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MODELLING SUBMARINE CONTROL SURFACE DEFLECTION DYNAMICS

George D. Watt

Defence Research Establishment Atlantic

Centre de Recherches pour la Défense Atlantique

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Approved by W.C.E. Nethercote
Acting Director / Technology Division

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Abstract

An algorithm is presented for modelling the time response of a lifting surface control system to an arbitrary set of commands. Response dynamics are governed by a linear, second order, ordinary differential equation. This allows the control system response frequency and damping to be modelled, as well as current control surface position and rate of change of position to be matched when a new command is issued. In order not to exceed the rate limit of the control system, yet maintain continuity in the response and its high order derivatives in time, the algorithm reduces response frequency when large deflection changes are required. This approach is compared to a common first order model, and to a second order model having piecewise continuous high order time derivatives but which always models both response frequency and rate limit.

Résumé

On présente un algorithme de modélisation de la réponse temporelle d'un système de commande de surface portante en réponse à un jeu arbitraire de commandes. La dynamique de la réponse est régie par une équation différentielle linéaire ordinaire du second degré. Il est donc possible de modéliser la fréquence de la réponse et l'amortissement du système de commande, et d'adapter la position de la surface de commande et le taux de variation de la position lorsqu'une nouvelle commande est introduite. Afin de ne pas dépasser la limite de variation du taux du système de commande tout en assurant la continuité de la réponse et de ses dérivées temporelles d'ordre supérieur, l'algorithme réduit la réponse en fréquence lorsque des variations importantes de la déflexion sont nécessaires. Cette approche est comparée à un modèle commun du premier degré et à un modèle du deuxième degré dont les dérivées temporelles de degré supérieur sont continues pièce par pièce mais qui modèle toujours la réponse en fréquence et la limite du taux de variation.

Contents

Nomenclature

t time.

t_0 time of issuance of a command.

β phase shift constant determined by the initial condition $\dot{\delta}_0 = \dot{\delta}(t_0)$.

δ the angular position of a control surface; a function of time.

$\dot{\delta}, \ddot{\delta}$ first and second time derivatives of δ.

$\delta_0, \dot{\delta}_0$ initial conditions; deflection and rate of deflection at $t = t_0$.

δ_c the current command; $\delta \rightarrow \delta_c$ as $t \rightarrow \infty$.

$\dot{\delta}_{\mathbf{max}}$ the value of $\dot{\delta}$ at the inflection point in the time segment of the response in which the initial conditions are applied (see Figure 1).

$\dot{\delta}_{\mathrm{RL}}$ control system characteristic, to be specified; a positive number giving the maximum possible control surface deflection rate.

ζ dimensionless control system damping, to be specified; $0 < \zeta < 1$.

ω frequency used in the algorithm time response; $\omega \leq \omega_{\mathbf{max}}$.

$\omega_{\mathbf{max}}$ response frequency of the control system, to be specified.

1 Introduction

This memorandum describes the initial version of an algorithm developed for modelling the control systems used by submarines for control surface deflections. The algorithm is to be used in numerical simulations of the six degree of freedom motion of a maneuvering submarine, and will be a component of the DREA Submarine Simulation Package currently under development. Compromises have been made in developing the algorithm, as discussed below. Increased effort could eliminate many of these compromises, but this awaits more in-house knowledge about the control systems themselves, as well as experience in using the algorithm in the simulation package.

The requirement is to model the dynamics of a control surface responding to a command (or a series of commands) to deflect to a certain angle (or angles) of attack. Two sets of parameters determine the response dynamics. First are the control system characteristics, which may be complex; the algorithm only considers control system damping, response frequency (determined by the control system time constant), and maximum rate of response. Second are the control surface initial conditions (the kinematic states when a new command is given) to which the response should be matched if an unrealistic discontinuity in the deflection time history is to be avoided.

To provide increased flexibility in modelling control system characteristics and to reduce the degree of discontinuity when a new command is issued, a second order model of the control system is chosen over a first order one. The dynamics are modelled with a linear, second order, ordinary differential equation in which two free parameters, system damping and response frequency, are determined by each individual control system. Two initial conditions allow continuity to be maintained in control surface position and deflection rate when the command is issued, even when a previous command is in the midst of being executed. Continuity cannot be maintained in second or higher time derivatives of control surface position.

The maximum rate at which a control surface deflects, its *rate limit*, is also an important characteristic to model. This is particularly true for large deflections where the rate limit dominates the nature of the response while the system response frequency governs only the transients leading up to and down from the maximum rate. However, given the limited number of free parameters in the second order model, enforcing a rate limit means the proper frequency cannot always be modelled (at least not without creating a response whose time derivatives are discontinuous, something which is desirable to avoid in this initial model).

With only two free parameters but three independent characteristics to model, compromises must be made. Following Campbell and Graham,[1] analytical solutions to the governing differential equation are first obtained by treating the damping and frequency parameters as constants. Then, if necessary, the value of the response frequency is set lower than its correct value so that the rate limit is not exceeded. The frequency is set only once, when the command is issued; the frequency does not change between commands.

Thus, the algorithm requires three control system characteristics to be specified: response frequency (equivalent to maximum frequency of response), rate limit (maximum deflection rate), and damping. Control system damping is assumed to be sub-critical and constant.

2 The Second Order Model

Consider the linear, second order, ordinary differential equation:

$$\ddot{\delta} + 2\zeta\omega\dot{\delta} + \omega^2\delta = \omega^2\delta_c \tag{1}$$

where δ is the control surface deflection at any point in time; δ_c is the commanded deflection angle; ζ is the dimensionless control system damping (assumed sub-critical, so $0 < \zeta < 1$); and ω is the response frequency of the control system. The first and second time derivatives of δ are $\dot{\delta}$ and $\ddot{\delta}$ respectively.

A general, exact, analytical solution to equation 1 is:

$$\delta = \delta_c + \alpha e^{-\zeta\omega(t-t_0)} \sin\left[\sqrt{1-\zeta^2}\,\omega(t-t_0) + \beta\right] \tag{2}$$

where t_0 is the time the command δ_c is given, and α and β are unknown constants to be determined by the initial conditions:

$$\delta_0 \equiv \delta(t_0)\,, \qquad \dot{\delta}_0 \equiv \dot{\delta}(t_0)\,. \tag{3}$$

Setting $t = t_0$ and $\delta = \delta_0$ in equation 2 gives $\alpha = -(\delta_c - \delta_0)/\sin\beta$. Thus, the solution and its first two time derivatives can be written:

$$\delta = \delta_c - \frac{\delta_c - \delta_0}{\sin\beta} e^{-\zeta\omega(t-t_0)} \sin\left[\sqrt{1-\zeta^2}\,\omega(t-t_0) + \beta\right] \tag{4a}$$

$$\dot{\delta} = \omega\frac{\delta_c - \delta_0}{\sin\beta} e^{-\zeta\omega(t-t_0)} \sin\left[\sqrt{1-\zeta^2}\,\omega(t-t_0) + \beta - \cos^{-1}\zeta\right] \tag{4b}$$

$$\ddot{\delta} = -\omega^2\frac{\delta_c - \delta_0}{\sin\beta} e^{-\zeta\omega(t-t_0)} \sin\left[\sqrt{1-\zeta^2}\,\omega(t-t_0) + \beta - 2\cos^{-1}\zeta\right] \tag{4c}$$

since $\sqrt{1-\zeta^2} = \sin(\cos^{-1}\zeta)$. Applying the final initial condition (equation 3) to equation 4b maintains continuity in rate:

$$\dot{\delta}_0 = \omega\frac{\delta_c - \delta_0}{\sin\beta} \sin(\beta - \cos^{-1}\zeta) \tag{5}$$

so that:

$$\tan\beta = \frac{\sqrt{1-\zeta^2}}{\zeta - \dot{\delta}_0/\left[\omega(\delta_c - \delta_0)\right]}\,, \qquad 0 \le \beta \le \pi\,. \tag{6}$$

The arctan function is multivalued, so β is determined uniquely by choosing it to be the principal value when $\dot{\delta}_0 = 0$; that is, $0 < \beta < \pi/2$ when $\beta = \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)$. Then, for $\dot{\delta}_0 \ne 0$, by considering how the term $\dot{\delta}_0/\left[\omega(\delta_c - \delta_0)\right]$ varies in equation 6 (for all possible combinations of the parameters), and in order for β to be continuous as the denominator in the RHS of the equation goes through zero, one can show that the range for β needs to be extended as shown.

Figure 1 shows an example of a solution to equation 1, where values for δ_0 and t_0 are not explicitly shown since placement of this curve in δ-t space depends on the initial conditions. For example, when $\delta_0 = \delta_c$, either $\beta = 0$ and the curve must be shifted so that $t_0 = t_a$, or

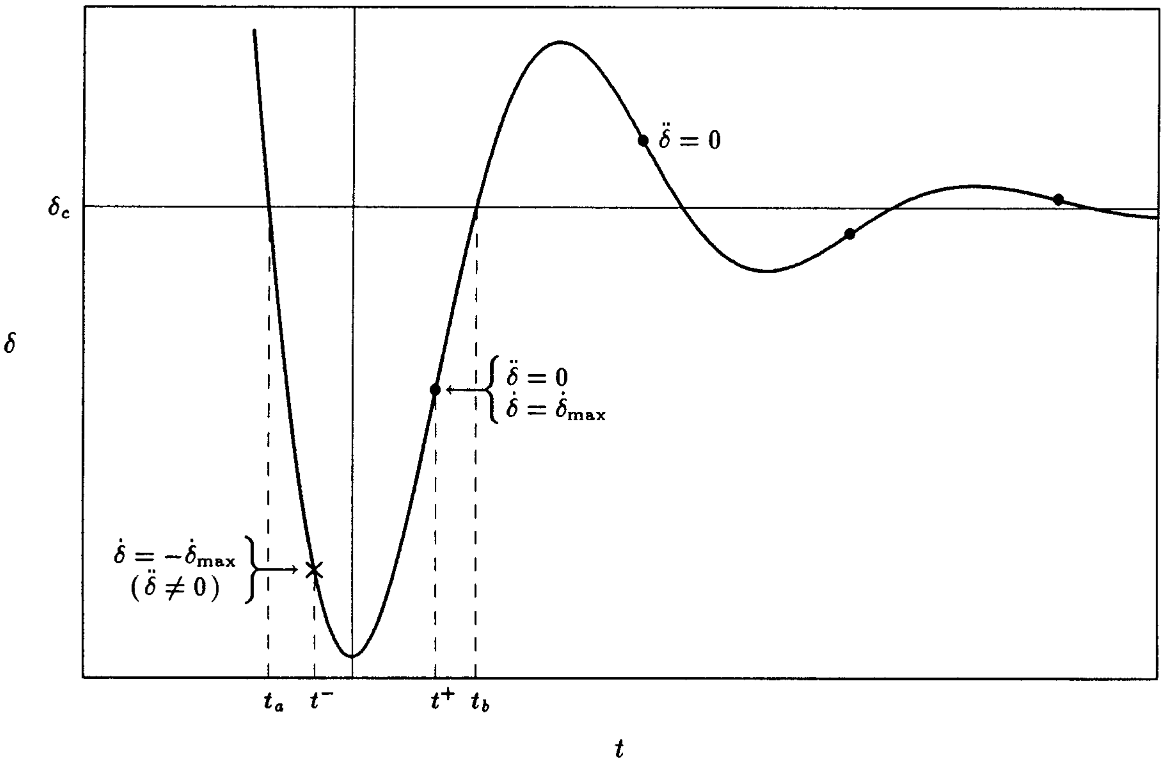


Figure 1 *Control surface time response (equation 4a) to command δ_c, assuming $\delta_c \geq \delta_0$. The command is issued at $t = t_0$, where $t_a \leq t_0 \leq t_b$; the portion of the curve before time t_0 is ignored.*

$\beta = \pi$ and $t_0 = t_b$. For the more general case that $\delta_0 < \delta_c$, t_0 will be between t_a and t_b. For $\delta_0 > \delta_c$, the correct solution is simply a reflection of the curve shown about the line $\delta = \delta_c$. In order to show better the qualitative nature of the solution, an unrealistically low damping value ($\zeta = 0.3$) has been used in the sketch.

Maximum deflection rates occur at $t = t^*$, when $\ddot{\delta}$ is zero:

$$\ddot{\delta} = 0 \quad \Longrightarrow \quad \sqrt{1-\zeta^2}\,\omega(t^* - t_0) + \beta - 2\cos^{-1}\zeta = n\pi; \quad n = 0, \pm 1, \pm 2, \ldots \tag{7}$$

There are an infinite number of values for t^*. However, the local maximum deflection rate of primary interest, $\dot{\delta}_{max}$, is the one immediately following the extreme in δ which occurs between t_a and t_b: that is, at $t^* = t^+$ (see Figure 1). Since β is defined between 0 and π, n must be zero when $t^* = t^+$, so:

$$\sqrt{1-\zeta^2}\,\omega(t^+ - t_0) = 2\cos^{-1}\zeta - \beta\,. \tag{8}$$

Substituting equation 8 into equation 4b gives the value of $\dot{\delta}_{max}$ for a given ω or, alternatively, the value of ω for a specified $\dot{\delta}_{max}$:

$$\omega = \frac{\dot{\delta}_{max}\sin\beta}{(\delta_c - \delta_0)\sqrt{1-\zeta^2}}\, e^{\zeta(2\cos^{-1}\zeta - \beta)/\sqrt{1-\zeta^2}}\,. \tag{9}$$

To evaluate this expression, β must first be obtained. This can be done using equation 6 if ω is known, or by eliminating ω using equation 5 if $\dot{\delta}_{max}$ is known. In the latter case, a trial and error solution of β is required and this is discussed in detail in a later section.

The response to a given command is determined by first assuming $\omega = \omega_{max}$ (the system response frequency), then checking to see if the maximum deflection rate is less than or equal to the rate limit, and then, if it is not, recalculating a solution in which the maximum rate is the rate limit. The following observations can be made about the nature of a solution constructed in this manner; they hold whether δ_0 is greater than or less than δ_c. (Recall from Figure 1 that $t_a \le t_0 \le t_b$ corresponds to $0 \le \beta \le \pi$.)

1) With ω initially set to ω_{max} and with β such that $t_a \le t_0 < t^-$, $|\dot{\delta}_{max}|$ will always be less than the rate limit since $|\dot{\delta}_0| \le$ rate limit, and over this portion of the Figure 1 curve $|\dot{\delta}_0| > |\dot{\delta}_{max}|$; thus, $\omega = \omega_{max}$ is the correct solution.

2) With ω initially set to ω_{max} and β such that $t^+ < t_0 \le t_b$, $|\dot{\delta}_{max}|$ can be greater than the rate limit because it is not part of the solution. In this case, $\omega = \omega_{max}$ is always the correct solution since it can be shown that all subsequent $|\dot{\delta}|$ values will be less than the initial $|\dot{\delta}_0|$ value, which is itself no bigger than the rate limit.

3) Given (1) and (2), exceeding the rate limit is only possible if $t^- < t_0 < t^+$; that is, if $\beta^- < \beta < 2\cos^{-1}\zeta$.

4) β^- can be shown to be purely a function of ζ by eliminating ω from equations 5 and 9 and setting $\dot{\delta}_0 = -\dot{\delta}_{max}$:

$$\sqrt{1-\zeta^2} - \sin(\cos^{-1}\zeta - \beta^-)e^{\zeta(2\cos^{-1}\zeta - \beta^-)/\sqrt{1-\zeta^2}} = 0\,.$$

β^- is solved by trial and error. Figure 2 shows it plotted against ζ.

5) Let:

$$R \equiv \frac{\delta_c - \delta^+}{\delta_c - \delta_0} = \frac{2\zeta\sqrt{1-\zeta^2}}{\sin\beta} e^{-\zeta(2\cos^{-1}\zeta - \beta)/\sqrt{1-\zeta^2}}$$

where $\delta^+ \equiv \delta(t^+)$. Although it may be obvious, one can rigorously show, for $t^- < t_0 < t^+$, that:

$$0 < R < 1 \quad \text{always.} \tag{10}$$

Clearly R is always positive. Further, when $t_0 = t^+$, $R = 1$. As t_0 decreases from t^+, β decreases from $2\cos^{-1}\zeta$. The derivative:

$$\frac{dR}{d\beta} = R\left[\cot(\cos^{-1}\zeta) - \cot\beta\right]$$

shows that R decreases monotonically until $\dot{\delta}_0 = 0$ (ie, $\beta = \cos^{-1}\zeta$). As t_0 decreases further to t^-, R increases monotonically to a local maximum (within the range of interest) at $t = t^-$. Thus, if $R > 1$, it must do so at $t_0 = t^-$: that is, at $\delta_0 = \delta^- \equiv \delta(t^-)$ and $\beta = \beta^-$. However, as shown in Figure 2:

$$0 < \frac{\delta_c - \delta^+}{\delta_c - \delta^-} = \frac{2\zeta\sin(\cos^{-1}\zeta - \beta^-)}{\sin\beta^-} < 1\,.$$

This result (equation 10) can be used to prove the second sentence of item (2).

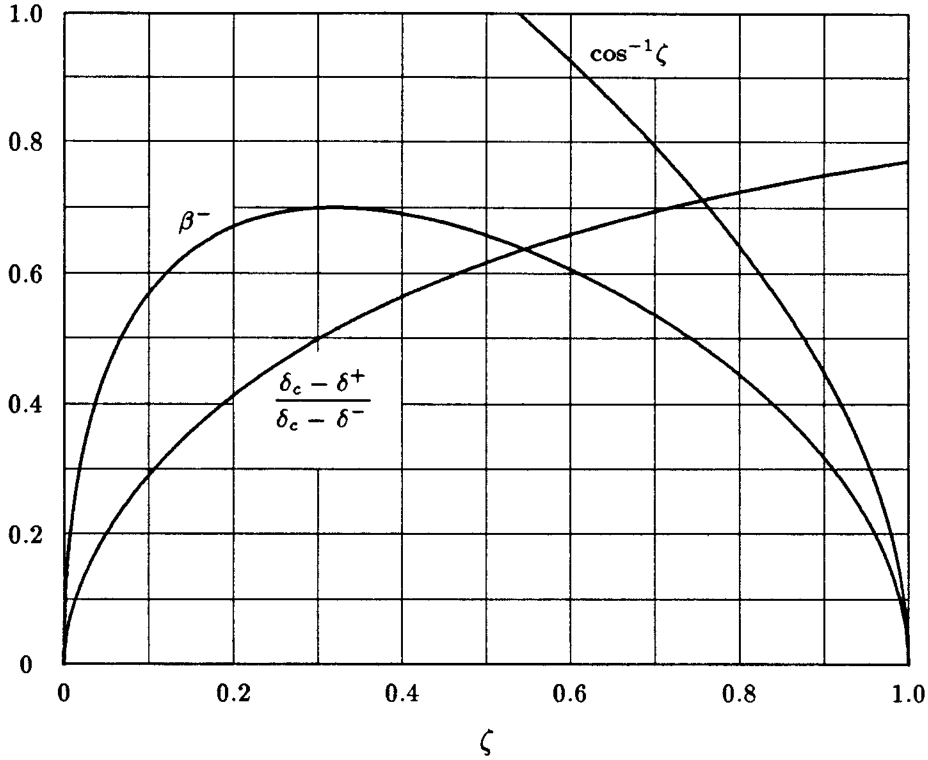


Figure 2 *Values of β (in radians) as a function of the damping: $\beta = \beta^-$ when $t_0 = t^-$ and $\beta = \cos^{-1}\zeta$ when $\dot{\delta}_0 = 0$. Also shown is the relative separation of δ_c from δ^+ and δ^-.*

And, finally, a major result which allows the sign of $\dot{\delta}_{\max}$ to be immediately established.

6) Equations 4a, 8, and 9 can be combined to give:

$$\delta_c - \delta^+ = \frac{2\zeta}{\omega}\,\dot{\delta}_{\max} \tag{11}$$

which, together with equation 10, leads to the following conclusion: When $t^- < t_0 < t^+$, the sign of $\delta_c - \delta_0$ equals the sign of $\dot{\delta}_{\max}$, for arbitrary δ_c, δ_0.

These observations are used in the analyses that follow.

3 Frequency or Rate Limited?

As discussed earlier, the system response is modelled in either a 'frequency limited' or rate limited mode. The frequency limited solution mode is always calculated first and uses the specified system response frequency ω_{max} to determine ω. This solution is correct if the magnitude of the associated maximum rate $\dot{\delta}_{max}$ is less than or equal to the specified rate limit. If not, the rate limited solution is calculated using the specified rate limit as the magnitude for $\dot{\delta}_{max}$. As will be shown, this results in an artificially low response frequency. In general, whether a solution should be frequency or rate limited is unknown until one of the solutions is calculated and the associated parameters examined. The frequency limited solution is calculated first because it involves no trial and error procedures, so less effort is expended if it turns out to be the wrong solution.

One proceeds as follows. Given ω_{max}, ζ, δ_c, δ_0, and $\dot{\delta}_0$, equation 6 is used to get β. Equation 9 is then used to calculate the value of $\dot{\delta}_{max}$ associated with this trial solution, unless β is too close to 0 or π, which results in the ratio $(\delta_c - \delta_0)/\sin\beta$ becoming indeterminate. In this case, a different formulation for this ratio, obtained from equation 5, must be used:

$$\frac{\delta_c - \delta_0}{\sin\beta} = \frac{\zeta(\delta_c - \delta_0) - \dot{\delta}_0/\omega}{\cos\beta\sqrt{1-\zeta^2}}\,. \tag{12}$$

If $|\dot{\delta}_{max}|$ is less than or equal to the specified rate limit, $\dot{\delta}_{RL}$ (always positive), or if $\beta \geq 2\cos^{-1}\zeta$, then the frequency limited solution is correct. Otherwise, a change is made to a rate limited solution.

As will now briefly be shown, when the change is made to a rate limited solution by reducing the magnitude of $\dot{\delta}_{max}$, the magnitude of ω is also reduced. To show this, the question to ask is 'What is the variational relationship between ω and $|\dot{\delta}_{max}|$ for a given command and set of initial conditions?'. This is most easily answered by considering how $|\dot{\delta}_{max}|$ varies with ω, or, how $\dot{\delta}_{max}/(\delta_c - \delta_0)$ (see item (6) above) varies with ω. It is convenient to define a new parameter γ using a version of equation 9:

$$\gamma \equiv \frac{\dot{\delta}_{max} e^{2\zeta\cos^{-1}\zeta/\sqrt{1-\zeta^2}}}{(\delta_c - \delta_0)\sqrt{1-\zeta^2}} = \frac{\omega e^{\zeta\beta/\sqrt{1-\zeta^2}}}{\sin\beta}$$

where, for a given problem, all the parameters on the LHS are constant except for $\dot{\delta}_{max}$. Thus, the question becomes 'What is $d\gamma/d\omega$?' where:

$$\frac{d\gamma}{d\omega} = \frac{\partial\gamma}{\partial\omega} + \frac{\partial\gamma}{\partial\beta}\frac{d\beta}{d\omega}\,.$$

Using equation 6 one can show that:

$$\frac{d\omega}{d\beta} = \frac{-\omega\sqrt{1-\zeta^2}}{\sin\beta\sin(\beta - \cos^{-1}\zeta)}$$

so that:

$$\frac{d\gamma}{d\omega} = \frac{e^{\zeta\beta/\sqrt{1-\zeta^2}}}{1-\zeta^2}\sin(2\cos^{-1}\zeta - \beta) > 0 \quad \text{for} \quad \beta^- < \beta < 2\cos^{-1}\zeta\,. \tag{13}$$

Thus, reducing the magnitude of $\dot{\delta}_{\max}$ simultaneously reduces the magnitude of ω; however, note that the opposite happens when $t_0 > t^+$, which shows that this question was not a trivial one.

This result allows one to proceed with the rate limited solution knowing the consequences of doing so.

4 The Rate Limited Solution

Here, it is assumed that the frequency limited solution was not acceptable and that the frequency must be altered to bring the magnitude of the maximum deflection rate down to $\dot{\delta}_{RL}$. The first step is to calculate a new value for β, knowing that $\beta^- < \beta < 2\cos^{-1}\zeta$.

After eliminating ω from equations 5 and 9, it is convenient to introduce F, a function defined to be zero when β takes on its correct value:

$$F(\beta) \equiv \frac{\dot{\delta}_0}{\dot{\delta}_{\max}} - \frac{\sin(\beta - \cos^{-1}\zeta)}{\sqrt{1-\zeta^2}} e^{\zeta(2\cos^{-1}\zeta - \beta)/\sqrt{1-\zeta^2}} . \tag{14}$$

Here, $\dot{\delta}_{\max}$ is determined by the rate limit and item (6) of Section 2:

$$\dot{\delta}_{\max} = \text{sign}(\delta_c - \delta_0)\, \dot{\delta}_{RL} . \tag{15}$$

4.1 Trial and Error Solution for β

The trial and error Newton-Raphson method is used to solve for β. For this, F is expanded in a Taylor series:

$$F(\beta) \sim F(\beta_0) + \frac{\partial F}{\partial \beta}(\beta_0)\,\Delta\beta + \cdots \qquad \text{as } \Delta\beta \to 0$$

where β_0 is an initial guess for β and $\Delta\beta$ is the error in the guess. Assuming β_0 is sufficiently close to β, this equation allows one to predict a new and improved value for β, namely $\beta_1 = \beta_0 + \Delta\beta$, by setting $F(\beta) = 0$. There results:

$$\Delta\beta = \frac{-F(\beta_0)}{\frac{\partial F}{\partial \beta}(\beta_0)} \tag{16}$$

where:

$$\frac{\partial F}{\partial \beta}(\beta) = \frac{-\sin(2\cos^{-1}\zeta - \beta)}{1-\zeta^2} e^{\zeta(2\cos^{-1}\zeta - \beta)/\sqrt{1-\zeta^2}} . \tag{17}$$

Thus:

$$\Delta\beta = \frac{\sqrt{1-\zeta^2}}{\sin(2\cos^{-1}\zeta - \beta_i)} \left[\sin(\cos^{-1}\zeta - \beta_i) + \frac{\dot{\delta}_0}{\dot{\delta}_{\max}} \sqrt{1-\zeta^2}\, e^{-\zeta(2\cos^{-1}\zeta - \beta_i)/\sqrt{1-\zeta^2}} \right] \tag{18}$$

is the correction to β_i, the i^{th} iteration; i is as large as necessary to give the required accuracy. The only difficulty with this approach is the indeterminateness in equation 18 as $\beta \to 2\cos^{-1}\zeta$; that is, as $\dot{\delta}_0/\dot{\delta}_{\max} \to 1$. This problem is resolved in the next subsection.

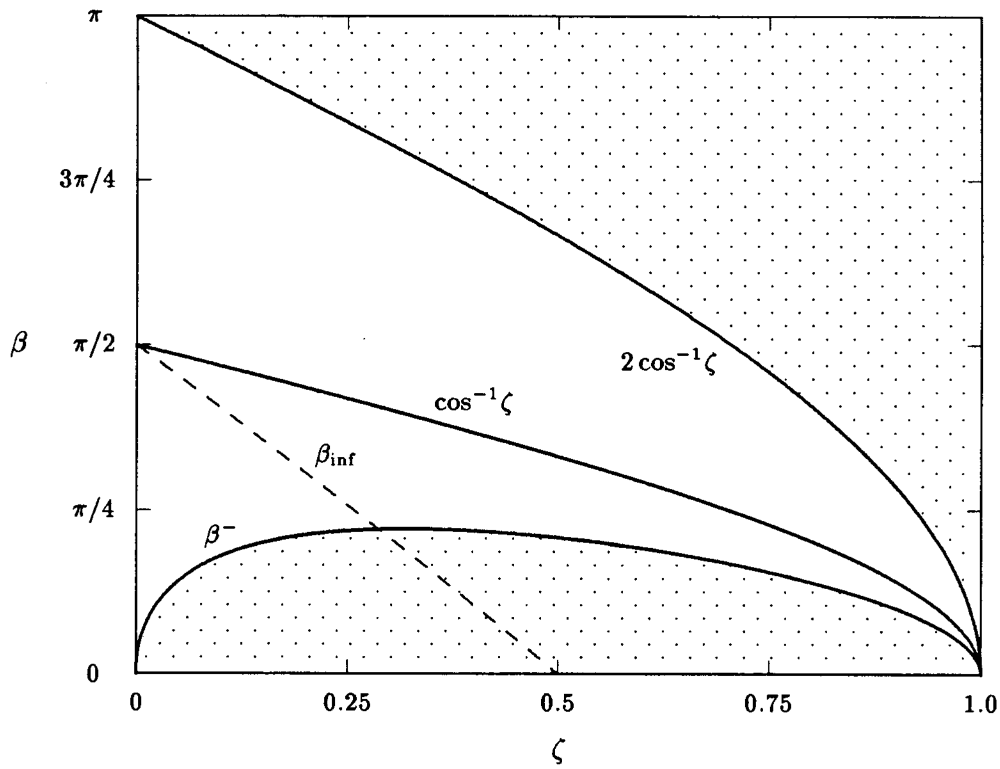


Figure 3 *The β region for the rate limited solution: $\beta^- < \beta < 2\cos^{-1}\zeta$.*

This iterative procedure should be examined further to ensure it has acceptable convergence properties. The Newton-Raphson method can be counted on to converge to the correct solution only if $\partial F/\partial\beta$ is nonzero everywhere within a range delimited by the extreme most β_i values encountered in the iterative process. Happily, equation 17 shows that:

$$\frac{\partial F}{\partial \beta} < 0 \text{ always for } 2\cos^{-1}\zeta - \pi < \beta < 2\cos^{-1}\zeta \tag{19}$$

(see Figure 3). Therefore, one must establish what β_0 values will ensure that the β_i remain within this range.

Consider the second derivative of F:

$$\frac{\partial^2 F}{\partial \beta^2}(\beta) = \frac{\sin(3\cos^{-1}\zeta - \beta)}{(1-\zeta^2)^{3/2}} e^{\zeta(2\cos^{-1}\zeta - \beta)/\sqrt{1-\zeta^2}} \tag{20}$$

which shows there is an inflection point in F at:

$$\beta_{\text{inf}} \equiv 3\cos^{-1}\zeta - \pi\,. \tag{21}$$

Furthermore, $F(\beta)$ is concave upwards for $\beta_{\text{inf}} < \beta < 3\cos^{-1}\zeta$ and concave downwards for $3\cos^{-1}\zeta - 2\pi < \beta < \beta_{\text{inf}}$. Thus, if one chooses β_0 such that:

$$\beta_{\text{correct}} < \beta_0 \leq \beta_{\text{inf}} \quad \text{or} \quad \beta_{\text{inf}} \leq \beta_0 < \beta_{\text{correct}}$$

as well as keeping β_0 within the range of equation 19, then the Newton-Raphson method will always march monotonically towards the correct β, be it greater than or less than $\beta_{\rm inf}$; that is:

$$\begin{aligned} &\beta_{\rm correct} < \cdots < \beta_{i+1} < \beta_i < \beta_{i-1} < \cdots < \beta_0 \quad \text{for} \quad \beta_{\rm correct} < \beta_0 \le \beta_{\rm inf} \\ &\beta_0 < \cdots < \beta_{i-1} < \beta_i < \beta_{i+1} < \cdots < \beta_{\rm correct} \quad \text{for} \quad \beta_{\rm inf} \le \beta_0 < \beta_{\rm correct} \,. \end{aligned} \tag{22}$$

In practice, since $\beta > 0$ always, one sets:

$$\beta_0 = \max(\beta_{\rm inf}, 0). \tag{23}$$

One could use $\beta_0 = \max(\beta_{\rm inf}, \beta^-)$, but β^- is too complicated a function for convenient use. When $\beta_{\rm correct} > \cos^{-1}\zeta$, the number of iterations can be reduced by increasing β_0:

$$\beta_0 = \cos^{-1}\zeta \quad \text{when} \quad \operatorname{sign}(\dot\delta_0) = \operatorname{sign}(\delta_c - \delta_0)\,. \tag{24}$$

This procedure has excellent convergence properties. Typically only 5 to 10 iterations are needed to give an accuracy greater than 1 part in 10^8.

4.2 Taylor Series Solution for β

The previous subsection provides a general numerical solution of $F(\beta) = 0$ (equation 14) for β. However, the solution cannot be used as $\dot\delta_0 \to \dot\delta_{\rm max}$ ($\beta \to 2\cos^{-1}\zeta$) because equation 18 becomes indeterminate. In this subsection, this situation is remedied by developing a solution of $F(\beta) = 0$ which gives β as a Taylor series in a suitably chosen small parameter.

Consider the small, but always positive parameter:

$$\lambda \equiv \frac{2\cos^{-1}\zeta - \beta}{\sqrt{1-\zeta^2}}; \qquad \lambda \to 0 \text{ as } \beta \to 2\cos^{-1}\zeta.$$

In terms of λ, $F(\beta) = 0$ becomes:

$$\begin{aligned} \frac{\dot\delta_0}{\dot\delta_{\rm max}} &= \frac{-e^{\zeta\lambda}}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\,\lambda - \cos^{-1}\zeta\right) \\ &= \frac{-e^{\zeta\lambda}}{\sqrt{1-\zeta^2}} \left(\zeta \sin\sqrt{1-\zeta^2}\,\lambda - \sqrt{1-\zeta^2}\cos\sqrt{1-\zeta^2}\,\lambda\right) \\ &\sim 1 - \frac{\lambda^2}{2} - \frac{\zeta\lambda^3}{3} - \frac{(4\zeta^2-1)\lambda^4}{24} - \cdots = 1 - \frac{\lambda^2}{2}\left[1 + \frac{2\zeta\lambda}{3} + \frac{(4\zeta^2-1)\lambda^2}{12} + \cdots\right] \end{aligned}$$

as $\lambda \to 0$. Rearranging and carrying out further expansions results in:

$$\lambda \sim \left[2\left(1 - \frac{\dot\delta_0}{\dot\delta_{\rm max}}\right)\right]^{1/2} \left(1 - \frac{\zeta\lambda}{3} + \frac{\lambda^2}{24} - \cdots\right). \tag{25}$$

Thus, the 'natural' small parameter in the sought after expansion has identified itself:

$$\epsilon \equiv \left[2\left(1 - \frac{\dot{\delta}_0}{\dot{\delta}_{\max}}\right)\right]^{1/2} . \tag{26}$$

Upon continually substituting equation 25 into itself, there results:

$$\begin{aligned}\beta \sim 2\cos^{-1}\zeta - \sqrt{1-\zeta^2}\Bigg[\epsilon - \frac{\zeta\epsilon^2}{3} + \frac{(3+8\zeta^2)\epsilon^3}{72} - \frac{(27\zeta+16\zeta^3)\epsilon^4}{540} \\ + \frac{(81+624\zeta^2+64\zeta^4)\epsilon^5}{17280} - \frac{(81\zeta+156\zeta^3-16\zeta^5)\epsilon^6}{8505} \\ + \frac{(30375+460728\zeta^2+260928\zeta^4-71168\zeta^6)\epsilon^7}{43545600} + \cdots\Bigg]\end{aligned} \tag{27}$$

as $\epsilon \to 0$. This number of terms was obtained by initially expanding $\dot{\delta}_0/\dot{\delta}_{\max}$ to terms $O(\lambda^8)$. All of these terms were then carried through the subsequent calculations.

Equation 27 presents no numerical problems provided ϵ is small enough. Although convergence has not been proved, at the very least the equation is an asymptotic expansion, so the error in it is of the order of the first neglected term. In practice, this error criterion is applied to the last term in the truncated expansion; ie, equation 27 is only used if:

$$\frac{30375+460728\zeta^2+260928\zeta^4-71168\zeta^6}{43545600}\epsilon^7 \leq E \tag{28}$$

where E is the level of relative error acceptable in β (noting that $2\cos^{-1}\zeta = O(\sqrt{1-\zeta^2})$). Actually, this expansion converges well for reasonable values of E (typically 10^{-5}).

Equation 28 provides the means for deciding whether the Taylor series or Newton-Raphson trial and error solution for β should be used, since ϵ is known from equations 15 and 26. Using double precision computer arithmetic (16–17 significant figures), the Newton-Raphson solution has been found to work well right up to the point where equation 27 takes over.

With β determined, calculation of the remainder of the rate limited solution is straightforward. Equation 9 gives ω and equation 4a gives the solution itself.

5 Some Examples

Figure 4 shows how the algorithm predicts the deflection time histories for both large and small deflections for different damping ratios. The lower the damping, the faster the commanded deflection is achieved, but at the cost of increasing overshoot (see Figure 5). For surface ship rudder roll stabilization studies, Campbell and Graham[1] decided on a damping ratio of 0.7 since this gave good response while keeping overshoot:

$$O = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \quad (\text{assuming } \dot{\delta}_0 = 0) \tag{29}$$

to less than 5 per cent of the commanded change. A damping ratio of 0.9 is preferred by others since it keeps overshoot to less than 0.2 per cent of the commanded change, which is zero for practical purposes.

Figures 5 and 6 show how the algorithm matches time responses to the proper initial conditions at each issuance of a new command. Keep in mind that second and higher order derivatives of these time responses are discontinuous at these command points.

As previously mentioned, the numerical implementation of the algorithm uses double precision arithmetic, and this helps provide a robust routine. Despite the presence of a singularity in the solution of β as $\zeta \to 1$ (see Figure 3), an accurate time response was still generated for $\zeta = 0.9999$ (Figure 5).

Figure 7 compares two other algorithms to the one being proposed. The first order model is governed by the first order differential equations:

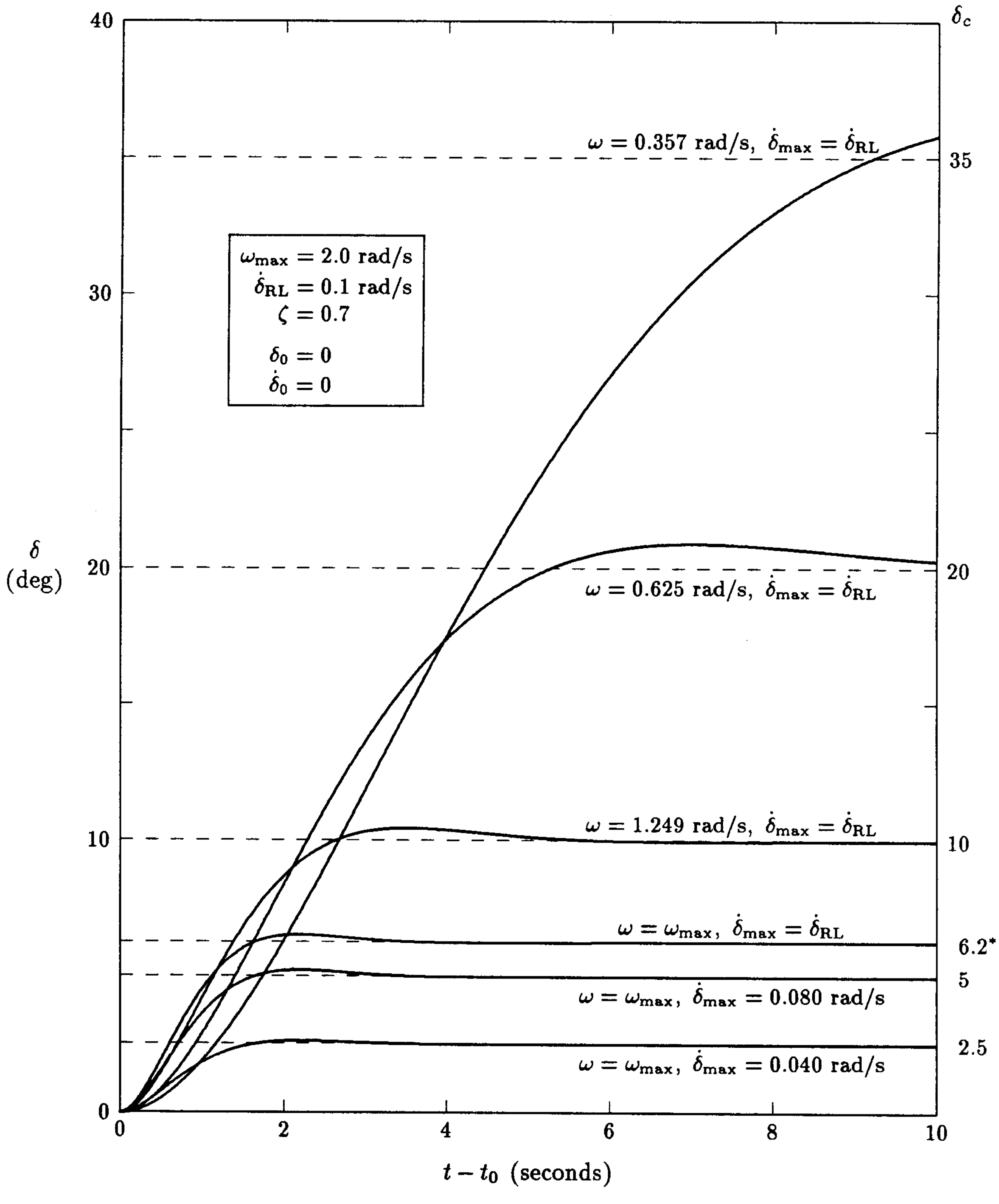
$$\begin{aligned} \dot{\delta} &= \omega_{\max}(\delta_c - \delta) && \text{for } |\dot{\delta}| < \dot{\delta}_{\text{RL}} \\ \dot{\delta} &= \dot{\delta}_{\text{RL}} && \text{otherwise.} \end{aligned} \tag{30}$$

There results:

$$\begin{aligned} \delta &= \delta_0 + \dot{\delta}_{\text{RL}}(t - t_0) && \text{for } t_0 \le t \le t' \\ \delta &= \delta_c - \frac{\dot{\delta}_{\text{RL}}}{\omega_{\max}} e^{-\omega_{\max}(t-t')} && \text{for } t' \le t \le \infty \end{aligned} \tag{31}$$

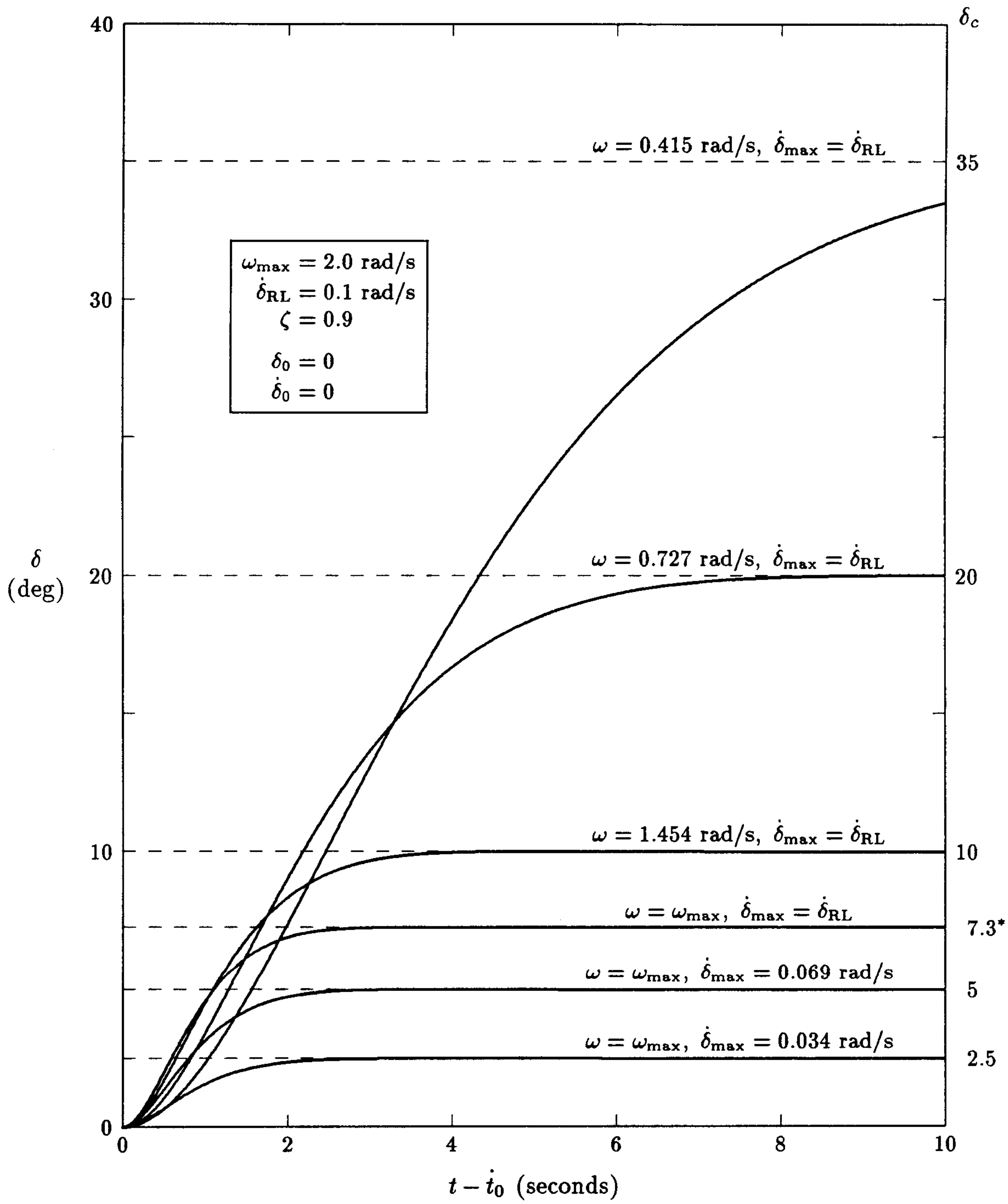
for $\delta_c > \delta_0$, and where t' is chosen to match equations 31 at $t = t'$. If $\delta_c - \delta_0$ is small enough, $t' \le t_0$ and only the last of equations 31 need be considered. In Figure 7, $t' = 2.99$ seconds, point A on the dotted curve. This first order model has a discontinuous slope at $t = t_0$, so it cannot match the initial condition $\dot{\delta}_0 = 0$. The model also has a discontinuous second derivative at $t = t'$.

The second order, 'piecewise continuous' model is probably closest to the actual response of the control system. It uses $\omega_{\max}$ to model the transient responses from $t = t_0$ until the rate limit is achieved (point B in Figure 7), and from termination of the rate limit (point C) to $t = \infty$. It is constructed from the proposed second order, continuous model's solution when $\omega = \omega_{\max}$ and $\dot{\delta}_{\max} = \dot{\delta}_{\text{RL}}$, which occurs when $\delta_c = 7.27$ degrees (see Figure 4b). This solution is simply pulled apart at its inflection point and a straight line (with slope $\dot{\delta}_{\text{RL}}$) of sufficient length inserted so that the required δ_c is achieved. Since $\ddot{\delta} = 0$ at the inflection point and, of course, for the straight line, the resulting solution is continuous through to and including its second derivatives at points B and C. The two second order models are identical for deflection changes less than the critical value of $\delta_c = 7.27$ degrees.



a) *Low damping: $\zeta = 0.7$.*

Figure 4 *Time responses of two control surfaces to six commands. In each case, the two smallest changes commanded are frequency limited, while the three largest are*



b) *High damping: $\zeta = 0.9$.*

rate limited. The commands (*) are those unique values at which the solution changes from one mode to the other.

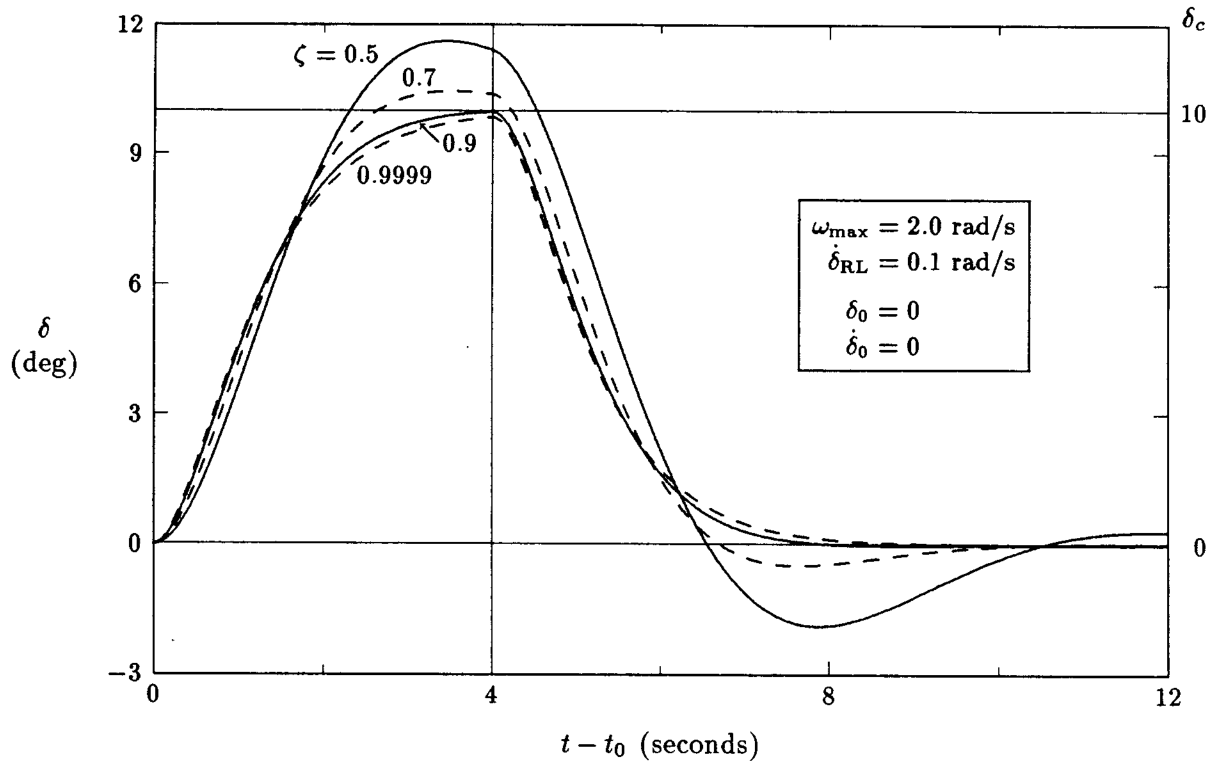


Figure 5 *Time responses to identical commands of four control systems with different damping ratios;* $\delta_c = 10$ *degrees issued at* $t = t_0$, $\delta_c = 0$ *issued at* $t - t_0 = 4$ *seconds.*

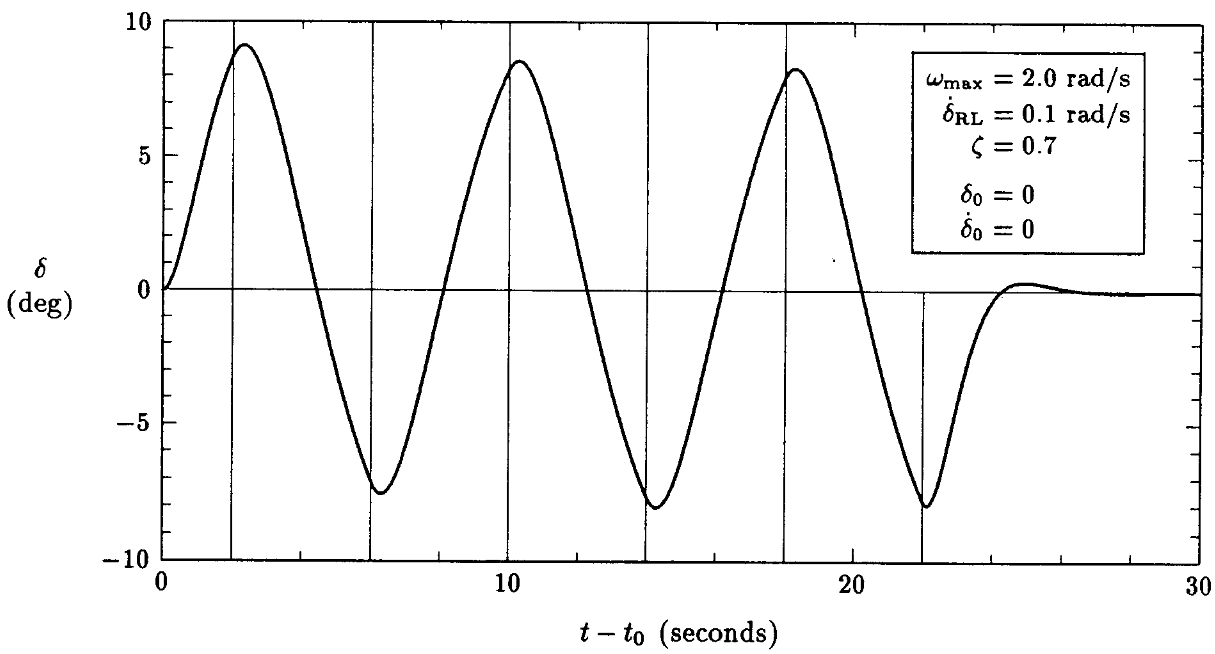


Figure 6 *Time response to a series of commands* $\delta_c = 10, -10, 10, -10, 10, -10,$ *and* 0 *degrees; commands are issued at* $t - t_0 = 0, 2, 6, 10, 14, 18,$ *and* 22 *seconds respectively.*

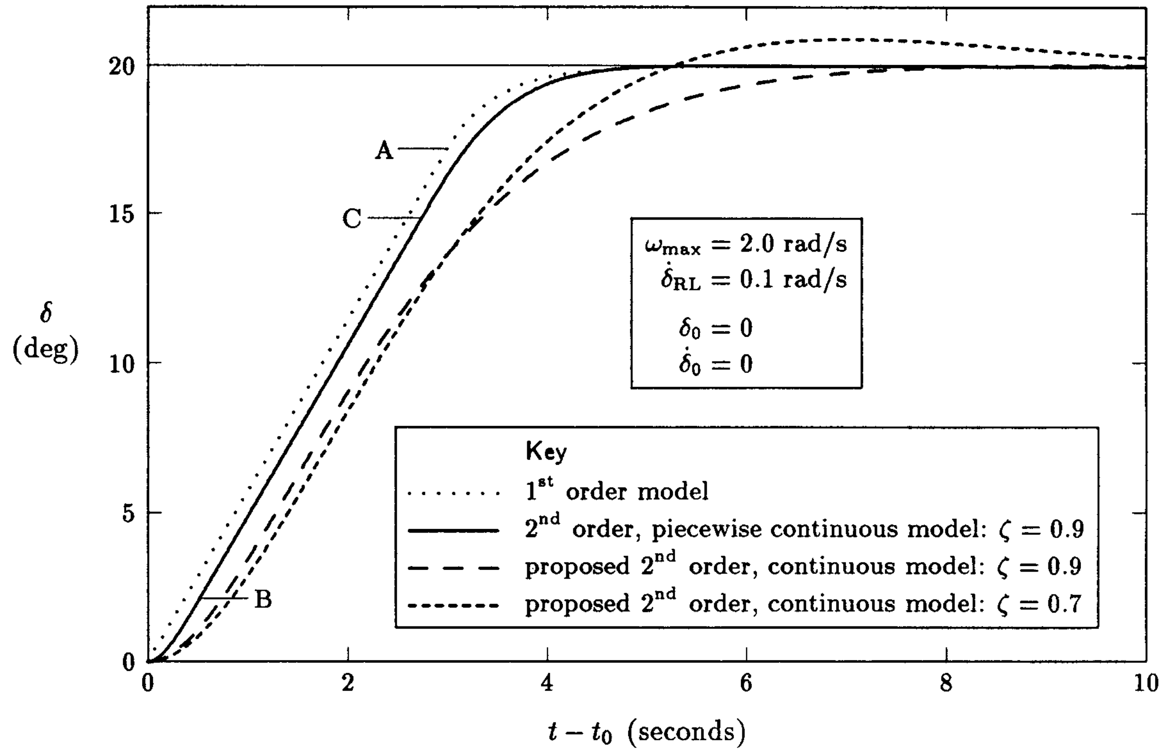


Figure 7 *Four time responses to the command $\delta_c = 20$ degrees issued at $t - t_0 = 0$. The first order model has a discontinuity in its slope at the origin, and in its second derivative at point* A. *The second order, piecewise continuous model has a discontinuous second derivative at the origin (as does the proposed model) and third derivative at points* B *and* C, *but always models both ω_{max} and $\dot{\delta}_{RL}$.*

Figure 4b shows how the proposed second order continuous model would tend toward the second order, piecewise continuous model as the commanded change decreases, and away from it as the commanded change increases. If the piecewise continuous model is indeed closest to the actual control system response, and if discontinuities in the response time derivatives are not of concern, then the preferred algorithm is the piecewise continuous model (even the first order model might be better for large deflections); however, if it is desirable to minimize discontinuities in the time derivatives of the equations of motion, the proposed second order continuous model may be best. Note that, for arbitrary initial conditions and large deflection changes, the second order, piecewise continuous model still requires the trial and error solution of equation 14 for the determination of β.

This concern over discontinuities is motivated by the fifth order numerical method being considered for integrating the submarine equations of motion. As is shown by Enright et al,[2] numerical integration of ordinary differential equations across discontinuities can be both inefficient and inaccurate unless special precautions are taken. These precautions involve stopping the integration at each discontinuity and restarting it on the other side. The precautions are mandatory if accurate estimates of the global errors[3] associated with the numerical integration procedure are required. Discontinuities in time derivatives as high as the sixth order (the order of the numerical method plus one) are of concern. Some discontinuities are inevitable (whenever a new command is issued for example), but by minimizing them the complexity of incorporating a control system model in the submarine equations of motion is also minimized.

Figure 7 also shows how lower damping allows the proposed control system model to achieve the commanded deflection sooner. This may be desirable if overshoot is not of concern or, indeed, if one is prepared to use overshoot to compensate empirically for the slow response of the proposed model.

6 Conclusion

A general algorithm has been presented for modelling the time response of a control system to an arbitrary set of commands. The algorithm considers the response frequency (time constant), rate limit, and damping of the control system in determining the response, but cannot always model response frequency if discontinuities in high order time derivatives of the response are to be minimized. The algorithm always matches the zeroth and first order time derivatives of the existing motion and response at the time the command is issued; subsequent motion is perfectly continuous.

This algorithm can be easily modified to provide a response which always models the specified response frequency, as well as the rate limit. This can be done without changing the characteristics of the match at the time the command is issued, but is at the expense of discontinuities in the third and higher order time derivatives of the subsequent motion.

Some time must now be spent evaluating this control system algorithm in conjunction with the proposed integration routine for the submarine equations of motion. Also, more knowledge is required about the relevant control systems before extensive effort is put into developing a control system model which minimizes discontinuities as well as closely reproducing control surface time responses.

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An algorithm is presented for modelling the time response of a lifting surface control system to an arbitrary set of commands. Response dynamics are governed by a linear, second order, ordinary differential equation. This allows the control system response frequency and damping to be modelled, as well as current control surface position and rate of change of position to be matched when a new command is issued. In order not to exceed the rate limit of the control system, yet maintain continuity in the response and its high order derivatives in time, the algorithm reduces response frequency when large deflection changes are required. This approach is compared to a common first order model, and to a second order model having piecewise continuous high order time derivatives but which always models both response frequency and rate limit.

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