    3011.    1818.    1005.     454.     125.
```

THE POPULATION REGENERATED FROM THE FOURTH ORDER SPLINE IS

```
1023102.1000000.  994230.  990114.  986766.  983816.  981102.  978541.  976124.  973869.
 971804.  969890.  968038.  966178.  964266.  962270.  960201.  958098.  955942.  953743.
 951483.  949171.  946789.  944337.  941806.  939197.  936492.  933692.  930788.  927763.
 924609.  921317.  917880.  914282.  910515.  906554.  902393.  898007.  893382.  888504.
 883342.  877883.  872098.  865967.  859464.  852554.  845214.  837413.  829114.  820292.
 810900.  800910.  790282.  778981.  766961.  754191.  740631.  726241.  710990.  694843.
 677771.  659749.  640761.  620782.  599824.  577882.  554975.  531133.  506403.  480850.
 454548.  427593.  400112.  372240.  344136.  315982.  287973.  260322.  233251.  206989.
 181765.  157799.  135297.  114440.   95378.   78221.   63036.   49838.   38593.   29215.
  21577.   15514.   11833.    7327.    4787.    3011.    1818.    1005.     454.     125.
```

THE POPULATION DENSITY   AT AGES   0,1,2,...,99   IS

```
1038460.1008672.  995297.  992581.  988103.  985351.  982383.  979819.  977299.  974970.
 972804.  970830.  968960.  967114.  965234.  963284.  961243.  959156.  957028.  954851.
 952623.  950337.  947992.  945576.  943085.  940516.  937861.  935108.  932259.  929297.
 926209.  922987.  919625.  916109.  912429.  908568.  904509.  900240.  895735.  890988.
 885972.  880664.  875047.  869092.  862781.  856079.  848958.  841394.  833348.  824794.
 815694.  806008.  795705.  784748.  773094.  760704.  747547.  733577.  718762.  703069.
 686464.  668919.  650419.  630936.  610466.  589017.  566587.  543207.  518911.  493758.
 467817.  441169.  413929.  386229.  358212.  330052.  301936.  274070.  246671.  219966.
 194185.  169555.  146288.  124581.  104599.   86475.   70297.   56107.   43896.   33602.
  25119.   18298.   12960.    8901.    5914.    3786.    2334.    1363.     688.     262.
```

CORE USAGE   OBJECT CODE=   6336 BYTES,ARRAY AREA=   12520 BYTES,TOTAL AREA

DIAGNOSTICS   NUMBER OF ERRORS=   0, NUMBER OF WARNINGS=   0, NUMBER

COMPILE TIME=   2.31 SEC.EXECUTION TIME=   12.33 SEC,   QUEEN'S WATFOR VERSION

COST FOR THIS PROGRAM IS $ 1.19   RUN IN HSC   MAR 19, 1974