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MULTI-CRITERIA DECISION PROBLEMS: ALTERNATIVES WITH UNCERTAIN SCORES

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OPERATIONAL RESEARCH DIVISION

DIRECTORATE OF MATHEMATICS AND STATISTICS

DMS RESEARCH NOTE RN 9602

MULTI-CRITERIA DECISION PROBLEMS: ALTERNATIVES WITH UNCERTAIN SCORES

by

D.J. Lamb

MAY 1996

OTTAWA, CANADA

National Defence Défense nationale

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ABSTRACT

Analytical expressions for an important class of multi-criteria decision problems are developed. This class represents a linear objective function of alternatives evaluated on a weighted set of numerical scores. The scores are assumed to be values from statistically independent uniform or triangular distributions. Some illustrative examples are given.

TABLE OF CONTENTS

I. STATEMENT OF PROBLEM

1. This paper presents a method and derives analytical expressions for an important class of multi-criteria decision problems. The general problem may be stated quite simply: the decision maker must select from a set of alternatives that alternative which best satisfies a set of criteria[1]. Formulating the problem in a manner which can be exploited mathematically however, presents a considerable challenge.

2. What issues must the analyst deal with to effect such a transformation? We need to decide whether or not the alternatives are functions of continuous variables representing the criteria. If this is the case and a single objective is applicable, it may be feasible to model the decision problem as (non-)linear mathematical program, utilizing well-known techniques for such cases. Some variables representing the criteria may be discrete, which adds to the complexity. Rather than a single objective, the decision problem may have several objectives, some of which may conflict. Much research has been undertaken on such problems and techniques developed for their solution[2].

3. In the present case, we are most interested in a finite set of alternatives which depend in some fashion on the criteria. This statement is meant to be vague, since most problems of interest in the defence community which relate to equipment acquisition are vague. In the sections to follow we comment on the nature of criteria, uncertainty and imprecision in evaluating the criteria, problems with hierarchical models, appropriate objective functions, and the statistics of a linear weighted objective.

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[1] Some of the literature refers to the alternatives as options and the criteria as attributes. We do not need to concern ourselves with this aspect of the language.

[2] The reader should consult Watson & Buede for more information and references to the literature.

II. NATURE OF CRITERIA

4. Since each alternative must be evaluated against each criteria, the analyst must ensure that the set of criteria is sufficient to the decision to be made. The criteria can be either quantitative or qualitative, and can be measured on a cardinal or an ordinal scale. Although the qualitative scale may simply express preference, we assume that we can at least rank order the preferences so that a partial ordering is in effect. Since these issues can become somewhat formidable in practice we discuss each in turn.

Qualitative Criteria

5. Qualitative criteria represent judgement and are non-metric by their very nature. This judgement can be obtained by an expert panel, by subject matter experts, or by the decision maker. The decision analyst should advise those providing the judgement with insight into the consequences but should not directly take part in the initial evaluation. These criteria are typically evaluated by stating that an alternative has good (or fair or excellent or very good or fair-to-good) performance in a certain criteria. An example of this may be a weapon system that has excellent reliability[3].

6. Although such criteria can be handled using ranking methods, most methods map such statements onto an interval on the real line. We assume that the non-metric qualitative criteria are ordered in some logical fashion - e.g. poor, poor-to-fair, fair, fair-to-good, good, very good, excellent. Such schema are readily mapped onto a finite set such as $\{1,2,\ldots,9\}$. Saaty has made much of rating (or evaluating) alternatives and criteria using such a scale. It is not an unreasonable way to proceed and we assume that our multi-criteria decision problem can be tackled in some such manner.

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[3] If we have quantitative data which measures *inter alia* MTBF and MTTR, say, we may use these in a quantitative manner and thus avoid problems interpreting a qualitative statement.

Quantitative Criteria

7. Although quantitative criteria appear simpler to consider we must be careful to avoid certain pitfalls. We can distinguish between hard and soft quantitative variables. A hard quantitative variable can be measured by scientific means, and, subject to measurement limitations and errors, is not subject to debate. A soft variable on the other hand could be estimated by simulation and does not necessarily lend itself to easy measurement, although it could, in theory if not in practice, be measured or calculated.

8. Suppose we are to select a fighter aircraft and that one of our major criteria is "performance". We can subdivide performance into a number of sub-criteria such as manouverability, speed, radius of action, weapons load, etc. Suppose we want to quantify speed. What could be easier? We can use knots, m/s, km/h, Mach no., or mph. This is clearly quantified although if the aircraft is only on the drawing boards, there will be some uncertainty as to its actual speed. Leaving that aside, should we use the fighter's maximum speed or speed carrying the maximum weapons load? Of course, we could use more than one variable (i.e. sub-criterion) to represent speed. Then comes the hard part - is a fighter with a top speed of Mach 2.4 twice as effective as one with a top speed of Mach 1.2. Mathematically the ratio of speeds is just 2:1. However, in our decision problem it is not at all evident that we should just accept the ratio (or difference for that matter) to represent the value of a higher speed to the choice of aircraft. To proceed we need to make a value judgement about not only speed, but typically all the other criteria and sub-criteria that make up our set of decision variables.

9. Although our quantitative variables are typically measured on continuous metric scales, such scales are not necessarily those we need to quantify their value when making a decision. In this paper we shall consider all variables (both quantitative and qualitative) to be represented on a finite interval. Note that judgement is required in all cases to affect such transformations and that, in the majority of cases, the transformations are non-linear.

III. UNCERTAINTY AND IMPRECISION

10. Once we have completed the task of quantifying all variables we need to examine how certain we are that the variables are correctly quantified. This is not the same as determining measurement errors in a test, trial, or exercise and performing subsequent statistical tests. Rather it is ascertaining two aspects of the variables which represent each criteria and sub-criteria, namely the uncertainty and imprecision of each. The distinction that we make with respect to these aspects is found in probability theory, which we use to represent uncertainty, and fuzzy set theory, which we use to represent imprecision.

11. Probability is not the only choice for representing uncertainty. Other suggestions have been the Dempster-Shafer theory of belief functions, the uncertainty calculus developed for early versions of MYCIN, and Cohen's theory of inference. We will not address these theories in this paper; only concerning ourselves with probability theory.

12. Although Zadeh originally developed fuzzy set theory as an alternative to probability theory for studying uncertainty, we believe that it more usefully applied to a concept of imprecision, *vis-a-vis*, uncertainty. It is important that the reader understand this distinction. If we make a precise statement about the probability distribution of a variable, i.e. we treat the variable as a random variable, then we are dealing with uncertainty. We may prefer to assert that a variable lies only within a certain range and consider its quantitative measure as that of membership in a fuzzy set. Then we are dealing with imprecision. We can mix the two concepts by being imprecise (fuzzy) about our uncertainty if we are uncertain which probability distribution represents a variable!

13. In this paper we restrict our attention to variables which have a probability distribution which represents the uncertain nature of our variables. This is not an unduly restrictive assumption and much progress and insight can be gained with this restricted viewpoint.

IV. HIERARCHICAL MODELS

14. Now that we have decided that all variables representing our criteria will be quantified, let us consider how we structure our evaluation and the various criteria. Many decision problems are represented using a multi-level hierarchical structure. For example, Buede and Bresnick show the evaluation hierarchy for the US FAADS. The authors were asked to compare and rank order several alternate mixes and each system was evaluated using a hierarchical structure.

15. At the highest level, operational effectiveness and cost were quantified. Operational effectiveness was subdivided into "mix effectiveness" and "other"(*sic*). Each of these (sub-) criteria were again subdivided several more times until no further subdivision was necessary. Both mix effectiveness and other were subdivided into at most three sublayers. However, not all sub-(sub-sub-)criteria were subdivided into an equal number of factors or layers.

16. Associated with each (sub-)criteria is the importance of each. Should we treat mix effectiveness and other as equally important? Are the sub-criteria of C^3I, flexibility, and sustainability equally important or is one more important than another? If the latter, how much more important is that factor? This leads directly into the weights which should be assigned to the (sub-)criteria (or factors) in order to quantify our decision problem.

17. Once we have assigned numerical values to the weight of each factor[4] we have to determine how each factor contributes to the final score of each alternative. The hierarchical structure developed for the analysis of FAADS alternatives illustrates the general problem associated with multi-level hierarchical structures; namely, how do we reasonably roll up lower level weights to the next highest level, and so forth until we reach the highest level (operational effectiveness in the FAADS example).

---

[4] Instead of refering to sub-criteria, sub-sub-criteria, etc, we will refer to any of these sub-levels as a factor.

18. The question is not academic since not all criteria have either the same number of levels nor the same number of factors at the same level. For example, under mission effectiveness, "other-flexibility-span of control" has three levels while "other-C^3I-degraded-warning/cueing" has four levels. Is the lowest level under other-flexibility to be treated as the lowest level under other-C^3I (i.e. level 4 for this factor is at the same level as level 3 for the other factor) or should we treat other-flexibility-span of control at the same level as other-C^3I-degraded? Although the hierarchical structure is a very useful way to evaluate complex systems, it clearly raises issues of its own regarding the meaning of the various levels within the hierarchy.

19. Not only must the meaning of the relative weight of each level be decided, the meaning of weights associated with unbalanced factors at the same level has to be resolved. The FAADS study shows that under mix effectiveness-close operation, "kill enemy" has three factors (armour, rotory, and fixed) while "protect forces" has two factors (CSS/C^3I and maneuver). Suppose for one alternative that we decide all factors are equally important. For "kill enemy" we can assign weights of 33.3% while for "protect forces" we would assign weights of 50%. Or should we? If we simply roll up the weights from the lowest to highest level by normalizing and multiplying, then the factors under "protect forces" will have higher final weights than those under "kill enemy", all other things being equal. We can adopt another view that says we must balance the factors at a given level and then proceed to evaluate the relative weights. In this example, we would assign weights of 20% fo the five stated factors. The decision analyst must ascertain what the decision maker really means in order to develop the final set of weights.

V. LINEAR WEIGHTED OBJECTIVES

20. Let us now assume that we have resolved the meaning of all weights associated with each criteria and their hierarchical sub-structure. We postulate that our decision problem can be formulated as a linear weighted objective (LWO). Suppose that we have K alternatives and that each of these alternatives is evaluted against N criteria. Asuume that we have assigned a score S_{jk} for alternative k against criteria j. Then our decision problem becomes

$$MAX \sum_{j=1}^{N} w_j S_{jk}$$

21. This formulation is deceptively simple for it hides the difficult issues of determining the weights of each criteria and the scores or value of each alternative against each criteria.[5] Once each alternative has been scored for each criteria, the 'best' alternative is that alternative which receives the highest weighted score. A rank ordering of alternatives is thus produced.

22. This rank ordering is the first step in our decision problem as the sensitivity of the solution to the weights as well as the scores should be determined. Brereton proposed a method which varies the weights to determine when the second best alternative interchanges with the best alternative - best in this case referring to our original assessment of the weights. Many authors perform standard sensitivity analysis by varying the scores over what is judged to be a reasonable range of values.

VI. PROBABILITY DISTRIBUTION OF LINEAR WEIGHTED OBJECTIVES

23. In most real decision problems the scores assigned to each alternative for each criterion can rarely be considered as constants. They are usually only approximately known. To proceed further, we assume that the said scores are represented by a random variable with a known distribution and that the random variables are assumed to be pairwise statistically independent. We restrict our analysis by assuming however, that the weights are known with certainty (i.e. they are neither fuzzy nor probabilistic).

24. To determine the distribution of the overall score of each alternative we initially restrict

[5] These issues are examined in greater detail in textbooks such as Watson and Buede, as well as in the literature (e.g. Buede and Maxwell).

our attention to the case where each score is represented by an independent random variable that is uniformly distributed over the same (finite) interval. The following points about this assumption are should be noted. First, we have assumed that a continuous random variable is a reasonable representation of each score for each variable.[6] Second, this represents a situation where no experts have evaluated an alternative and the overall score comes from randomly choosing scores in each criteria.

25. In this case, the LWO represents the weighted sum of identically distributed, independent random variables. If the weights are all equal (i.e. the criteria are equally important to the decision), the central limit theorem may be applied. In this case the distribution of the LWO is approximately normal (or Gaussian) with mean and variance simply related to the mean and variance of the score distributions.

26. In general however, the weights are not all equal and the distribution of the LWO must be determined for each particular case. Annexes A through D show the methodology adopted in this paper. If we represent the LWO sum in section V by the random variable z, then the probability density function (pdf) for the LWO with each score S_{jk} given by a uniform density $(0,1)$ and normalized weights w_j is

$$
\begin{aligned}
f(z; w_1, w_2, \ldots, w_N) = & \frac{1}{W(N-1)!} z^{N-1} H(z) \\
& + \frac{(-1)^1}{W(N-1)!} \sum_j (z - w_j)^{N-1} H(z - w_j) \\
& + \frac{(-1)^2}{W(N-1)!} \sum_j \sum_{k \neq j} (z - \overline{w_j + w_k})^{N-1} H(z - \overline{w_j + w_k}) \\
& + \ldots \\
& + \frac{(-1)^N}{W(N-1)!} (z - \sum_j w_j)^{N-1} H(z - \sum_j w_j) ,
\end{aligned}
$$

where $W = \prod_j w_j$, i.e. the product of the weights, and the summations are over all the weights.

[6] This is to be contrasted with, say, Saaty, who uses the discrete set *{1,2,3,4,5,6,7,8,9}* in his analytic hierarchy approach.

$H(x)$ is the Heaviside unit step function, defined by

$$H(x) = \begin{cases} 0 & x \leq 0, \\ 1 & x > 0. \end{cases}$$

(This expression is derived in Annex C.) The reader should note that the above expression[7] is symmetric about $z=½$ and that $\sum_j w_j = 1$.

VII. EXAMPLES

27. The expression of the pdf of the LWO of identical uniformly distributed independent random variables looks rather daunting at first glance. For a few criteria (5 or less) it is readily derived and plotted, either directly using the method of Annex B or by simply inserting in the formula of Section VI. For more criteria, it can be plotted with the aid of modern software such as MAPLE. We shall give examples of both.

28. Suppose we have five criteria and the weights associated with the criteria are 1:1:1:2:3. This allows for a number of degeneracies in the above formula as we see the impact of various combinations of criteria. Assume that the criteria are measured over the unit interval *(0,1)* but that the weights are not normalized. The sum of the weights $\sum_j w_j = 8$, and the product $W = \prod_j w_j = 6$. Using formulae and techniques from Annexes B and C we find that the pdf is proportional to

$$f(z) \propto z^4 H(z) - 3(z-1)^4 H(z-1) + 2(z-2)^4 H(z-2) + (z-3)^4 H(z-3) \\ - (z-5)^4 H(z-5) - 2(z-6)^4 H(z-6) + 3(z-7)^4 H(z-7) - (z-8)^4 H(z-8).$$

Dividing this expression by W and $4!$ yields the pdf for this decision problem. Notice that the pdf is symmetric about the mid-point $z=4$ in this case. This function was plotted with the aid of MAPLE and is shown as Figure 1.

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[7] An alternative form of the above is given in Springer.

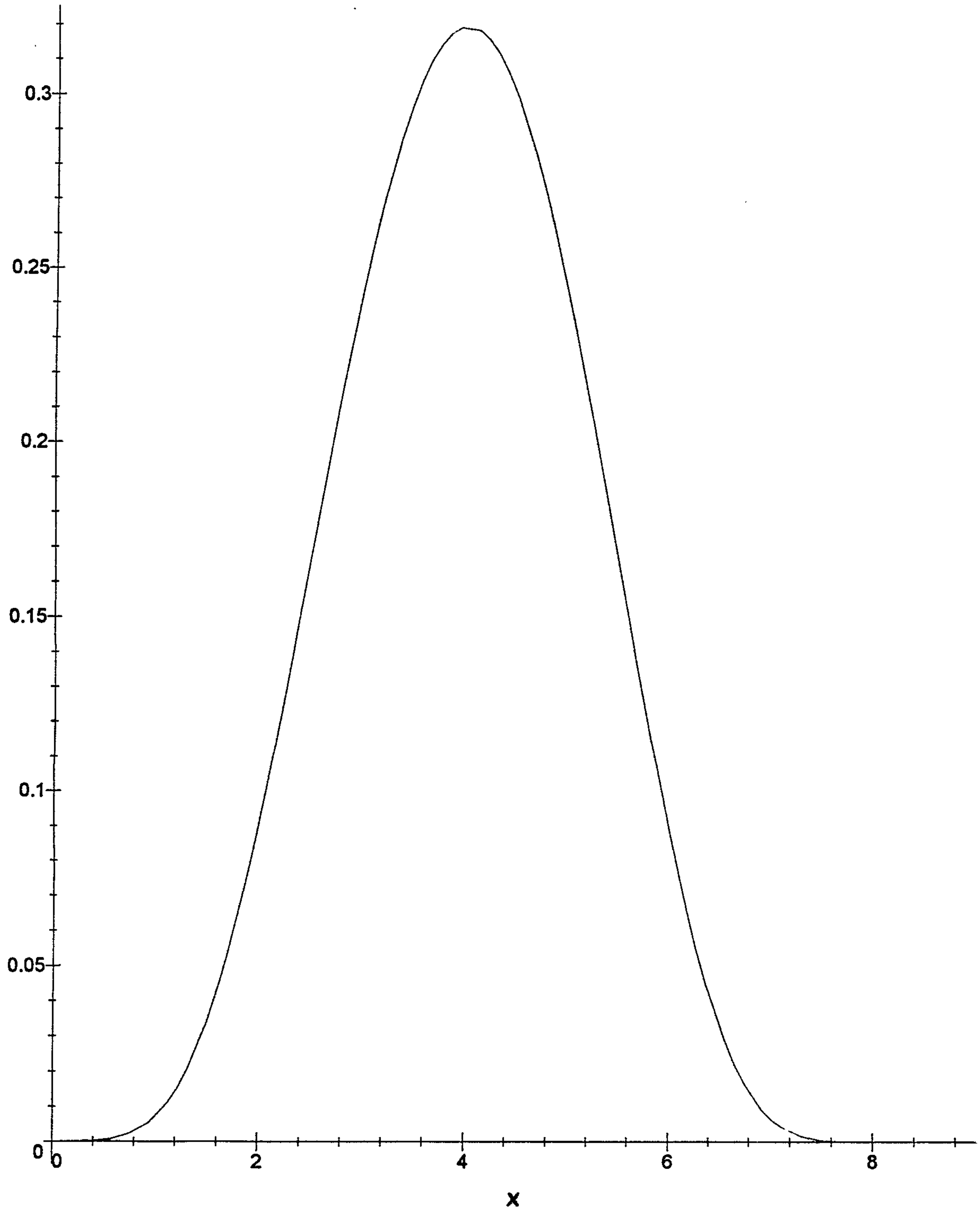

Figure 1: Probability Density (Weights 1:1:1:2:3)

29. A more complex example is due to Gass. For this decision problem, nine alternative diesel generators were evaluated against 15 criteria. It was decided to compare their scores with that obtained by a purely random ranking against each criterion. The weight structure of the criteria was *(1,2,1,2,1,3,4,1,1,1,1,2,1,1,3)*. We see that nine criteria have unit weight, three have weight 2, two have weight 3, while the remaining criterion has weight 4. The scores were originally taken over *(0,100)*. If we assume that each criterion is sampled from independent uniform density functions over the *(0,1)* interval[8], then using the method described above, we find that the pdf for the LWO is proportional to

$$
\begin{aligned}
f(z) \propto\, & z^{14} H(z) \\
& - 9(z-1)^{14} H(z-1) + 33(z-2)^{14} H(z-2) - 59(z-3)^{14} H(z-3) \\
& + 38(z-4)^{14} H(z-4) + 42(z-5)^{14} H(z-5) - 105(z-6)^{14} H(z-6) \\
& + 107(z-7)^{14} H(z-7) - 105(z-8)^{14} H(z-8) + 97(z-9)^{14} H(z-9) \\
& - 6(z-10)^{14} H(z-10) - 126(z-11)^{14} H(z-11) + 196(z-12)^{14} H(z-12) \\
& - 196(z-13)^{14} H(z-13) + 125(z-14)^{14} H(z-14) + 6(z-15)^{14} H(z-15) \\
& - 97(z-16)^{14} H(z-16) + 105(z-17)^{14} H(z-17) - 107(z-18)^{14} H(z-18) \\
& + 105(z-19)^{14} H(z-19) - 42(z-20)^{14} H(z-20) - 38(z-21)^{14} H(z-21) \\
& + 59(z-22)^{14} H(z-22) - 33(z-23)^{14} H(z-23) + 9(z-24)^{14} H(z-24) \\
& - (z-25)^{14} H(z-25)\,.
\end{aligned}
$$

Dividing this expression by W and *14!* yields the pdf for this decision problem. This function has been plotted using MAPLE and is shown as Figure 2. Once again the symmetry about the mode should be noted.

30. We can use the above pdf (or the cumulative distribution function obtained by integration) to determine the 'distance' between the overall scores for each alternative. In the above example, the alternatives were scored as follows: *5.39, 5.55, 11.66, 13.08, 14.75, 15.61, 19.01, 19.24,* and *21.81*. We can advise the decision maker on these scores by determining where these values lie in the pdf.

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[8] This is done by mapping the original score onto the unit interval for each criterion, assigning *0* to the lowest score and *1* to the highest.

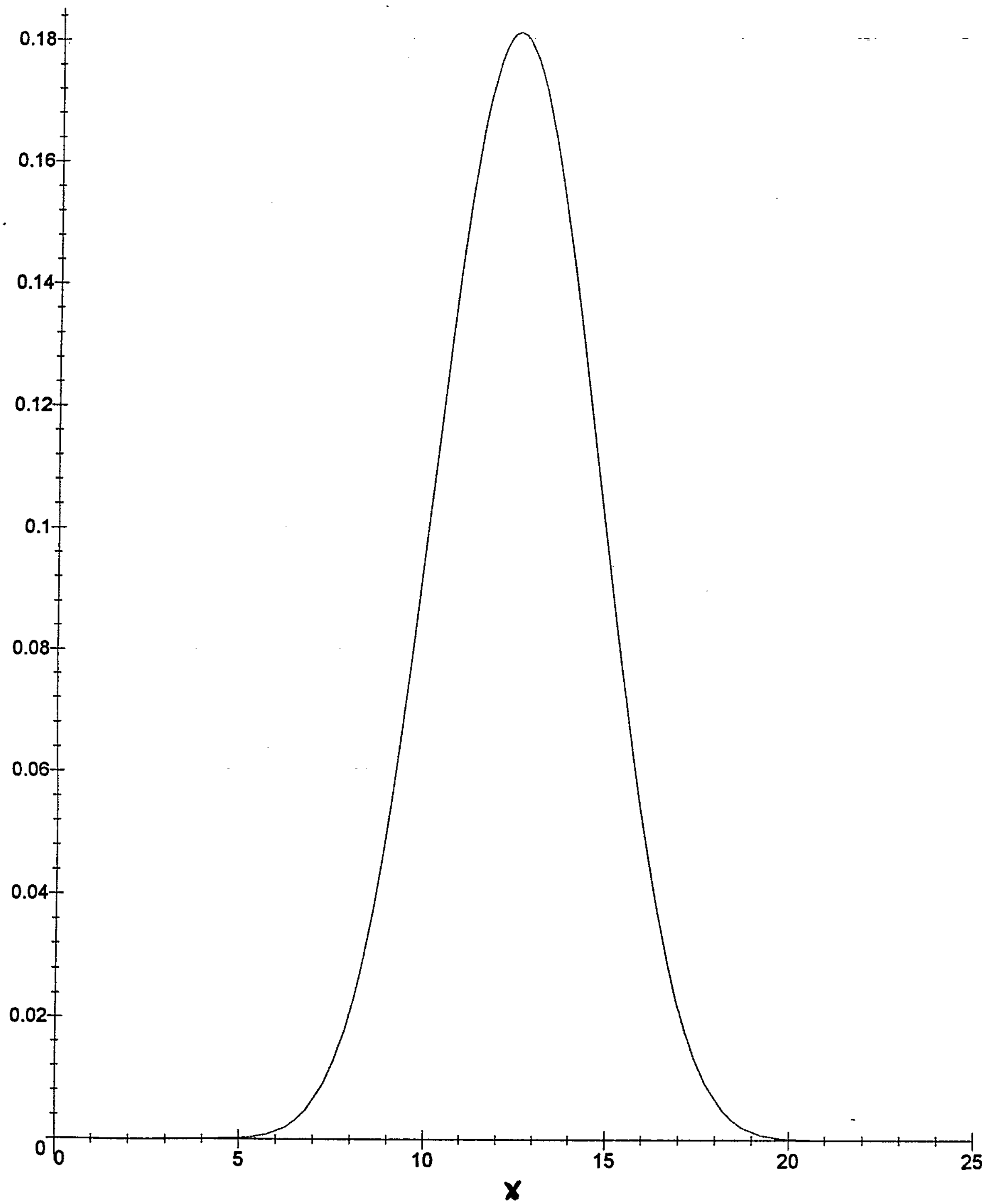

Figure 2: Probability Density (Diesel Generator Weights)

31. In the above examples we were given no information concerning the uncertainty of each score S_{jk}. Often we may have a good idea regarding this uncertainty. If this is so, then we can derive pdfs for each alternative and get more information on which to base our judgement. The following two examples[9] make this clear.

32. Our third example is based on a very simple case. This allows us to see our way clearly yet illustrates the methodology. For this case we have only two alternatives scored agaisnt two criteria. The ratings are shown in Table 1.

Table 1. Scores and Weights For Example 3

	Weight	Alternative 1 Scores	Alternative 2 Scores
Criterion 1	Very Important	Good	Fair
Criterion 2	Rather Unimportant	Fair	Good

33. Notice the deliberate vagueness in our scores and weights. For the present case suppose we represent the vagueness in the scores as follows. "Good" is interpreted as any value from *U(0.6,1.0)* while "Fair" is interpreted as any value form *U(0.4,0.8)*. We chose the "Very Important" weight as *0.9* and the "Rather unimportant" weight as *0.2*. Then the LWO for each alternative can be written

$$R_{A_1} \sim 0.9*U(0.6,1.0) + 0.2*U(0.4,0.8)$$
$$R_{A_2} \sim 0.9*U(0.4,0.8) + 0.2*U(0.6,1.0)$$

[9] These are based on examples 1 and 4 found in Baas and Kwakernaak. Their examples use fuzzy set theory. They have been slightly modified to suit our purpose by replacing the imprecision in scoring alternatives with uncertainty, and interpreting the membership function as a pdf. In addition we fix the weights.

34. Using the methods of Annexes A and B we find that the pdfs for the alternatives are

$$f_{A_1}(x) = \frac{1}{0.08 * 0.36}((x-0.62)H(\leftarrow)-(x-0.70)H(\leftarrow)-(x-0.98)H(\leftarrow)+(x-1.06)H(\leftarrow))$$

$$f_{A_2}(x) = \frac{1}{0.08 * 0.36}((x-0.48)H(\leftarrow)-(x-0.56)H(\leftarrow)-(x-0.84)H(\leftarrow)+(x-0.92)H(\leftarrow))$$

where $H(\leftarrow)$ is the Heaviside step function with argument the same as the expression preceding it. These pdfs are trapezoidal in shape and a plot (Figure 3) shows that the two alternatives overlap considerably. Based on this evaluation, we must conclude that either alternative provides an acceptable choice.

35. An example with more alternatives *(3)* and criteria *(4)* is given in Table 2.

Table 2. Scores and Weights for Example 4.

	Weight	Alt 1 Scores	Alt 2 Scores	Alt 3 Scores
Criterion 1	Very Important	Good	Very Good	Fair
Criterion 2	Moderately Important	Poor	Poor	Poor
Criterion 3	Moderately Important	Poor	Fair to Good	Fair
Criterion 4	Rather Unimportant	Good	Not Clear	Fair

36. As in Example 3 we have interpreted the vague assessments of the various alternatives and replaced them with probability distributions. For the above case "Good" and "Fair" are interpreted as before (para 33). "Moderately important" is *0.5*, "Very Good" becomes *U(0.8,1.0)*, "Poor" becomes *U(0,0.4)*, "Fair to Good" becomes *U(0.4,1.0)* and "Not Clear" becomes *U(0,1)*. This allows to write the LWO for each alternative as

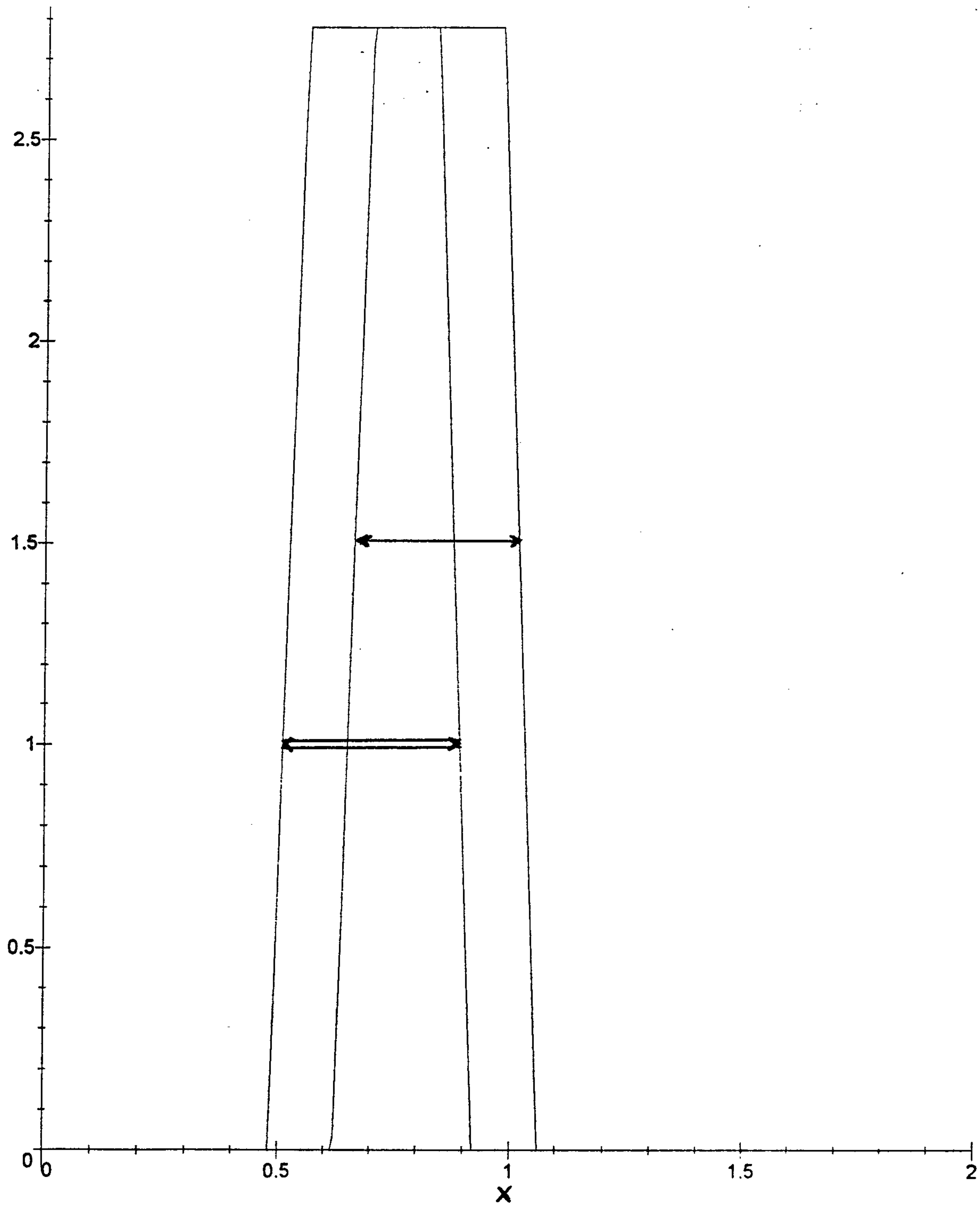

Figure 3: Probability Densities (Final Ranking of 2 Alternatives)

$$R_{A_1} \sim 0.9*U(.6, 1) + 0.5*U(0, .4) + 0.5*U(0, .4) + 0.2*U(.6, 1)$$
$$R_{A_2} \sim 0.9*U(.8, 1) + 0.5*U(0, .4) + 0.5*U(.4, 1) + 0.2*U(0, 1)$$
$$R_{A_3} \sim 0.9*U(.4, .8) + 0.5*U(0, .4) + 0.5*U(.4, .8) + 0.2*U(.4, .8)$$

37. Using the methods of Annexes B and C we can show that the pdfs of the alternatives are

$$\begin{aligned} n_1*f_{A_1} = {} & (x-.66)^3H(x-.66) - (x-.74)^3H(x-.74) - 2(x-.86)^3H(x-.86) \\ & + 2(x-.94)^3H(x-.94) - (x-1.02)^3H(x-1.02) + (x-1.06)^3H(x-1.06) \\ & + (x-1.1)^3H(x-1.1) - (x-1.14)^3H(x-1.14) + 2(x-1.22)^3H(x-1.22) \\ & - (x-1.3)^3H(x-1.3) - (x-1.42)^3H(x-1.42) + (x-1.5)^3H(x-1.5) \end{aligned}$$

$$\begin{aligned} n_2*f_{A_2} = {} & (x-.92)^3H(x-.92) - (x-1.1)^3H(x-1.1) - 2(x-1.12)^3H(x-1.12) \\ & - (x-1.22)^3H(x-1.22) + 2(x-1.3)^3H(x-1.3) + (x-1.32)^3H(x-1.32) \\ & + (x-1.4)^3H(x-1.4) + 2(x-1.42)^3H(x-1.42) - (x-1.5)^3H(x-1.5) \\ & - 2(x-1.6)^3H(x-1.6) - (x-1.62)^3H(x-1.62) + (x-1.8)^3H(x-1.8) \end{aligned}$$

$$\begin{aligned} n_3*f_{A_3} = {} & (x-.64)^3H(x-.64) - (x-.72)^3H(x-.72) - 2(x-.84)^3H(x-.84) \\ & + 2(x-.92)^3H(x-.92) - (x-1.0)^3H(x-1.0) + (x-1.04)^3H(x-1.04) \\ & + (x-1.08)^3H(x-1.08) - (x-1.12)^3H(x-1.12) + 2(x-1.2)^3H(x-1.2) \\ & - (x-1.28)^3H(x-1.28) - (x-1.4)^3H(x-1.4) + (x-1.48)^3H(x-1.48) \end{aligned}$$

where $n1 = 3! \times 0.36 \times 0.2 \times 0.2 \times 0.08$,

$n2 = 3! \times 0.18 \times 0.2 \times 0.3 \times 0.2$, and

$n3 = 3! \times 0.36 \times 0.2 \times 0.2 \times 0.08$.

Each normalization constant has the form $3! \times \prod_j w_j\,(b_{jk} - a_{jk})$. A plot (Figure 4) of these distributions shows that alternative 2 is preferred to the other two although there is some overlap in the range between *1.0* and *1.5*.

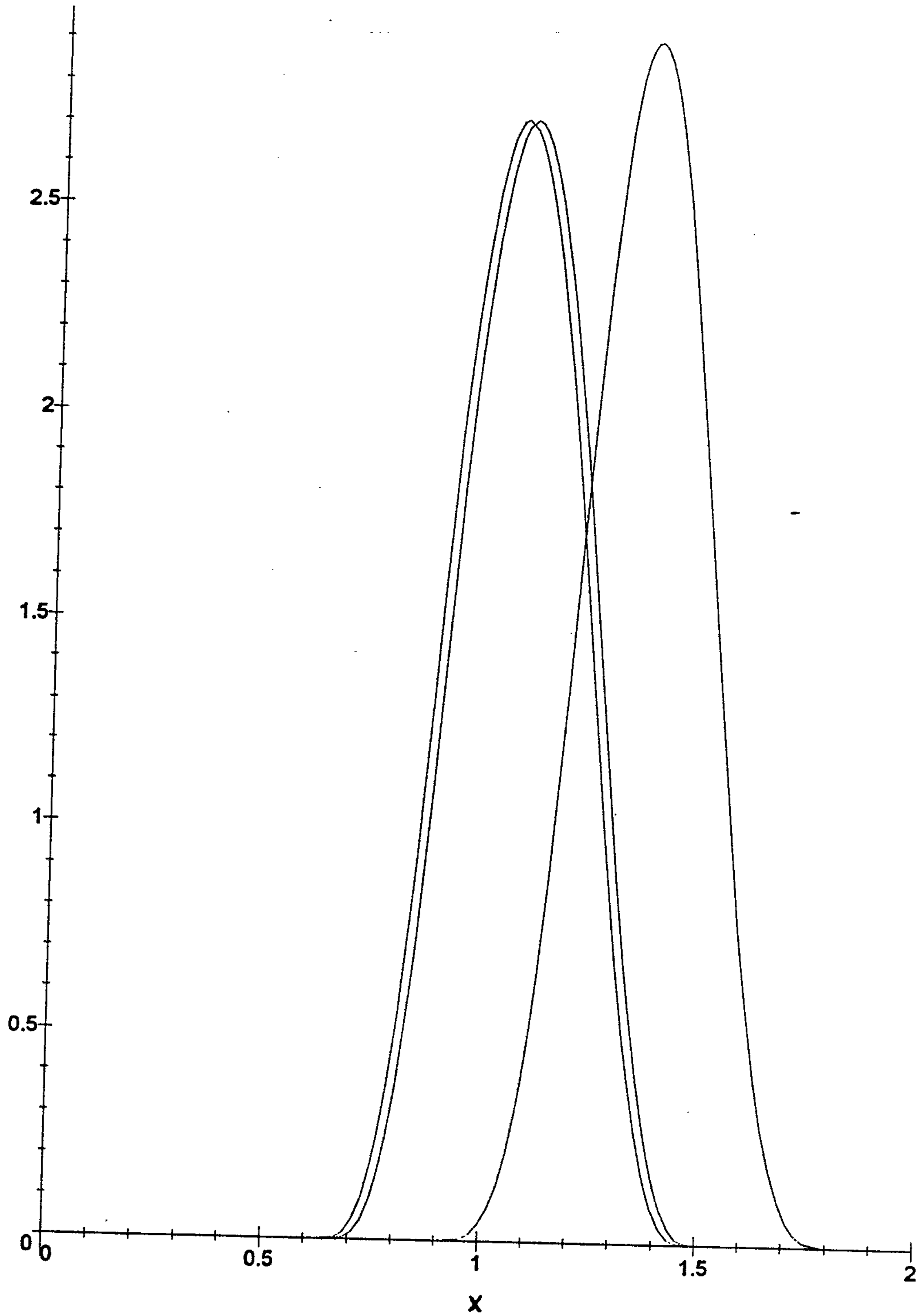

Figure 4: Probability Densities (Final Rankings for 3 Alternatives)

38. For problems which require more than 20 major criteria, the computation of analytical expressions such as the above is not particularly helpful. In these cases a good procedure is to generate the pdf by Monte Carlo simulation. If the decision problem is placed in a spreadsheet with the alternatives as columns and the criteria as rows, say, then @Risk[10] is adequate for finding the pdf of a purely random ranking or for studying the effect of uncertainty of each alternative in turn.

VII. CONCLUSIONS

39. We have derived expressions of the probability density function of a linear weighted objective function for cases when scores are uniformly distributed. When scores are modelled using triangular density functions, the method of Annex D may be used. The former case implies no knowledge about an alternative (i.e. complete uncertainty for each criterion's score) while the latter implies limited knowledge about criteria scores. The latter case often relects the nature of uncertainty in a decision problem.

40. This work has been extended to deal with uncertainty in both the weights as well as the criteria. Not unexpectedly, this complicates our analytical approach but it need not completely overwhelm the analyst. A separate paper will address such issues as well as deal with imprecision using fuzzy sets.[11]

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[10] This is a software package from Palisade Corp which is added into a spreadsheet and allows random samples to be drawn from some twenty probability distributions.

[11] See Kahne for a probability approach, and Baas and Kwakernaak for a fuzzy approach.

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ANNEX A TO
DMS RESEARCH NOTE 9602
MAY 1996

LAPLACE TRANSFORM AND INVERSE OF THE UNIFORM DISTRIBUTION

In this Annex we determine the Laplace Transform (LT) and its inverse, of a random variable uniformly distributed over $(0,w_j)$. This determination will prove useful before attempting to determine the distribution of the weighted sum of n identically-distributed uniform distributions.

Recall that the density of the uniform distribution over (a,b) is defined by

$$f(x;a,b) = \begin{cases} 0 & x<a, \\ 1 & a \le x \le b, \\ 0 & x>b. \end{cases}$$

The uniform distribution over $(0,w_j)$ is identical to the distribution of $w_j u(x)$ where $u(x)$ is the density with parameters $a=0$ and $b=1$. We denote this distribution by $f(x;w_j)$. Although the subscript on w is not necessary in this Annex it is useful to introduce it here for comparison with both the main text and subsequent Annexes.

The LT for a generic probability density $f(x)$ restricted to the non-negative reals has the following form

$$L_f(s) = \int_0^{\infty} e^{-sx} f(x)\, dx$$

with its inverse

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} L_f(s)\, ds\,.$$

The latter integral is a contour integral in the complex s-plane. The (real) constant c is chosen so that all the poles of $L_f(s)$ lie to the left of c.

The LT of the uniform distribution over (a,b) is easily shown to be

$$L_{f(x;a,b)}(s) = \frac{1}{b-a}\left(\frac{e^{-as} - e^{-bs}}{s}\right).$$

Then the inverse transform is found by evaluating

$$f(x;a,b) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx}\,\frac{1}{b-a}\left(\frac{e^{-as} - e^{-bs}}{s}\right) ds\,.$$

Using Cauchy's theorem we get

$$f(x;a,b) = \frac{1}{b-a}H(x-a) - \frac{1}{b-a}H(x-b),$$

where $H(x)$ is the Heaviside unit step function, defined by

$$H(x) = \begin{cases} 0 & x \le 0, \\ 1 & x > 0. \end{cases}$$

The LT of the uniform distribution over $(0,w_j)$ is

$$L(s;w_j) = \frac{1}{w_j}\left(\frac{1 - e^{-w_j s}}{s}\right).$$

ANNEX B TO
DMS RESEARCH NOTE 9602
MAY 1996

WEIGHTED SUM OF TWO INDEPENDENT UNIFORM DISTRIBUTIONS

It is insightful to examine the weighted sum of two independent uniform distributions (IUDs). This prepares us for the general case of n identically distributed IUDs, each with weight w_j.

We proceed as follows. Consider $y = w_1x_1+w_2x_2$, where x_1 and x_2 are IUDs and $x_1{\sim}U(a_1', b_1')$ and $x_2{\sim}U(a_2', b_2')$. w_1 and w_2 positive real numbers representing the weights, which may or may not be normalized. To simplify our notation set $a_j = w_j a_j'$ and $b_j = w_j b_j'$.

To determine the density of y recall that the density of the sum of independent random variables (RVs) may be found from the n-fold convolution of the individual density functions. This is equivalent to inverting the product of the Laplace transform of each RV. If we can do this successfully, the problem is solved; otherwise we may be faced with an equally difficult contour integration. For the two RVs under consideration we get

$$L_y(s) = \left(\frac{e^{-a_1 s} - e^{-b_1 s}}{(b_1 - a_1)\, s}\right)\left(\frac{e^{-a_2 s} - e^{-b_2 s}}{(b_2 - a_2)\, s}\right).$$

The probability density for y is found by evaluating

$$g(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sy}\, L_y(s)\, ds\ .$$

To proceed, we note the existence of a double pole at the origin. This pole has contributions from four pieces - each of the form $e^{s(y-W)}$. The residue from each contribution to the double pole is

$$\frac{d}{ds} e^{s(y-W)} \Big|_{s=0} = y - W \quad for \quad y \geq W,$$
$$= 0 \quad for \quad y < W.$$

where $W \in \{a_1 + a_2, a_1 + b_2, a_2 + b_1, b_1 + b_2\}$.

Evaluating the residues gives us

$$g(y) = \frac{1}{(b_1 - a_1)(b_2 - a_2)} X$$
$$(y - \overline{a_1 + a_2}) H(\leftarrow) - (y - \overline{a_1 + b_2}) H(\leftarrow)$$
$$-(y - \overline{a_2 + b_1}) H(\leftarrow) + (y - \overline{b_1 + b_2}) H(\leftarrow),$$

where ($\leftarrow$) represents the expression appearing in front of the Heaviside function and the scaling factor is common to all terms.

When $w_1 = w_2 = 1$ and $x_1 \sim U(0, 1)$ and $x_2 \sim U(0, 1)$ this reduces to

$$g(y) = y H(y) - 2(y-1) H(y-1) + (y-2) H(y-2),$$

or, in a more familiar form,

$$g(y; 1, 1) = \begin{cases} 0 & y < 0 \\ y & 0 \leq y \leq 1 \\ 2-y & 1 \leq y \leq 2 \\ 0 & y > 2. \end{cases}$$

This is the triangular distribution $f_T(y; 0, 1, 2)$ which is discussed in Annex D.

WEIGHTED SUM OF N INDEPENDENT UNIFORM DISTRIBUTIONS

The weighted sum of N independent uniform distributions may be found using the techniques of Annex A and B.

We assume that the uniform distributions are over the unit interval *(0,1)*. Suppose that $z = \sum_{j=1}^{N} w_j x_j$ where x_j are independent random variables uniformly distributed over the unit interval and w_j are the weights.

As noted in Annex B, the Laplace transform (LT) of z is found by multiplying the LT of each x_j together as follows

$$L(s\,;w_1, w_2, \ldots, w_N) = \prod_{j=1}^{N} \left(\frac{1 - e^{-w_j s}}{w_j\, s} \right).$$

Expansion of this product results in a power series in terms of e^{-s}. The denominator is the product of the weights and s^N. Calculation of the inverse LT by performing the contour integration results in the following:

$$\begin{aligned} f(z\,;w_1, w_2, \ldots, w_N) = {} & \frac{1}{W(N-1)!} z^{N-1} H(z) \\ & + \frac{(-1)^1}{W(N-1)!} \sum_j (z - w_j)^{N-1} H(z - w_j) \\ & + \frac{(-1)^2}{W(N-1)!} \sum_j \sum_{k \neq j} (z - \overline{w_j + w_k})^{N-1} H(z - \overline{w_j + w_k}) \\ & + \ldots \\ & + \frac{(-1)^N}{W(N-1)!} (z - \sum_j w_j)^{N-1} H(z - \sum_j w_j)\,, \end{aligned}$$

where $W = \prod_j w_j$, i.e. the product of the weights, and the summations are over all the weights. Although this expression is somewhat ungainly, by examining each term we see that we have a contribution from each weight, then each pair of weights, followed by each triple of weights, ending with the contribution from the sum of all the weights. The resulting pdf ranges from *0* to $\sum_j w_j$ and is symmetric about $\frac{1}{2}\sum_j w_j$. If all of the weights are equal, we can evoke the central limit theorem and note that the resulting distribution is approximately Gaussian (i.e. normal).

The cumulative distribution function (CDF) is found easily by integrating the pdf. Its form is very similar to the pdf and may be written by inspection when we realize that

$$\frac{1}{(n-1)!}\int_a^x (t-a)^{n-1}dt = \frac{(x-a)^n}{n!}.$$

Hence the CDF for the weighted sum over standard uniform distributions is simply

$$\begin{aligned} F(z; w_1, w_2, \ldots, w_N) &= \frac{1}{WN!} z^N H(z) \\ &+ \frac{(-1)^1}{WN!}\sum_j (z-w_j)^N H(z-w_j) \\ &+ \frac{(-1)^2}{WN!}\sum_j \sum_{k\neq j} (z-\overline{w_j+w_k})^N H(z-\overline{w_j+w_k}) \\ &+ \ldots \\ &+ \frac{(-1)^N}{WN!}(z-\sum_j w_j)^N H(z-\sum_j w_j), \end{aligned}$$

As an example, consider a case where we have *N-1* unit weights and one large weight M where $M >> (N-1)$. Using the notation above, $w_j = 1$ for $j = 1,2,\ldots,N-1$ and $w_N = M$. The sum of the weights is $M+N-1$. Then the Laplace transform of the probability density is

$$\left(\frac{1-e^{-s}}{s}\right)^{N-1}\left(\frac{1-e^{-Ms}}{Ms}\right).$$

Expanding the term representing the unit weights using the binomial theorem and setting $n=N-1$ to ease the notation gives

$$\frac{1}{M}\left(1+\sum_{j=1}^{n}(-1)^{j}\binom{n}{j}e^{-s}-e^{-Ms}-\sum_{j=1}^{n}(-1)^{j}\binom{n}{j}e^{-(M+j)s}\right).$$

We perform the inverse Laplace transform by observing that there is a pole of order N (i.e. $n+1$) at the origin, obtaining

$$\begin{aligned} f(z;1,\ldots,1,M) &= \frac{z^{n}}{Mn!}H(z)+\sum_{j=1}^{n}\binom{n}{j}\frac{(z-j)^{n}}{Mn!}H(z-j) \\ &- \frac{(z-M)^{n}}{Mn!}H(z-M)-\sum_{j=1}^{n}\binom{n}{j}\frac{(z-\overline{M+j})^{n}}{Mn!}H(z-\overline{M+j}). \end{aligned}$$

This distribution resembles a table-top, the flat top of the distribution being the contribution from the dominant weight and the table 'legs' being the contribution from the n unit weights. The observant reader will note that the equal weights do not have to be unity although we should ensure that the dominant weight is greater than the sum of all the other (equal) weights. If this is not so, we will not see much of a top on the table.

As an example, consider the situation where we have six criteria *(N=6)*, five of which have unit weight, and the remaining one having a weight of 10. For this case $M=10$ and $n=5$. Substituting into the above expression yields

$$\begin{aligned} f(z) \propto\ & z^{5}H(z)-5(z-1)^{5}H(z-1)+10(z-2)^{5}H(z-2) \\ & -10(z-3)^{5}H(z-3)+5(z-4)^{5}H(z-4)-(z-5)^{5}H(z-5) \\ & -(z-10)^{5}H(z-10)+5(z-11)^{5}H(z-11)-10(z-12)^{5}H(z-12) \\ & +10(z-13)^{5}H(z-13)-5(z-14)^{5}H(z-14)+(z-15)^{5}H(z-15). \end{aligned}$$

after using the binomial theorem. Dividing by $M\,(N-1)!$ $(= 10\times5!)$ yields the pdf. A distinctive table top is evident when plotted (Figure C-1).

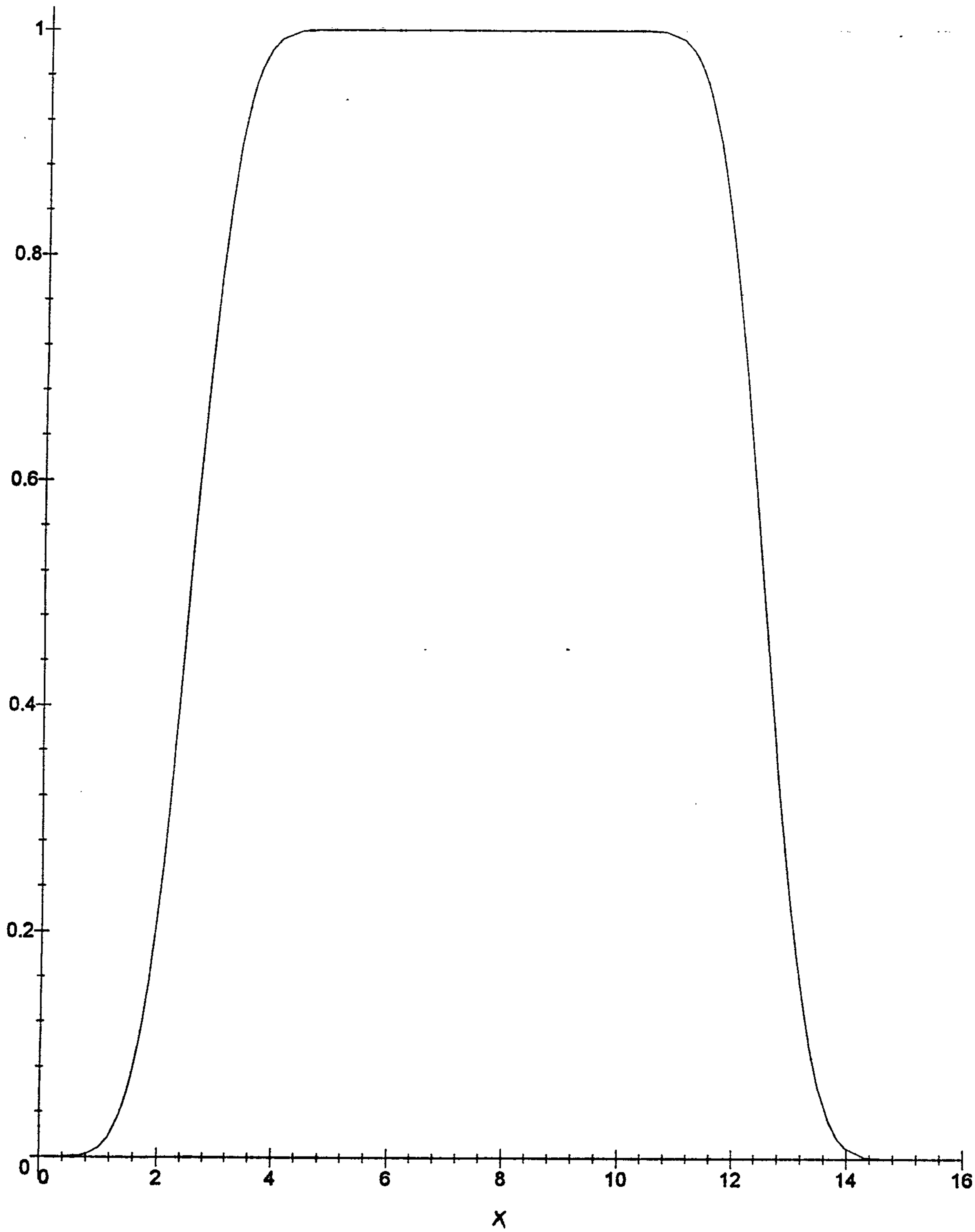

Figure C-1: Probability Density (Weights 1:1:1:1:1:10)

ANNEX D TO
DMS RESEARCH NOTE 9602
MAY 1996

WEIGHTED SUM OF TRIANGULAR DISTRIBUTIONS

As noted in the paper, criteria scores are sometimes better represented by triangular distributions (or 3-point estimates). The triangular distribution is specified by only three parameters - its minimum a, its most likely value (i.e. its mode) m, and its maximum b. We assume that $a<b$. However m may be equal to either a or b. The general triangular distribution has the probability density

$$f_T(x;a,m,b) = \begin{cases} \frac{2}{b-a}\frac{x-a}{m-a} & a<x\leq m \\ \frac{2}{b-a}\frac{b-x}{b-m} & m\leq x<b. \end{cases}$$

and zero elsewhere. This may also be written

$$\frac{2}{(b-a)(m-a)}(x-a)\,H(x-a) - \frac{2}{(m-a)(b-m)}(x-m)\,H(x-m) + \frac{2}{(b-a)(b-m)}(x-b)\,H(x-b).$$

Setting $a=0$, $m=1$, $b=2$ yields the distribution of the sum of two independent $(0,1)$ uniform distributions given on page 2 of Annex B.

If $m=a$ we have

$$f_T(x;a,a,b) = \frac{2}{b-a}\frac{x-a}{b-a}$$

and if $m=b$

$$f_T(x;a,b,b) = \frac{2}{b-a}\frac{b-x}{b-a}.$$

The Laplace transform of the general triangular density is given by

$$L_T(s\,;a,m,b) = \int_0^\infty e^{-sx} f_T(x)\,dx$$

$$= \frac{2}{b-a}\int_a^m e^{-sx}\,\frac{x-a}{m-a}\,dx + \frac{2}{b-a}\int_m^b e^{-sx}\,\frac{b-x}{b-m}\,dx\,.$$

The integration is straightforward and yields

$$L_T(s\,;a,m,b) = \frac{2}{(b-a)(b-m)(m-a)}\,\frac{(b-m)e^{-as} - (b-a)e^{-ms} + (m-a)e^{-bs}}{s^2}\,.$$

If $m=\frac{1}{2}(a+b)$ this reduces to the LT of the sum of two independent, identical uniform distributions over (a,b). Setting $a=0$, $m=1$, $b=2$ yields the LT of the sum of two standard uniform distributions (Annex B).

The arguments of Annex C can be applied to determine the LT of the weighted sum of independent triangular distributions. For this case we no longer assume that the distributions have the same parameters however. Once again the LT will be the product of terms similar to that above.

For example, suppose that we wish to find the pdf of the weighted sum of two triangular distributions with parameters $\{a_j,m_j,b_j\}$. To simplify our notation we assume that the weights have been absorbed into the definitions of the triangular parameters. We can then show that the LT of two triangular distributions is

$$c_1 e^{-(a_1+a_2)s} + c_2 e^{-(a_1+m_2)s} + c_3 e^{-(a_2+m_1)s} + c_4 e^{-(m_1+m_2)s} + c_5 e^{-(a_1+b_2)s}$$
$$+ c_6 e^{-(a_2+b_1)s} + c_7 e^{-(m_1+b_2)s} + c_8 e^{-(m_2+b_1)s} + c_9 e^{-(b_1+b_2)s}$$

where

$$
\begin{aligned}
c_1 &= (b_1-m_1)(b_2-m_2) \\
c_2 &= (b_1-m_1)(b_2-a_2) \\
c_3 &= (b_1-a_1)(b_2-m_2) \\
c_4 &= (b_1-a_1)(b_2-a_2) \\
c_5 &= (b_1-m_1)(m_2-a_2) \\
c_6 &= (m_1-a_1)(b_2-m_2) \\
c_7 &= (b_1-a_1)(m_2-a_2) \\
c_8 &= (m_1-a_1)(b_2-a_2) \\
c_9 &= (m_1-a_1)(m_2-a_2)
\end{aligned}
$$

By inverting the LT we find that the pdf is given by

$$
\begin{aligned}
f_T(x;0,m_j,b_j) = {} & c_1 \frac{(x-\overline{a_1+a_2})^3}{3!} H(.) + c_2 \frac{(x-\overline{a_1+m_2})^3}{3!} H(.) + c_3 \frac{(x-\overline{a_2+m_1})^3}{3!} H(.) \\
& + c_4 \frac{(x-\overline{m_1+m_2})^3}{3!} H(.) + c_5 \frac{(x-\overline{a_1+b_2})^3}{3!} H(.) + c_6 \frac{(x-\overline{a_2+b_1})^3}{3!} H(.) \\
& + c_7 \frac{(x-\overline{m_1+b_2})^3}{3!} H(.) + c_8 \frac{(x-\overline{m_2+b_1})^3}{3!} H(.) + c_9 \frac{(x-\overline{b_1+b_2})^3}{3!} H(.)
\end{aligned}
$$

where *H(.)* indicates that the Heaviside function has the same arguement as the expression immediately preceding it.

If we have *N* variables then there will be a pole of order *2N-1* at the origin when we invert the LT. The calculation is straightforward, albeit messy.[1] The resulting expression is not particularly useful for our purposes.

[1] There are 3^N terms in the numerator, for example. Unless some parameters are identical, all terms will differ.

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Multi-Criteria Decision Problems: Alternatives with Uncertain Scores

4. AUTHORS (last name, first name, middle initial)

Lamb, D.J.

5. DATE OF PUBLICATION (month Year of Publication of document) May 1996	6a. NO OF PAGES (total containing information. Include Annexes, Appendices, etc.) 35	6b. NO OF REFS (total cited in document) 9

7. DESCRIPTIVE NOTES (the category of document, e.g. technical report, technical note or memorandum. If appropriate, enter the type of report e.g. interim, progress, summary, annual or final. Give the inclusive dates when a specific reporting period is covered.)

RESEARCH NOTE

8. SPONSORING ACTIVITY (the name of the department project office or laboratory sponsoring the research and development. Include the address).

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9a. PROJECT OR GRANT NO. (if appropriate, the applicable research and development project or grant number under which the document was written. Please specify whether project or grant.) 3563-41303	9b. CONTRACT NO. (if appropriate, the applicable number under which the document was written.) ---
10a. ORIGINATOR's document number (the official document number by which the document is identified by the originating activity. This number must be unique to this document.) DMS Research Note RN 9602	10b. OTHER DOCUMENT NOS. (Any other numbers which may be assigned this document either by the originator or by the sponsor.) ---

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Analytical expressions for an important class of multi-criteria decision problems are developed. This class represents a linear objective function of alternatives evaluated on a weighted set of numerical scores. The scores are assumed to be values from statistically evaluated on a weighted set of numerical scores. The scores are assumed to be values from statistically independent uniform or triangular distributions. Some illustrative examples are given.

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